INTEGRAL MEANS OF CERTAIN MULTIVALENT FUNCTIONS

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For analytic and multivalent functions $f(z)$ and $p(z)$ in the open unit disk $\mathbb{U}$, a subordination theorem due to Littlewood (1925) which was called *the integral mean* is applied. Some simple examples for our results are also considered.

1. Introduction

Let $\mathcal{A}_{p,n}$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic and multivalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ belonging to $\mathcal{A}_{p,n}$ is called multivalently starlike of order α in $\mathbb{U}$ if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.2}$$

for some $\alpha(0 \leq \alpha < p)$. Also, a function $f(z) \in \mathcal{A}_{p,n}$ is said to be multivalently convex of order α in $\mathbb{U}$ if it satisfies the inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.3}$$

for some $\alpha(0 \leq \alpha < p)$. We denote by $\mathcal{S}^*_{p,n}(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$ the class of functions $f(z) \in \mathcal{A}_{p,n}$ which are multivalently starlike of order α and multivalently convex of order α, respectively. We note that

$$f \in \mathcal{K}_{p,n}(\alpha) \Longleftrightarrow \frac{zf'}{p} \in \mathcal{S}^*_{p,n}(\alpha). \tag{1.4}$$

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For functions $f(z)$ belonging to the classes $\mathcal{S}^*_{p,n}(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$, Owa [4] has shown the following coefficient inequalities.

Theorem 1.1. *If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} (k-\alpha)|a_k| \leq p-\alpha \tag{1.5}$$

for some $\alpha(0 \leq \alpha < p)$, then $f(z) \in \mathcal{S}^*_{p,n}(\alpha)$.

Theorem 1.2. *If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies*

$$\sum_{k=p+n}^{\infty} k(k-\alpha)|a_k| \leq p(p-\alpha) \tag{1.6}$$

for some $\alpha(0 \leq \alpha < p)$, then $f(z) \in \mathcal{K}_{p,n}(\alpha)$.

For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, $f(z)$ is said to be *subordinate* to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad \text{(cf. Duren [1])}. \tag{1.7}$$

The following subordination theorem will be required in our present investigation.

Theorem 1.3 (Littlewood [3]). *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |g(z)|^{\mu} d\theta. \tag{1.8}$$

Applying Theorem 1.3 by Littlewood [3], Sekine et al. [5] have considered some integral means inequalities for certain analytic functions. Furthermore, Güney et al. [2] have studied integral means inequalities for multivalent functions.

In the present paper, we discuss the integral means inequalities for multivalent functions which are the generalization of the paper by Güney et al. [2].

2. Integral means inequalities for $f(z)$ and $p(z)$

In this section, we discuss the integral means inequalities for $f(z) \in \mathcal{A}_{p,n}$ and $p(z)$ defined by

$$p(z) = z^p + \sum_{s=1}^{m} b_{sj-(s-1)p} z^{sj-(s-1)p} \quad (j \geqq n+p,\ n \in \mathbb{N},\ m \geq 2). \tag{2.1}$$

We begin by giving the following lemma due to Sekine et al. [5].

Lemma 2.1. *Let $P_m(t)$ denote the polynomial of degree m of the form*

$$P_m(t) = c_1 t^m - c_2 t^{m-1} - \cdots - c_{m-1} t^2 - c_m t - d \quad (t \geqq 0), \tag{2.2}$$

where c_i $(i = 1,2,\dots,m)$ *are arbitrary positive constants and* $d \geqq 0$*. Then* $P_m(t) = 0$ *has unique solution for* $t > 0$*. If the solution is given by* t_0*,* $P_m(t) < 0$ *for* $0 < t < t_0$ *and* $P_m(t) > 0$ *for* $t > t_0$.

Our first result for integral means is contained in the following.

