ON A CHARACTERIZATION OF THE LATTICE OF SUBSYSTEMS OF A TRANSITION SYSTEM

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It was first proved by Birkhoff and Frink, and the result now belongs to the folklore, that any algebraic lattice is up to isomorphism the lattice of subuniverses of a universal algebra. A study of subsystems of a transition system yields a new algebraic concept, that of a strongly algebraic lattice. We give here a representation theorem to the manner of Birkhoff and Frink of such lattices.

A transition system is a pair $(S, \longrightarrow_S)$, where

(i) S is a set of states,

(ii) $\longrightarrow_S \subseteq S \times S$ is the transition relation.

We write $s \longrightarrow_S s'$ for $(s,s') \in \longrightarrow_S$.

Nondeterministic transition systems, those $(S, \longrightarrow_S)$ for which the set of successors of any element $s \in S$ is an arbitrary set, are easily seen to be coalgebras of the covariant powerset functor $\mathcal{P}$: Sets → Sets from the category of sets to itself.

Observe that any unary algebra $(S, \mathcal{F})$ gives rise to a unique transition system $(S, \longrightarrow_S)$, but the converse in the general case is false.

A subsystem of a transition system $(S, \longrightarrow_S)$ is a subset X of S which has the following stability property: $s \longrightarrow_S s'$ and $s \in X$ imply $s' \in X$. The empty set and the universe S are subsystems of $(S, \longrightarrow_S)$, they are said to be trivial. It is straightforward to check that the set Subs(S) of subsystems of $(S, \longrightarrow_S)$ is stable for arbitrary unions and intersections. Given a subset X of S, we denote by $\langle X \rangle$ the subsystem of $(S, \longrightarrow_S)$ generated by X. It is the intersection of all subsystems of $(S, \longrightarrow_S)$ containing X. The notation $\overset{*}{\longrightarrow}_S$ will be used to denote the reflexive and transitive closures of the binary relation $\longrightarrow_S$ on S. The

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subsystem $\langle X\rangle$ is then characterized as follows:

$$\langle X\rangle = \{x' \in X : \exists x \in X,\ x \xrightarrow[S]{*} x'\}. \tag{1}$$

Hence for $s \in S$, writing $\langle s\rangle$ the subsystem $\langle\{s\}\rangle$, we get

$$\langle s\rangle = \{s' \in S : s \xrightarrow[S]{*} s'\}. \tag{2}$$

The mapping $\langle -\rangle : \mathcal{P}(S) \to \mathcal{P}(S)$ defined from the set of subsets of S to itself is a closure operator on S. The previous characterization of $\langle X\rangle$ permits to see that

$$\langle X\rangle = \{x' \in S : \exists x \in X,\ x' \in \langle x\rangle\} = \bigcup_{x\in X} \langle x\rangle. \tag{3}$$

We say that the closure operator $\langle -\rangle$ is completely additive. One can notice that

(i) subsystems $\langle s\rangle$ of $(S, \xrightarrow[S]{})$, $s \in S$, satisfy the following finiteness condition: for all families $(X_i,\ i \in I)$ of subsystems of $(S, \xrightarrow[S]{})$ if $\langle s\rangle \subseteq \bigcup_{i\in I} X_i$, then there exists an index $i_0 \in I$ such that $\langle s\rangle \subseteq X_{i_0}$,

(ii) $\langle s'\rangle \subseteq \langle s\rangle$ if and only if $s \xrightarrow[S]{*} s'$.

These observations prompt us to initiate the following definitions.

Definition 1. Let $(E, \leq)$ be an ordered set which admits arbitrary suprema. An element a in E is called s-compact (s for strongly compact), if for all covering $a \leq \bigvee_{i\in I} a_i$ of a there exists an index i for which $a \leq a_i$.

Consider a sup-complete lattice $(E, \leq)$ (i.e., an ordered set admitting arbitrary suprema). As a poset, $(E, \leq)$ can be viewed as a cocomplete category. Let a be in E, it is equivalent to say that a is s-compact or in categorical terms, every morphism $f : a \to \mathrm{colim}_I a_i$ factors uniquely into a morphism $\overline{f} : a \to a_i$ (for some $i \in I$). This means that the covariant hom-functor $[a, -]$ preserves all (small) colimits. Such an object a is called absolutely presentable (see [2]).

Definition 2. A sup-complete lattice $(L, \leq)$ is called s-algebraic (or strongly algebraic), if each element a of L can be written as supremum of s-compact elements less than a.

