ON WEAK-OPEN COMPACT IMAGES OF METRIC SPACES

ZHIYUN YIN

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We give some characterizations of weak-open compact images of metric spaces.

1. Introduction and definitions

To find internal characterizations of certain images of metric spaces is one of central problems in general topology. Arhangel'skiĭ [1] showed that a space is an open compact image of a metric space if and only if it has a development consisting of point-finite open covers, and some characterizations for certain quotient compact images of metric spaces are obtained in [3, 5, 8]. Recently, Xia [12] introduced the concept of weak-open mappings. By using it, certain g-first countable spaces are characterized as images of metric spaces under various weak-open mappings. Furthermore, Li and Lin in [4] proved that a space is g-metrizable if and only if it is a weak-open σ-image of a metric space.

The purpose of this paper is to give some characterizations of weak-open compact images of metric spaces, which showed that a space is a weak-open compact image of a metric space if and only if it has a weak development consisting of point-finite *cs*-covers.

In this paper, all spaces are Hausdorff, all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers. $\tau(X)$ denotes the topology on a space X. For the usual product space $\prod_{i\in\mathbb{N}} X_i$, π_i denotes the projection $\prod_{i\in\mathbb{N}} X_i$ onto X_i. For a sequence $\{x_n\}$ in X, denote $\langle x_n\rangle = \{x_n : n \in \mathbb{N}\}$.

Definition 1.1 [1]. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a collection of subsets of a space X. $\mathcal{P}$ is called a weak base for X if

(1) for each $x \in X$, $\mathcal{P}_x$ is a network of x in X,

(2) if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$,

(3) $G \subset X$ is open in X if and only if for each $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

$\mathcal{P}_x$ is called a weak neighborhood base of x in X, every element of $\mathcal{P}_x$ is called a weak neighborhood of x in X.

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Definition 1.2. Let $f : X \to Y$ be a mapping.

(1) f is called a weak-open mapping [12], if there exists a weak base $\mathcal{B} = \cup\{\mathcal{B}_y : y \in Y\}$ for Y, and for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ satisfying the following condition: for each open neighborhood U of x_y, $B_y \subset f(U)$ for some $B_y \in \mathcal{B}_y$.

(2) f is called a compact mapping, if $f^{-1}(y)$ is compact in X for each $y \in Y$.

It is easy to check that a weak-open mapping is quotient.

Definition 1.3 [2]. Let X be a space, and $P \subset X$. Then the following hold.

(1) A sequence $\{x_n\}$ in X is called eventually in P, if the $\{x_n\}$ converges to x, and there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \geq m\} \subset P$.

(2) P is called a sequential neighborhood of x in X, if whenever a sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is eventually in P.

(3) P is called sequential open in X, if P is a sequential neighborhood at each of its points.

(4) X is called a sequential space, if any sequential open subset of X is open in X.

Definition 1.4 [7]. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a *cs*-cover for X, if every convergent sequence in X is eventually in some element of $\mathcal{P}$.

(2) $\mathcal{P}$ is called an *sn*-cover for X, if every element of $\mathcal{P}$ is a sequential neighborhood of some point in X, and for any $x \in X$, there exists a sequential neighborhood P of x in X such that $P \in \mathcal{P}$.

Definition 1.5 [7]. Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X.

(1) $\{\mathcal{P}_n\}$ is called a point-star network for X, if for each $x \in X$, $\langle st(x,\mathcal{P}_n)\rangle$ is a network of x in X.

(2) $\{\mathcal{P}_n\}$ is called a weak development for X, if for each $x \in X$, $\langle st(x,\mathcal{P}_n)\rangle$ is a weak neighborhood base for X.

2. Results

Theorem 2.1. *The following are equivalent for a space X.*

(1) X is a weak-open compact image of a metric space.

(2) X has a weak development consisting of point-finite cs-covers.

(3) X has a weak development consisting of point-finite sn-covers.

Proof. (1)⇒(2). Suppose that $f : M \to X$ is a weak-open compact mapping with M a metric space. Let $\{\mathcal{U}_n\}$ be a sequence consisting of locally finite open covers of M such that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ and $\langle st(K,\mathcal{U}_n)\rangle$ forms a neighborhood base of K in M for each compact subset K of M (see [7, Theorem 1.3.1]). For each $n \in \mathbb{N}$, put $\mathcal{P}_n = f(\mathcal{U}_n)$. Since f is compact, then $\{\mathcal{P}_n\}$ is a point-finite cover sequence of X.

