$\mathbb{C}$-CONVEXITY IN INFINITE-DIMENSIONAL BANACH SPACES AND APPLICATIONS TO KERGIN INTERPOLATION

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We investigate the concepts of linear convexity and $\mathbb{C}$-convexity in complex Banach spaces. The main result is that any $\mathbb{C}$-convex domain is necessarily linearly convex. This is a complex version of the Hahn-Banach theorem, since it means the following: given a $\mathbb{C}$-convex domain Ω in the Banach space X and a point $p \notin \Omega$, there is a complex hyperplane through p that does not intersect Ω. We also prove that linearly convex domains are holomorphically convex, and that Kergin interpolation can be performed on holomorphic mappings defined in $\mathbb{C}$-convex domains.

1. Introduction

The objective of this paper is to study properties of $\mathbb{C}$-convex and linearly convex sets in general complex Banach spaces. These notions of convexity are natural complex generalizations of ordinary convexity and have been studied quite extensively in $\mathbb{C}^n$.

Recall that there are several equivalent ways to define convexity for a domain $\Omega \subset \mathbb{R}^n$. One can, for instance, require Ω to have contractible intersections with lines, or one can require that through each point in the complement of Ω there passes a hyperplane that does not intersect the set, or that through each point in the boundary of Ω there passes such a hyperplane.

Obviously, domains in $\mathbb{C}^n$ can be convex in this sense, since $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$. However, when the conditions above are given a complex interpretation, new notions of convexity arise.

(1) A domain $\Omega \subset \mathbb{C}^n$ is said to be $\mathbb{C}$-*convex* if its intersection with any complex line is contractible (or empty).

(2) A domain $\Omega \subset \mathbb{C}^n$ is said to be *linearly convex* if through each point in the complement of Ω there passes a complex hyperplane that does not intersect Ω.

(3) A domain $\Omega \subset \mathbb{C}^n$ is said to be *weakly linearly convex* if through each boundary point of Ω there passes a complex hyperplane that does not intersect Ω.

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A quite extensive theory of these complex notions of convexity in $\mathbb{C}^n$ has been developed (cf. the books by Andersson et al. [2] and Hörmander [6]).

It is obvious that ordinary convexity implies ℂ-convexity, and ℂ-convexity has been shown to imply linear convexity. Clearly, linear convexity implies weak linear convexity, which in turn has been shown to imply pseudoconvexity. Since in general none of these implications is an equivalence, there is a whole scale of different notions of convexity in $\mathbb{C}^n$.

Since linear convexity and ℂ-convexity are defined by means of complex lines and hyperplanes, these concepts are equally natural in a general complex Banach space (or, for that matter, any complex locally convex topological vector space, but we will not stress this point). No study of this kind has previously been conducted.

Our aim is to study these convexity notions in this more general setting and to show that to a large extent, the aforementioned implications remain true. The main result is Theorem 4.4, stating that ℂ-convex domains are necessarily linearly convex. This can be thought of as a complex Hahn-Banach theorem; cf. Corollary 4.5. We also show that, as in the finite-dimensional case, Kergin interpolation, a generalized Lagrange-Hermite-type polynomial interpolation, can be performed on holomorphic mappings defined in a ℂ-convex domain.

The organization of the paper is as follows. In Section 2, we make some notational conventions and record a few basic definitions and facts. In Section 3, we discuss linear convexity in the Banach space setting. Section 4 deals with ℂ-convexity in Banach spaces and contains the main results of the paper. Finally, we give applications to Kergin interpolation in Section 5.

2. Notational conventions and fundamental definitions

Throughout this paper, X and Y will denote complex Banach spaces and X^* and Y^* are their dual counterparts. A mapping $P : X \to Y$ is said to be a polynomial of degree d if for all $x \in X$,

$$P(x) = L_0 + L_1x + L_2x^2 + \cdots + L_dx^d, \tag{2.1}$$

where each L_j is a j-linear map from X to Y, and L_jx^j is shorthand notation for $L_j(x,x,\ldots,x)$, the x occurring j times. We let $\mathcal{P}(X,Y)$ denote the space of continuous polynomials; in case $Y = \mathbb{C}$, we simply write $\mathcal{P}(X)$.