Any s-algebraic lattice is obviously algebraic, but the converse is not true. In fact given a group $(G, *)$, the lattice $(\mathrm{Sg}(G), \subseteq)$ of subgroups of G is algebraic (see [1]). Further algebraic elements in $(\mathrm{Sg}(G))$ are finitely generated subgroups of G. It is easy to verify that $(\mathrm{Sg}(\mathbb{Z}, +), \subseteq)$ the lattice of subgroups of the additive group $(\mathbb{Z}, +)$ is not s-algebraic.

Consider the sup-complete lattice $(L, \leq)$ as a cocomplete category; it will be called s-algebraic if every element in L is a colimit of absolutely presentable objects in L. Hence an s-algebraic lattice viewed as a category is locally absolutely presentable with the set of s-compact elements as set of absolutely presentable objects.

The basic example is that of a complete lattice of subsystems of a transition system; this seems also to be a generic s-algebraic lattice as shown by the following representation theorem.

Theorem 3. *Let* $(L, \leq)$ *be an s-algebraic lattice. There exist a transition system* $(S, \underset{S}{\longrightarrow})$ *and an isomorphism from L onto the lattice* $\mathrm{Subs}(S)$ *of subsystems of* $(S, \underset{S}{\longrightarrow})$.

Proof. We denote by S the set of s-compact elements of L. Define on S a binary relation $\underset{S}{\longrightarrow}$ as follows: for all $a, b \in S$, $a \underset{S}{\longrightarrow} b$ if and only if $b \leq a$. Let $\downarrow x$ be the set of elements $x' \in L$ such that $x' \leq x$. For all x in L, the set $S \cap \downarrow x$ of s-compact elements less than x is a subsystem of S. In fact if $a \underset{S}{\longrightarrow} b$ and $a \in S \cap \downarrow x$, then we have $b \leq a$, hence $b \in S \cap \downarrow x$. On deduces the mapping

$$\psi : L \longrightarrow \mathrm{Subs}(S), \qquad x \longmapsto S \cap \downarrow x. \tag{4}$$

Let us check that ψ is order preserving and reflecting. To this end, let us consider x and x' in L. If $x \leq x'$, then $\downarrow x \subseteq \downarrow x'$ and therefore $S \cap \downarrow x \subseteq S \cap \downarrow x'$, that is, $\psi(x) \subseteq \psi(x')$. Conversely if $\psi(x) \subseteq \psi(x')$, since each element of L can be written as a supremum of s-compact elements less than itself, we have $x = \bigvee \psi(x) \leq \bigvee \psi(x') = x'$.

Finally let us show that ψ is a one-to-one mapping by exhibiting its inverse. For that set the mapping

$$\phi : \mathrm{Subs}(S) \longrightarrow L, \qquad X \longmapsto \bigvee X. \tag{5}$$

For all $x \in L$, we have $\phi\psi(x) = \bigvee\{a \mid a \text{ is } s\text{-compact and } a \leq x\} = x$. Further, for all subsystem X of $(S, \underset{S}{\longrightarrow})$,

$$\psi\phi(X) = \psi(\vee X) = S \bigcap \downarrow \bigvee X. \tag{6}$$

It is clear that $X \subseteq S \bigcap \downarrow \bigvee X$. Let $a \in L$ such that $a \leq \vee X$ and $a \in S$. By s-compacity of a, there exists $x \in X$ such that $a \leq x$, that is, $x \underset{S}{\longrightarrow} a$ by definition. Since X is a subsystem of $(S, \underset{S}{\longrightarrow})$ and $x \in X$, we obtain $a \in X$. One deduces the inclusion $S \cap \downarrow \bigvee X \subseteq X$ which induces the equality $S \cap \downarrow \bigvee X = X$, hence $\psi\phi(X) = X$.

The fact that ψ preserves arbitrary suprema follows from the fact that each order isomorphism between complete lattices is automatically an isomorphism of complete lattice. The theorem is proved. □

Since the s-algebraic lattice $(L, \leq)$ as a poset is a locally absolutely presentable category, it is isomorphic to the free cocompletion $[S^0, \mathrm{Set}]$ of the set S of s-compact elements, under all (small) colimits. This free cocompletion is, of course, isomorphic to the lattice of down-closed subsets of S which are precisely the subsystems of S. Therefore Theorem 3 gives a theoretical lattice version of the categorical well-known result stating that: every locally absolutely presentable category is isomorphic to the presheaf category.

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References

[1] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics, vol. 78, Springer, New York, 1981.
[2] C. Centazzo, J. Rosický, and E. M. Vitale, *A characterization of locally D-presentable categories*, Cahiers de Topologie et Géométrie Différentielle Catégoriques **45** (2004), no. 2, 141–146.

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