Theorem 2.2. *Let* $f(z) \in \mathcal{A}_{p,n}$ *and let* $p(z)$ *be given by (2.1). If* $f(z)$ *satisfies*

$$\sum_{k=p+n}^{\infty} |a_k| \leqq |b_{mj-(m-1)p}| - \sum_{s=1}^{m-1} |b_{sj-(s-1)p}| \tag{2.3}$$

with

$$\sum_{s=1}^{m-1} |b_{sj-(s-1)p}| < |b_{mj-(m-1)p}|, \tag{2.4}$$

and there exists an analytic function $w(z)$ *such that*

$$\sum_{s=1}^{m} b_{sj-(s-1)p} w(z)^{s(j-p)} - \sum_{k=p+n}^{\infty} a_k z^{k-p} = 0, \tag{2.5}$$

then, for $\mu > 0$ *and* $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leqq \int_0^{2\pi} |p(z)|^{\mu} d\theta. \tag{2.6}$$

Proof. Putting $z = re^{i\theta}$ $(0 < r < 1)$, it follows that

$$\begin{aligned} \int_0^{2\pi} |f(z)|^{\mu} d\theta &= r^{p\mu} \int_0^{2\pi} \left| 1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} \right|^{\mu} d\theta, \\ \int_0^{2\pi} |p(z)|^{\mu} d\theta &= r^{p\mu} \int_0^{2\pi} \left| 1 + \sum_{s=1}^{m} b_{sj-(s-1)p} z^{sj-sp} \right|^{\mu} d\theta. \end{aligned} \tag{2.7}$$

Applying Theorem 1.3, we have to show that

$$1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} \prec 1 + \sum_{s=1}^{m} b_{sj-(s-1)p} z^{s(j-p)}. \tag{2.8}$$

Let us define the function $w(z)$ by

$$1 + \sum_{s=1}^{m} b_{sj-(s-1)p} \{w(z)\}^{s(j-p)} = 1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} \tag{2.9}$$

or by

$$\sum_{s=1}^{m} b_{sj-(s-1)p}\{w(z)\}^{s(j-p)} = \{w(z)\}^{j-p}\left(\sum_{s=1}^{m} b_{sj-(s-1)p}\{w(z)\}^{(s-1)(j-p)}\right) = \sum_{k=p+n}^{\infty} a_k z^{k-p}. \tag{2.10}$$

Thus, it follows that

$$\{w(0)\}^{j-p}\left(\sum_{s=1}^{m} b_{sj-(s-1)p}\{w(z)\}^{(s-1)(j-p)}\right) = 0. \tag{2.11}$$

Therefore, if there exists an analytic function $w(z)$ which satisfies the equality (2.5), we can consider an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$.

Further, we need to prove that this analytic function $w(z)$ satisfies $|w(z)| < 1$ $(z \in \mathbb{U})$ for

$$\sum_{k=p+n}^{\infty} |a_k| \leqq |b_{mj-(m-1)p}| - \sum_{s=1}^{m-1} |b_{sj-(s-1)p}| \qquad \left(\sum_{s=1}^{m-1} |b_{sj-(s-1)p}| < |b_{mj-(m-1)p}|\right). \tag{2.12}$$

From the equality (2.5), we know that

$$\begin{aligned} &|b_{mj-(m-1)p}w(z)^{m(j-p)} + b_{(m-1)j-(m-2)p}w(z)^{(m-1)(j-p)} + b_{(m-2)j-(m-3)p}w(z)^{(m-2)(j-p)} \\ &+ \cdots + b_{2j-p}w(z)^{2(j-p)} + b_j w(z)^{j-p}| \leqq \sum_{k=p+n}^{\infty} |a_k z^{k-p}| < \sum_{k=p+n}^{\infty} |a_k| \end{aligned} \tag{2.13}$$

for $z \in \mathbb{U}$, so that

$$|b_{mj-(m-1)p}|\,|w(z)^{m(j-p)}| - \sum_{s=1}^{m-1} |b_{sj-(s-1)p}|\,|w(z)|^{s(j-p)} - \sum_{k=p+n}^{\infty} |a_k| < 0 \tag{2.14}$$

for $z \in \mathbb{U}$.