If Ω is an open subset of X, then a mapping $f : \Omega \to Y$ is said to be holomorphic if for each $\xi \in \Omega$ there exist an open ball $B_r(\xi)$ of radius r around ξ and a sequence of j-linear maps L_j such that

$$f(x) = \sum_{j=0}^{\infty} L_j(x-\xi)^j, \tag{2.2}$$

uniformly for $x \in B_r(\xi)$. We let $\mathcal{O}(\Omega, Y)$ denote the space of holomorphic mappings of Ω into Y; in case $Y = \mathbb{C}$, we simply write $\mathcal{O}(\Omega)$.

A complex line in X is a set of the kind

$$\ell = \{x \in X : x = a + bz\}, \tag{2.3}$$

where a and b are fixed vectors in X and z ranges over $\mathbb{C}$. A complex hyperplane in X is an affine subspace of complex codimension one, or equivalently, the level set of a continuous linear functional $\alpha \in X^*$, that is, a set of the type

$$H = \{x \in X : \alpha(x) = c\}, \tag{2.4}$$

where c is some fixed complex number. If $c \neq 0$ and β is the functional $c^{-1}\alpha$, then

$$H = \{x \in X : \beta(x) = 1\}, \tag{2.5}$$

and so we can identify hyperplanes in X that do not pass through the origin with points in the dual space X^*. As in the finite-dimensional case, we now introduce the concept of dual complement as follows.

Definition 2.1. Let Ω be a subset of X. The dual complement Ω^* of Ω is defined to be the set of all hyperplanes not intersecting Ω. Assuming that $0 \in \Omega$, Ω^* can be viewed as a subset of X^*, and then

$$\Omega^* = \{\alpha \in X^* : \alpha(x) \neq 1,\ \forall x \in \Omega\}. \tag{2.6}$$

Clearly, if Ω is open, then Ω^* is closed, and if Ω is closed, then Ω^* is open.

3. Linear convexity in Banach spaces

We begin our study of complex convexity notions in Banach spaces by the following definition.

Definition 3.1. An open set $\Omega \subset X$ is said to be linearly convex if for each $x \in X\backslash\Omega$, there exists an affine complex hyperplane H such that $x \in H \subset X\backslash\Omega$. The set Ω is said to be weakly linearly convex if the condition is fulfilled for each $x \in \partial\Omega$.

Since every real hyperplane contains a complex one, every convex set is linearly convex. The converse is not true. The fact that a complex hyperplane has a connected complement makes linear convexity a weaker condition than ordinary convexity.

Linear convexity is preserved under intersections, and so one can define the linearly convex hull of a set Ω as the smallest linearly convex set containing Ω. If X is a reflexive Banach space, then for any set $\Omega \subset X$ we can form the dual complement of the dual complement and end up with a set $\Omega^{**} \subset X$. In this situation, the dual complement of a set is always linearly convex, and Ω^{**} is the linearly convex hull of Ω.

An open set $\Omega \subset X$ is said to be holomorphically convex if for each compact set $K \subset \Omega$, the holomorphic hull

$$\widehat{K}_{\mathbb{O}(\Omega)} := \left\{x \in \Omega : |f(x)| \leq \sup_K |f|,\ \forall f \in \mathbb{O}(\Omega)\right\} \tag{3.1}$$

is compact. Note that in a general Banach space, any domain of holomorphy is holomorphically convex, but the converse is not true. We also introduce the concept of affine hull as follows. Let $\mathbb{C} \oplus X^*$ denote the vector space of all continuous affine forms on X. The affine hull of $K \subset X$ is then defined to be the set

$$\widehat{K}_{\mathbb{C}\oplus X^*} := \left\{ x \in X : |f(x)| \leq \sup_K |f|,\ \forall f \in \mathbb{C} \oplus X^* \right\}. \tag{3.2}$$

This concept will be of use to us in the proof of the following result. The proof is very similar to the proof of the corresponding property in the finite-dimensional case (cf. [2]).