If $x \in V$ with V open in X, then $f^{-1}(x) \subset f^{-1}(V)$. Since $f^{-1}(x)$ compact in M, then $st(f^{-1}(x),\mathcal{U}_n) \subset f^{-1}(V)$ for some $n \in \mathbb{N}$, and so $st(x,\mathcal{P}_n) \subset V$. Hence $\langle st(x,\mathcal{P}_n)\rangle$ forms a network of x in X. Therefore, $\{\mathcal{P}_n\}$ is a point-star network for X.

We will prove that every $\mathcal{P}_k$ is a *cs*-cover for X. Since f is weak-open, there exists a weak base $\mathcal{B} = \cup\{\mathcal{B}_x : x \in X\}$ for X, and for each $x \in X$, there exists $m_x \in f^{-1}(x)$

satisfying the following condition: for each open neighborhood U of m_x in M, $B \subset f(U)$ for some $B \in \mathcal{B}_x$.

For each $x \in X$ and $k \in \mathbb{N}$, let $\{x_n\}$ be a sequence converging to a point $x \in X$. Take $U \in \mathcal{U}_k$ with $m_x \in U$. Then $B \subset f(U)$ for some $B \in \mathcal{B}_x$. Since B is a weak neighborhood of x in X, then B is a sequential neighborhood of x in X by [6, Corollary 1.6.18], so $f(U) \in \mathcal{P}_k$ is also. Thus $\{x_n\}$ is eventually in $f(U)$. This implies that each $\mathcal{P}_k$ is a *cs*-cover for X. Since $f(U)$ is a sequential neighborhood of x in X, then $st(x,\mathcal{P}_k)$ is also. Obviously, X is a sequential space. So $\langle st(x,\mathcal{P}_k)\rangle$ is a weak neighborhood base of x in X.

In words, $\{\mathcal{P}_n\}$ is a weak development consisting of point-finite *cs*-covers for X.

(2)⇒(3). By Theorem A in [5], X is a sequential space. It suffices to prove that if $\mathcal{P}$ is a point-finite *cs*-cover for X, then some subset of $\mathcal{P}$ is an *sn*-cover for X. For each $x \in X$, denote $(\mathcal{P})_x = \{P_i : i \le k\}$, where $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$. If each element of $(\mathcal{P})_x$ is not a sequential neighborhood of x in X, then for each $i \le k$, there exists a sequence $\{x_{in}\}$ converging to x such that $\{x_{in}\}$ is not eventually in P_i. For each $n \in \mathbb{N}$ and $i \le k$, put $y_{i+(n-1)k} = x_{in}$, then $\{y_m\}$ converges to x and is not eventually in each P_i, a contradiction. Thus there exists $P_x \in \mathcal{P}$ such that P_x is a sequential neighborhood of x in X. Put $\mathcal{F} = \{P_x : x \in X\}$, then $\mathcal{F}$ is an *sn*-cover for X.

(3)⇒(1). Suppose $\{\mathcal{P}_n\}$ is a weak development consisting of point-finite *sn*-covers for X. For each $i \in \mathbb{N}$, let $\mathcal{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$, endow Λ_i with the discrete topology, then Λ_i is a metric space. Put

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \langle P_{\alpha_i}\rangle \text{ forms a network at some point } x_\alpha \text{ in } X\right\}, \tag{2.1}$$

and endow M with the subspace topology induced from the usual product topology of the collection $\{\Lambda_i : i \in \mathbb{N}\}$ of metric spaces, then M is a metric space. Since X is Hausdorff, x_α is unique in X. For each $\alpha \in M$, we define $f : M \to X$ by $f(\alpha) = x_\alpha$. For each $x \in X$ and $i \in \mathbb{N}$, there exists $\alpha_i \in \Lambda_i$ such that $x \in P_{\alpha_i}$. From $\{\mathcal{P}_i\}$ being a point-star network for X, $\{P_{\alpha_i} : i \in \mathbb{N}\}$ is a network of x in X. Put $\alpha = (\alpha_i)$, then $\alpha \in M$ and $f(\alpha) = x$. Thus f is surjective. Suppose $\alpha = (\alpha_i) \in M$ and $f(\alpha) = x \in U \in \tau(X)$, then there exists $n \in \mathbb{N}$ such that $P_{\alpha_n} \subset U$. Put

$$V = \{\beta \in M : \text{the } n\text{th coordinate of } \beta \text{ is } \alpha_n\}. \tag{2.2}$$

Then $\alpha \in V \in \tau(M)$, and $f(V) \subset P_{\alpha_n} \subset U$. Hence f is continuous.