Putting $t = |w(z)|^{j-p}$ $(t \geqq 0)$, we define the polynomial $P(t)$ of degree m by

$$P(t) = |b_{mj-(m-1)p}|\,t^m - \sum_{s=1}^{m-1} |b_{sj-(s-1)p}|\,t^s - \sum_{k=p+n}^{\infty} |a_k|. \tag{2.15}$$

By means of Lemma 2.1, if $P(1) \geqq 0$, we have $t < 1$ for $P(t) < 0$. Hence for $|w(z)| < 1$ $(z \in \mathbb{U})$, we need the following inequality:

$$P(1) = |b_{mj-(m-1)p}| - \sum_{s=1}^{m-1} |b_{sj-(s-1)p}| - \sum_{k=p+n}^{\infty} |a_k| \geqq 0 \tag{2.16}$$

so that

$$\sum_{k=p+n}^{\infty} |a_k| \leqq |b_{mj-(m-1)p}| - \sum_{s=1}^{m-1} |b_{sj-(s-1)p}|. \tag{2.17}$$

Therefore, the subordination in (2.8) holds true, and this evidently completes the proof of Theorem 2.2. □

Theorem 2.2 gives us the following corollary.

Corollary 2.3. *Let $f(z) \in \mathcal{A}_{p,n}$ and let $p(z)$ be given by (2.1). If $f(z)$ satisfies the conditions of Theorem 2.2, then for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leqq 2\pi r^{p\mu} \left(1 + \sum_{s=1}^{m} |b_{sj-(s-1)p}|^2 r^{2s(j-p)}\right)^{\mu/2} < 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-(s-1)p}|^2\right)^{\mu/2}. \tag{2.18}$$

Furthermore, $f(z) \in \mathcal{H}^q(\mathbb{U})$ for $0 < q \leqq 2$, where $\mathcal{H}^q$ denotes the Hardy space.

Proof. Since

$$\int_0^{2\pi} |p(z)|^{\mu} d\theta = \int_0^{2\pi} |z^p|^{\mu} \left|1 + \sum_{s=1}^{m} b_{sj-(s-1)p} z^{s(j-p)}\right|^{\mu} d\theta, \tag{2.19}$$

applying Hölder inequality for $0 < \mu < 2$, we obtain that

$$\begin{aligned}
\int_0^{2\pi} |p(z)|^{\mu} d\theta &\leqq \left(\int_0^{2\pi} (|z|^{p\mu})^{2/(2-\mu)} d\theta\right)^{(2-\mu)/2} \left\{\int_0^{2\pi} \left(\left|1+\sum_{s=1}^{m} b_{sj-(s-1)p} z^{s(j-p)}\right|^{\mu}\right)^{2/\mu} d\theta\right\}^{\mu/2} \\
&= \left(r^{2p\mu/(2-\mu)} \int_0^{2\pi} d\theta\right)^{(2-\mu)/2} \left(\int_0^{2\pi} \left|1 + \sum_{s=1}^{m} b_{sj-(s-1)p} z^{s(j-p)}\right|^2 d\theta\right)^{\mu/2} \\
&= \left(2\pi r^{2p\mu/(2-\mu)}\right)^{(2-\mu)/2} \left\{2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-(s-1)p}|^2 r^{2s(j-p)}\right)\right\}^{\mu/2} \\
&= 2\pi r^{p\mu} \left(1 + \sum_{s=1}^{m} |b_{sj-(s-1)p}|^2 r^{2s(j-p)}\right)^{\mu/2} < 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-(s-1)p}|^2\right)^{\mu/2}.
\end{aligned} \tag{2.20}$$

Further, it is easy to see that, for $\mu = 2$,

$$\int_0^{2\pi} |f(z)|^2 d\theta \leqq 2\pi r^{2p} \left(1 + \sum_{s=1}^{m} |b_{sj-(s-1)p}|^2 r^{2s(j-p)}\right) < 2\pi \left(1 + \sum_{s=1}^{m} |b_{sj-(s-1)p}|^2\right). \tag{2.21}$$

From the above, we also have that, for $0 < \mu \leqq 2$,

$$\sup_{z\in U}\frac{1}{2\pi}\int_0^{2\pi}|f(z)|^{\mu}\,d\theta \leqq \left(1+\sum_{s=1}^{m}|b_{sj-(s-1)p}|^2\right) < \infty \tag{2.22}$$

which observes that $f(z)\in H^2(U)$. Noting that $H^q \subset H^r$ $(0<r<q<\infty)$, we complete the proof of Corollary 2.3. □