Proposition 3.2. *Every weakly linearly convex open set $\Omega \subset X$ is holomorphically convex.*

Proof. Let K be a compact subset of Ω. We must prove that the holomorphic hull $\widehat{K}_{\mathbb{O}(\Omega)}$ is compact in Ω. Since $\widehat{K}_{\mathbb{O}(\Omega)}$ is clearly contained in $\widehat{K}_{\mathbb{C}\oplus X^*}$ which is easily seen to be compact, it is enough to prove that the closure of $\widehat{K}_{\mathbb{O}(\Omega)}$ is contained in Ω. Let $x_0 \in \partial\Omega$. Since Ω is weakly linearly convex, there is an affine complex linear function f vanishing at x_0 with the property that the zero set of f is contained in the complement of Ω. Now the function $1/f$ is holomorphic in Ω and $|1/f(x)| \leq \sup_K |1/f|$ if $x \in \widehat{K}_{\mathbb{O}(\Omega)}$, and so it follows that x_0 is not in the closure of $\widehat{K}_{\mathbb{O}(\Omega)}$. □

4. ℂ-convexity in Banach spaces

In this section, we turn to ℂ-convexity, which is the principal concept under study in this paper. The definition is as follows.

Definition 4.1. An open set Ω of a complex Banach space X is said to be ℂ-convex if $\Omega \cap \ell$ is a simply connected subset of ℓ for each affine complex line ℓ.

Clearly, all convex sets in X are ℂ-convex, but there are ℂ-convex sets which are not convex. See [2, 6] for (finite-dimensional) examples. We now set out to prove some basic facts about ℂ-convex open sets in Banach spaces.

Proposition 4.2. *Every open ℂ-convex set Ω of a complex Banach space X is simply connected.*

Proof. Ω is connected almost by definition, since any two points x_1 and x_2 in Ω can be joined by an arc contained in the complex line spanned by these two points. To prove that Ω is simply connected, we must take an arbitrary closed curve $\gamma \subset \Omega$ and show that it is homotopic to a point. Since γ is compact, it can be covered by finitely many open balls contained in Ω. Inside each of these balls, we can make a small deformation of γ, and so we may assume that γ is piecewise linear with vertices $v_0, v_1, v_2, \dots, v_N = v_0$. By ℂ-convexity, such a γ is homotopic to any curve passing through these vertices in order and which between v_j and v_{j+1} is contained in the line ℓ_j spanned by these two points.

Clearly, it is enough to show that γ is homotopic to a point in Ω intersected with the finite-dimensional subspace of X spanned by 0 and the vertices $v_0, v_1, \dots, v_{N-1}$. For the rest of the proof, we may therefore assume that X is finite dimensional and use the following argument given by Hörmander [6].

The homotopy class is not changed if the points v_j are moved a little bit to the points v_j' sufficiently close. Indeed, if we insert a path from v_j to v_j' and back to v_j, we get a homotopic path, and the path from v_j' to v_j to v_{j+1} to v_{j+1}' is homotopic to its orthogonal projection on the line L_j' spanned by v_j' and v_{j+1}', if $v_k - v_k'$ is sufficiently small for all k. Thus, the homotopy class is independent of $(v_0, v_1, \dots, v_{N-1}) \in \Omega^N$, for Ω^N is connected. Since we can choose all points in a convex subset of Ω, we conclude that it is equal to 0. □

Proposition 4.3. *Let Ω be a $\mathbb{C}$-convex open set in X. If $T : X \to Y$ is a continuous surjective complex affine map, then $T(\Omega)$ is a $\mathbb{C}$-convex open set in Y. If $S : W \to X$ is a continuous complex affine map and W is a Banach space, then $S^{-1}(\Omega)$ is a $\mathbb{C}$-convex open set in W.*

Proof. It is clear that $S^{-1}(\Omega) \cap \ell$ is connected and simply connected for any complex line ℓ in W, since ℓ is either mapped to a point by S or it is mapped bijectively onto a line. Also, by continuity, $S^{-1}(\Omega)$ is open.

The mapping T is surjective, and so $T(\Omega)$ is open and obviously connected. Let us prove that $T(\Omega)$ is simply connected. Let γ be an arbitrary closed curve in $T(\Omega)$. By Michael's selection theorem (cf. [10]), we can find a closed curve Γ in Ω such that $T(\Gamma) = \gamma$. Since Ω is simply connected by Proposition 4.2, there is a homotopy from Γ to a point in Ω. If we apply T to such a homotopy, we get a homotopy from γ to a point in $T(\Omega)$, and so $T(\Omega)$ is indeed simply connected. Now let ℓ be any complex line in Y. Then $\Omega \cap T^{-1}(\ell)$ is $\mathbb{C}$-convex as a subset of $T^{-1}(\ell)$, hence simply connected. Since $T : T^{-1}(\ell) \to \ell$ is surjective, it follows from what we just proved that $T(\Omega \cap T^{-1}(\ell)) = T(\Omega) \cap \ell$ is connected and simply connected and the proof is complete. □

The fact that $\mathbb{C}$-convexity is preserved under affine maps is, while important in itself, also one of the main ingredients in the proof of the fact that $\mathbb{C}$-convex domains are linearly convex. Since this can be thought of as a complex Hahn-Banach theorem, it is perhaps not surprising that part of the argument is mimicked on the proof of a geometrical version of the classical Hahn-Banach theorem. The proof also uses the corresponding result in finite dimensions: in $\mathbb{C}^n$ all $\mathbb{C}$-convex open sets are linearly convex (cf. [2] or [6]).