For each $x \in X$ and $i \in \mathbb{N}$, put

$$B_i = \{\alpha_i \in \Lambda_i : x \in P_{\alpha_i}\}, \tag{2.3}$$

then $\prod_{i\in\mathbb{N}} B_i$ is compact in $\prod_{i\in\mathbb{N}} \Lambda_i$. If $\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} B_i$, then $\langle P_{\alpha_i}\rangle$ is a network of x in X. So $\alpha \in M$ and $f(\alpha) = x$. Hence $\prod_{i\in\mathbb{N}} B_i \subset f^{-1}(x)$; If $\alpha = (\alpha_i) \in f^{-1}(x)$, then $x \in \bigcap_{i\in\mathbb{N}} P_{\alpha_i}$, so $\alpha \in \prod_{i\in\mathbb{N}} B_i$. Thus $f^{-1}(x) \subset \prod_{i\in\mathbb{N}} B_i$. Therefore, $f^{-1}(x) = \prod_{i\in\mathbb{N}} B_i$. This implies that f is a compact mapping.

We will prove that f is weak-open. For each $x \in X$, since every $\mathcal{P}_i$ is an sn-cover for X, then there exists $\alpha_i \in \Lambda_i$ such that P_{α_i} is a sequential neighborhood of x in X. From $\{\mathcal{P}_i\}$ a point-star network for X, $\langle P_{\alpha_i} \rangle$ is a network of x in X. Put $\beta_x = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i$, then $\beta_x \in f^{-1}(x)$.

Let $\{U_{m\beta_x}\}$ be a decreasing neighborhood base of β_x in M, and put

$$\mathcal{B}_x = \{f(U_{m\beta_x}) : m \in \mathbb{N}\},$$
$$\mathcal{B} = \bigcup \{\mathcal{B}_x : x \in X\}, \tag{2.4}$$

then $\mathcal{B}$ satisfies (1), (2) in Definition 1.1. Suppose G is open in X. For each $x \in G$, from $\beta_x \in f^{-1}(x)$, $f^{-1}(G)$ is an open neighborhood of β_x in M. Thus $U_{m\beta_x} \subset f^{-1}(G)$ for some $m \in \mathbb{N}$, so $f(U_{m\beta_x}) \subset G$ and $f(U_{m\beta_x}) \in \mathcal{B}_x$. On the other hand, suppose $G \subset X$ and for $x \in G$, there exists $B \in \mathcal{B}_x$ such that $B \subset G$. Let $B = f(U_{m\beta_x})$ for some $m \in \mathbb{N}$, and let $\{x_n\}$ be a sequence converging to x in X. Since P_{α_i} is a sequential neighborhood of x in X for each $i \in \mathbb{N}$, then $\{x_n\}$ is eventually in P_{α_i}. For each $n \in \mathbb{N}$, if $x_n \in P_{\alpha_i}$, let $\alpha_{in} = \alpha_i$; if $x_n \notin P_{\alpha_i}$, pick $\alpha_{in} \in \Lambda_i$ such that $x_n \in P_{\alpha_i n}$. Thus there exists $n_i \in \mathbb{N}$ such that $\alpha_{in} = \alpha_i$ for all $n > n_i$. So $\{\alpha_{in}\}$ converges to α_i. For each $n \in \mathbb{N}$, put

$$\beta_n = (\alpha_{in}) \in \prod_{i\in\mathbb{N}} \Lambda_i, \tag{2.5}$$

then $f(\beta_n) = x_n$ and $\{\beta_n\}$ converges to β_x. Since $U_{m\beta_x}$ is an open neighborhood β_x in M, then $\{\beta_n\}$ is eventually in $U_{m\beta_x}$, so $\{x_n\}$ is eventually in G. Hence G is a sequential neighborhood of x. So G is sequential open in X. By X being a sequential space, G is open in X. This implies $\mathcal{B}$ is a weak base for X.

By the idea of $\mathcal{B}$, f is weak-open. □

We give examples illustrating Theorem 2.1 of this note.

Example 2.2. Let X be the Arens space S_2 (see [6, Example 1.8.6]). It is not difficult to see that the space is a weak-open compact image of a metric space. But X is not an open compact image of a metric space, because X is not developable. Thus the following holds.

A weak-open compact image of a metric space is not always an open compact image of a metric space.

Example 2.3. Let Y be the weak Cauchy space in [10, Example 2.14(3)]. By the construction, Y is a quotient compact image of a metric space. But Y is not Cauchy, Y is not a weak-open compact image of a metric space by Theorem 2.1. Thus the following holds:

A quotient compact image of a metric space is not always a weak-open compact image of a metric space.

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Zhiyun Yin: Department of Information, Hunan Business College, Changsha,
Hunan 410205, China
E-mail address: yzy8846@163.com