Example 2.4. Let $f(z)\in\mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.5) and for $m=2$,

$$p(z)=z^p+\frac{n}{p+n-\alpha}\varepsilon_1 z^j+\varepsilon_2 z^{2j-p} \tag{2.23}$$

and for $m \geq 3$,

$$p(z)=z^p+\frac{nt^{m-2}}{p+n-\alpha}\varepsilon_1 z^j+\sum_{s=2}^{m-1}\frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha}\varepsilon_s z^{sj-(s-1)p}+\varepsilon_m z^{mj-(m-1)p}, \tag{2.24}$$

where $|\varepsilon_i|=1$ for all $i\in\mathbb{N}$ and $0\leq\alpha<p$. Then, for $m=2$,

$$b_j=\frac{n}{p+n-\alpha}\varepsilon_1,\qquad b_{2j-p}=\varepsilon_2 \tag{2.25}$$

and for $m\geq 3$,

$$b_j=\frac{nt^{m-2}}{p+n-\alpha}\varepsilon_1,\qquad b_{sj-(s-1)p}=\frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha}\varepsilon_s\quad\{s=2,3,\dots,m-1\}, \tag{2.26}$$

and $b_{mj-(m-1)p}=\varepsilon_m$.

Güney et al. [2] have shown the above example for the case $m=2$. By virtue of (1.5), we observe that, for $m\geqq 3$,

$$\sum_{k=p+n}^{\infty}|a_k|\leqq\frac{p-\alpha}{p+n-\alpha}=1-\sum_{s=2}^{m-1}\frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha}-\frac{nt^{m-2}}{p+n-\alpha} \tag{2.27}$$

$$=|b_{mj-(m-1)p}|-\sum_{s=2}^{m-1}|b_{sj-(s-1)p}|-|b_j|. \tag{2.28}$$

Therefore, $f(z)$ and $p(z)$ satisfy the conditions in Theorem 2.2. Thus, we have for $0<\mu\leqq 2$ and $z=re^{i\theta}$ $(0<r<1)$,

$$\begin{aligned}&\int_0^{2\pi}|f(z)|^{\mu}d\theta\\&\quad\leqq 2\pi r^{p\mu}\left\{1+\left(\frac{nt^{m-2}}{p+n-\alpha}\right)^2 r^{2(j-p)}+\sum_{s=2}^{m-1}\left(\frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha}\right)^2 r^{2s(j-p)}+r^{2m(j-p)}\right\}^{\mu/2}\\&\quad<2\pi\left\{2+\left(\frac{nt^{m-2}}{p+n-\alpha}\right)^2+\sum_{s=2}^{m-1}\left(\frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha}\right)^2\right\}^{\mu/2}.\end{aligned}\tag{2.29}$$

3. Integral means for $f'(z)$ and $p'(z)$

Using the same techniques in Theorem 2.2, we obtain the following theorem.

Theorem 3.1. *Let $f(z)\in\mathcal{A}_{p,n}$ and let $p(z)$ be given by (2.1). If $f(z)$ satisfies*

$$\sum_{k=p+n}^{\infty}k|a_k|\leqq(mj-(m-1)p)|b_{mj-(m-1)p}|-\sum_{s=1}^{m-1}(sj-(s-1)p)|b_{sj-(s-1)p}|\tag{3.1}$$

with

$$(mj-(m-1)p)|b_{mj-(m-1)p}|>\sum_{s=1}^{m-1}(sj-(s-1)p)|b_{sj-(s-1)p}|,\tag{3.2}$$

and there exists an analytic function $w(z)$ such that

$$\sum_{s=1}^{m}(sj-(s-1)p)b_{sj-(s-1)p}\{w(z)\}^{s(j-p)}-\sum_{k=p+n}^{\infty}ka_kz^{k-p}=0,\tag{3.3}$$

then, for $\mu>0$ and $z=re^{i\theta}$ $(0<r<1)$,

$$\int_0^{2\pi}|f'(z)|^{\mu}d\theta\leqq\int_0^{2\pi}|p'(z)|^{\mu}d\theta.\tag{3.4}$$

Further, with the help of Hölder inequality, we have the following.