Theorem 4.4. *Every open $\mathbb{C}$-convex set Ω of a complex Banach space X is linearly convex.*

Proof. Let a be any point in $X \setminus \Omega$. We need to prove the existence of a complex hyperplane through a that does not cut Ω. We may clearly assume that $a = 0$. Let

$$L := \Big\{M : M \text{ subspace of } X,\ 0 \in M,\ M \cap \Omega = \varnothing\Big\}. \tag{4.1}$$

The set L is partially ordered by inclusion, and if $\{M_\alpha\}$ is a totally ordered subset of L, then $\bigcup_\alpha M_\alpha$ is an upper bound for $\{M_\alpha\}$. Zorn's lemma guarantees the existence of a maximal element $N \in L$. By construction, $0 \in N$ and $N \cap \Omega = \varnothing$. It remains to prove that N is a hyperplane. Suppose it is not, then it is a subspace of X of codimension at least two. Let Z be a two-dimensional linear subspace of X such that $N \cap Z = 0$. By Proposition 4.3,

$\tilde{\Omega} := \Omega \cap (N \oplus Z)$ is ℂ-convex. Also, by Proposition 4.3, the projection

$$\pi : N \oplus Z \longrightarrow Z \tag{4.2}$$

maps $\tilde{\Omega}$ to a ℂ-convex set $\pi(\tilde{\Omega})$ that avoids the origin. Since Z is two dimensional, all its ℂ-convex subsets are linearly convex (cf. [2]), and so there is a complex hyperplane, that is, a complex line ℓ through the origin that does not cut $\pi(\tilde{\Omega})$. But then $(N \cup \pi^{-1}(\ell)) \cap \Omega = \varnothing$ which contradicts the maximality of N. The theorem is proved. □

Considering the definitions of ℂ-convexity and linear convexity, we immediately get the following purely complex version of the Hahn-Banach theorem.

Corollary 4.5. *Given a ℂ-convex domain Ω and a point $p \notin \Omega$, there exists a complex hyperplane passing through p that does not intersect Ω.*

In $\mathbb{C}^n$, it has been proved that all ℂ-convex open sets are polynomially convex (cf. [2, 6]). To generalize this result to the present setting, we first need some definitions.

For any set $K \subset X$, we define its polynomial hull to be the set

$$\widehat{K}_{\mathcal{P}(X)} := \left\{ x \in X : |f(x)| \leq \sup_K |f|,\ \forall f \in \mathcal{P}(X) \right\}, \tag{4.3}$$

and an open set Ω is said to be *polynomially convex* if $\widehat{K}_{\mathcal{P}(X)} \cap \Omega$ is compact for every compact $K \subset \Omega$. An open set $\Omega \subset X$ is said to be *Runge* if $\mathcal{P}(X)$ is dense in $\mathcal{O}(\Omega)$ with respect to the compact-open topology. Also, Ω is said to be *finitely polynomially convex (resp., finitely Runge)* if $\Omega \cap M$ is polynomially convex (resp., Runge) for each finite-dimensional subspace M. In general, these concepts are different, but Proposition 4.6 gives a sufficient condition for them to coincide.

Recall that the Banach space X is said to have *the approximation property* if for each compact set $K \subset X$ and $\epsilon > 0$, there is a continuous linear operator T on X with finite-dimensional range such that $\|T(x) - x\| < \epsilon$ for every $x \in K$.

Proposition 4.6. *Let X be a Banach space with the approximation property and Ω a holomorphically convex open subset of X. Then the following conditions are equivalent.*

(1) Ω *is polynomially convex.*
(2) Ω *is finitely polynomially convex.*
(3) Ω *is finitely Runge.*
(4) Ω *is Runge.*

Proof. This was proved by Aron and Schottenloher, extending results by Dineen and Noverraz (cf. [9]). □

Now we are ready to prove the last result of this section.

Proposition 4.7. *Let X be a complex Banach space with the approximation property. Then every ℂ-convex open set $\Omega \subset X$ is polynomially convex and Runge.*

Proof. Clearly, $\Omega \cap M$ is ℂ-convex for each finite-dimensional subspace M of X. Since in the finite-dimensional case ℂ-convexity implies polynomial convexity, our set Ω is finitely polynomially convex. By Proposition 3.2, Ω is also holomorphically convex, and so by an application of Proposition 4.6, we are done. $\square$

5. Applications to Kergin interpolation

In [5], it is proved that if f is a C^k mapping from an open convex subset U of a Banach space X into a Banach space Y, for each sequence of points $p = (p_0, p_1, \dots, p_k)$ in U, one can associate to f a polynomial $K_p f$ of degree at most k with the following properties.

(1) $K_p f(p_j) = f(p_j)$, $j = 0, 1, \dots, k$.
(2) If $p \subset q$, then $K_p f = K_p K_q f$.
(3) For polynomials f of degree at most k, $K_p f = f$.
(4) For any continuous affine map A, $K_p(g \circ A) = (K_{Ap} g) \circ A$.

Moreover, the mapping $f \mapsto K_p f$ is continuous and independent of the ordering of the points in the sequence p. The polynomial $K_p f$ is called the Kergin polynomial and is given by the explicit formula

$$K_p f(x) = f(p_0) + \int_{[p_0, p_1]} D_{x-p_0} f + \cdots + \int_{[p_0, p_1, \dots, p_k]} D_{x-p_{k-1}} \cdots D_{x-p_0} f, \tag{5.1}$$

where D_y denotes the directional derivative of f in the direction y, and the integrals are the so-called simplex functionals defined by

$$\int_{[p_0, p_1, \dots, p_j]} g := \int_{S_j} g(p_0 + s_1(p_1 - p_0) + \cdots + s_j(p_j - p_0)) \, ds_1 \cdots ds_j. \tag{5.2}$$

Here $S_j = \{(s_1, \dots, s_j) : s_i \geq 0, \sum s_i \leq 1\}$ is the standard j-simplex in $\mathbb{R}^j$ and the integral on the right is a Bochner integral with respect to Lebesgue measure. Note that these simplex functionals take values in a Banach space, hence they are not functionals in the usual sense (cf. [5] for more details about this). See also [1, 3, 7, 8] for finite-dimensional results.

We will now prove that, in the complex case, Kergin interpolation can be extended to mappings holomorphic in $\mathbb{C}$-convex domains. This was done in [1] for the finite-dimensional case, and we will adopt their method to the present situation.

To begin with, we need to define an extension of the simplex functional to any $\mathbb{C}$-convex domain Ω. The construction is as follows.

Let $\Omega \subset X$ be a $\mathbb{C}$-convex domain and let $p = (p_0, p_1, \dots, p_k)$ be a sequence of points in Ω. Denote the standard j-simplex by S_j, its vertices by $v_0, v_1, \dots, v_k$, and for each $j \leq k$, let Ω_j be the intersection of Ω with the complex affine space spanned by $p_0, p_1, \dots, p_j$. Also, let ω_j be the preimage of Ω_j under the complex affine map $\mathbb{C}^j \to X$ taking each v_i to p_i (we use the standard inclusion $\mathbb{R}^j \subset \mathbb{C}^j$). It turns out that ω_j is again $\mathbb{C}$-convex. Finally, introduce singular chains $\gamma_j : S_j \to \omega_j$ mapping every face of S_j into the complex $(j-1)$-plane which it spans. This is possible by $\mathbb{C}$-convexity, and each v_i remains fixed.

Definition 5.1. With the notation introduced above, the complex simplex functional is defined to be

$$g \longmapsto \int_{[p_0, p_1, \dots, p_j]} g := \int_{\gamma_j} g(p_0 + \lambda_1(p_1 - p_0) + \cdots + \lambda_j(p_j - p_0)) \, d\lambda_1 \wedge \cdots \wedge d\lambda_j. \tag{5.3}$$

Observe that the complex simplex functional depends on the domain Ω. The following properties of the complex simplex functional are useful.