Corollary 3.2. *Let $f(z)\in\mathcal{A}_{p,n}$ and let $p(z)$ $(m\geqq 2)$ be given by (2.1). If $f(z)$ satisfies the conditions in Theorem 3.1, then for $0<\mu\leqq 2$ and $z=re^{i\theta}$ $(0<r<1)$,*

$$\begin{aligned}\int_0^{2\pi}|f'(z)|^{\mu}d\theta&\leqq 2\pi r^{(p-1)\mu}\left(p^2+\sum_{s=1}^{m}(sj-(s-1)p)^2|b_{sj-(s-1)p}|^2r^{2s(j-p)}\right)^{\mu/2}\\&<2\pi\left(p^2+\sum_{s=1}^{m}(sj-(s-1)p)^2|b_{sj-(s-1)p}|^2\right)^{\mu/2}.\end{aligned}\tag{3.5}$$

Example 3.3. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.6) and for $m = 2$,

$$p(z) = z^p + \frac{n}{j(p+n-\alpha)}\varepsilon_1 z^j + \frac{1}{2j-p}\varepsilon_2 z^{2j-p} \tag{3.6}$$

and for $m \geq 3$,

$$\begin{aligned} p(z) &= z^p + \frac{nt^{m-2}}{j(p+n-\alpha)}\varepsilon_1 z^j \\ &\quad + \sum_{s=2}^{m-1} \frac{nt^{(m-1)-s}(1-t)}{[sj-(s-1)p](p+n-\alpha)}\varepsilon_s z^{sj-(s-1)p} + \frac{\varepsilon_m}{mj-(m-1)p} z^{mj-(m-1)p}, \end{aligned} \tag{3.7}$$

where $|\varepsilon_i| = 1$ for all $i \in \mathbb{N}$ and $0 \leq \alpha < p$. Then for $m = 2$,

$$b_j = \frac{n}{j(p+n-\alpha)}\varepsilon_1, \qquad b_{2j-p} = \frac{1}{2j-p}\varepsilon_2 \tag{3.8}$$

and for $m \geq 3$,

$$\begin{gathered} b_j = \frac{nt^{m-2}}{j(p+n-\alpha)}\varepsilon_1, \qquad b_{sj-(s-1)p} = \frac{nt^{(m-1)-s}(1-t)}{[sj-(s-1)p](p+n-\alpha)}\varepsilon_s, \\ b_{mj-(m-1)p} = \frac{\varepsilon_m}{mj-(m-1)p}. \end{gathered} \tag{3.9}$$

Güney et al. [2] have shown the above example for the case $m = 2$. Since, for $m \geqq 3$,

$$\begin{aligned} \sum_{k=p+n}^{\infty} k\,|a_k| &\leqq \frac{p-\alpha}{p+n-\alpha} = 1 - \sum_{s=2}^{m-1} \frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha} - \frac{nt^{m-2}}{p+n-\alpha} \\ &= (mj-(m-1)p)\,|b_{mj-(m-1)p}| - (sj-(s-1)p)\sum_{s=2}^{m-1} |b_{sj-(s-1)p}| - j\,|b_j|, \end{aligned} \tag{3.10}$$

$f(z)$ and $p(z)$ satisfy the conditions in Theorem 3.1. Thus, by Corollary 3.2, we have for $0 < \mu \leqq 2$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\begin{aligned} &\int_0^{2\pi} |f'(z)|^{\mu} d\theta \\ &\quad \leqq 2\pi r^{(p-1)\mu}\left\{p^2 + \left(\frac{nt^{m-2}}{p+n-\alpha}\right)^2 r^{2(j-p)} + \sum_{s=2}^{m-1}\left(\frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha}\right)^2 r^{2s(j-p)} + r^{2m(j-p)}\right\}^{\mu/2} \\ &\quad < 2\pi\left\{p^2 + 1 + \left(\frac{nt^{m-2}}{p+n-\alpha}\right)^2 + \sum_{s=2}^{m-1}\left(\frac{nt^{(m-1)-s}(1-t)}{p+n-\alpha}\right)^2\right\}^{\mu/2}. \end{aligned} \tag{3.11}$$

Remark 3.4. In the above theorems, if we take that $p = 1$, we obtain the results by Sekine et al. [5]. Furthermore, if we take that $m = 2, 3$ in the above theorems and examples, we obtain the results of Güney et al. [2]. Therefore, the results of our paper are a generalization of the results in [2].

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