Proposition 5.2. *For $p = (p_0, p_1, \ldots, p_j)$ in the $\mathbb{C}$-convex domain $\Omega \subset X$, the complex simplex functional*

$$\int_{[p]} : \mathcal{O}(\Omega, Y) \longrightarrow Y, \tag{5.4}$$

defined above, is independent of the particular choice of chain γ_j in ω_j. Moreover,

(1) *it is independent of the order of the points in the sequence p,*

(2) *it is invariant under complex affine mappings, that is, if $A : W \to X$ is such a map, W is another Banach space, then $\int_{[p]} g \circ A = \int_{[Ap]} g$, where $Ap = (A(p_0), A(p_1), \ldots, A(p_j))$.*

Proof. If the target space Y is $\mathbb{C}$, then everything follows from the corresponding result in [1], since when we evaluate the simplex functional, we only take into account the behavior of g on a subspace of (the finite-dimensional) affine subspace of X spanned by the points in the sequence p. If Y is any Banach space, then the proposition is true if we replace g by $\psi \circ g$, where ψ is any continuous linear functional on Y. Since the Bochner integral has the property that, for all $\psi \in Y^*$,

$$\psi\left(\int_E g\, d\mu\right) = \int_E \psi \circ g\, d\mu, \tag{5.5}$$

where E is any measurable set (cf. [4]), and the continuous linear functionals separate points, the proposition follows in the general case. □

Now we can define the Kergin polynomial in this setting and state and prove our theorem on Kergin interpolation of holomorphic mappings on $\mathbb{C}$-convex domains in Banach spaces.

Definition 5.3. Let Ω be a $\mathbb{C}$-convex domain in X and $p = (p_0, p_1, \ldots, p_k)$ a sequence of points in Ω. Then, for any $f \in \mathcal{O}(\Omega)$, the Kergin polynomial $K_p f$ of f with respect to the points p is defined to be

$$K_p f(x) = f(p_0) + \int_{[p_0, p_1]} D_{x-p_0} f + \cdots + \int_{[p_0, p_1, \ldots, p_k]} D_{x-p_{k-1}} \cdots D_{x-p_0} f. \tag{5.6}$$

Theorem 5.4. *Let Ω be a $\mathbb{C}$-convex domain in X and $p = (p_0, p_1, \ldots, p_k)$ a sequence of points in Ω. Then, for any $f \in \mathcal{O}(\Omega)$, the Kergin polynomial $K_p f$ defined above is a polynomial such that*

$$K_p f(p_j) = f(p_j), \quad j = 0, 1, \ldots, k. \tag{5.7}$$

Moreover, the mapping $f \mapsto K_p f$ is continuous in the compact-open topology and has the following properties.

(1) *It is independent of the ordering of the points in the sequence.*

(2) *It is associative, in the sense that, if $p \subset q$, $K_p f = K_p K_q f$.*

(3) *It is affine invariant, meaning that if* $f = g \circ A$ *for some continuous affine map* A *and holomorphic function* g, *then* $K_p f = K_p(g \circ A) = (K_{Ap} g) \circ A$, *where as before* $Ap = (A(p_0), \dots, A(p_k))$.
(4) *It is a projection, that is, if* f *is itself a polynomial of degree at most* k, *then* $K_p f = f$.

Proof. In case $Y = \mathbb{C}$, all assertions follow from the results in [1], since everything takes place in the convex hull of the points in the sequence p which is contained in a finite-dimensional subspace of X. Suppose that Y is any complex Banach space. Then $K_p f$ is a polynomial of degree k if and only if $\psi \circ K_p f$ is a polynomial of degree at most k for each $\psi \in Y^*$. Also, for any such ψ,

$$\psi \circ K_p f(p_j) = K_p(\psi \circ f)(p_j) = \psi \circ f(p_j), \quad j = 0, 1, \dots, k. \tag{5.8}$$

It follows that $K_p f(p_j) = f(p_j)$ for all j since the functionals separate points of Y. The properties (1)–(4) follow in a similar manner from their classical counterparts and the properties of the continuous linear functionals in Y^*. □

Remark 5.5. We want to point out that there is a concept called c-convexity in the theory of normed linear spaces. Such a space X is said to be uniformly c-convex if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|y\| < \epsilon$ whenever $x, y \in X$, $\|x\| = 1$, $\|x + \lambda y\| \le 1$ for all complex numbers λ with $|\lambda| \le 1$. This concept of c-convexity is not related to our concept of $\mathbb{C}$-convexity.

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