# BIHARMONIC CURVES IN MINKOWSKI 3-SPACE. PART II

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We give a differential geometric characterization for biharmonic curves with null principal normal in Minkowski 3-space.

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## 1. Introduction

This is a supplement to our previous research note [3]. In [3], we gave a characterization of biharmonic curves in Minkowski 3-space. More precisely, we pointed out that every biharmonic curves with *nonnull* principal normal in Minkowski 3-space is a helix, whose curvature $\kappa$ and torsion $\tau$ satisfy $\kappa^2 = \tau^2$. In the classification of biharmonic curves in Minkowski 3-space due to Chen and Ishikawa [1], there exist biharmonic spacelike curves with *null* principal normal. In this supplement, we give a characterization of biharmonic curves with null principal normal.

## 2. Preliminaries

Let $\mathbb{E}^3_1$ be the Minkowski 3-space with natural Lorentz metric $\langle \cdot, \cdot \rangle = -dx^2 + dy^2 + dz^2$. Let $\gamma = \gamma(s)$ be a spacelike curve parametrized by the arclength parameter; that is, $\gamma$ satisfies $\langle \gamma', \gamma' \rangle = 1$. A spacelike curve $\gamma$ is said to be a *Frenet curve* if its acceleration vector field $\gamma''$ satisfies the condition $\langle \gamma'', \gamma'' \rangle \neq 0$. Every spacelike Frenet curve admits an orthonormal frame field along it (see [3]). Since biharmonicity for spacelike Frenet curves is studied in [3], hereafter we restrict our attention to spacelike curves with *null acceleration vector field*. Note that spacelike curves with zero acceleration vector field are lines. There are no timelike curves with null acceleration vector field.

Lemma 2.1. *Let $\gamma(s)$ be a spacelike curve parametrized by arclength such that $\langle \gamma'', \gamma'' \rangle = 0$. Then there exists a matrix-valued function $F(s) = (\mathbf{f}_1(s), \mathbf{f}_2(s), \mathbf{f}_3(s))$, which satisfies the following ordinary differential equation:*

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$$\nabla_{\gamma'} F = F \begin{pmatrix} 0 & 0 & -1 \\ 1 & k & 0 \\ 0 & 0 & -k \end{pmatrix}, \qquad \mathbf{f}_1 = \gamma'. \tag{2.1}$$

*Here $\nabla$ is the Levi-Civita connection of $\mathbb{E}^3_1$.*

*Conversely, let $F(s) = (\mathbf{f}_1(s), \mathbf{f}_2(s), \mathbf{f}_3(s))$ be a solution to (2.1). Then there exists a spacelike curve $\gamma(s)$ with arclength parameter $s$ such that*

$$\gamma' = \mathbf{f}_1, \qquad \langle \gamma'', \gamma'' \rangle = 0. \tag{2.2}$$

*Proof.* By the assumption, $\mathbf{f}'_1 = \gamma''$ is a null vector field. We set $\mathbf{f}_2 = \mathbf{f}'_1$. Since $\mathbf{f}_1 = \gamma'$ is a unit spacelike vector field, there exists a unique null vector field $\mathbf{f}_3$ along $\gamma$ such that (cf. [2])

$$\langle \mathbf{f}_2, \mathbf{f}_3 \rangle = 1, \qquad \langle \mathbf{f}_1, \mathbf{f}_3 \rangle = 0. \tag{2.3}$$

One can check that $F = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ satisfies (2.1). For instance, expand $\mathbf{f}_2$ as $\mathbf{f}_2 = a\mathbf{f}_1 + b\mathbf{f}_2 + c\mathbf{f}_3$. Then

$$a = \langle \mathbf{f}'_2, \mathbf{f}_1 \rangle = -\langle \mathbf{f}_2, \mathbf{f}'_1 \rangle = 0, \qquad c = \langle \mathbf{f}'_2, \mathbf{f}_2 \rangle = \langle \mathbf{f}_2, \mathbf{f}_2 \rangle' = 0. \tag{2.4}$$

Hence $\mathbf{f}'_2 = b\mathbf{f}_2$. By similar computations, we get

$$\mathbf{f}'_3 = -\mathbf{f}_1 - b\mathbf{f}_3. \tag{2.5}$$

Thus $F$ satisfies (2.1) with $k = b$.

Conversely, let $F$ be a solution to (2.1). Then $F$ satisfies the following conditions (*cf.* [2, Section 2]):

$$\begin{aligned} \langle \mathbf{f}_1, \mathbf{f}_1 \rangle = 1, \qquad \langle \mathbf{f}_2, \mathbf{f}_2 \rangle = \langle \mathbf{f}_3, \mathbf{f}_3 \rangle = 0, \\ \langle \mathbf{f}_2, \mathbf{f}_3 \rangle = 1, \qquad \langle \mathbf{f}_1, \mathbf{f}_2 \rangle = \langle \mathbf{f}_1, \mathbf{f}_3 \rangle = 0. \end{aligned} \tag{2.6}$$

Integrating $\mathbf{f}_1(s)$ by $s$, we obtain a spacelike curve $\gamma(s)$ with null acceleration, since $\gamma'' = \mathbf{f}'_1 = \mathbf{f}_2$. □

We call the matrix-valued function $F$, the *null frame* of $\gamma$. We call $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$, the *tangent vector field*, *principal normal vector field*, and *binormal vector field* of $\gamma$, respectively. We call the function $k$ the *curvature function* of $\gamma$. Note that both principal normal and binormal are null.

*Example 2.2.* Let us consider $\gamma$ with $k = 0$. Since $\mathbf{f}'_2 = 0$, we have

$$\mathbf{f}_1 = s\mathbf{n} + \mathbf{u}, \tag{2.7}$$

where the constant vectors $\mathbf{n}$ and $\mathbf{u}$ satisfy the relation

$$\langle \mathbf{n}, \mathbf{n} \rangle = \langle \mathbf{n}, \mathbf{u} \rangle = 0, \qquad \langle \mathbf{u}, \mathbf{u} \rangle = 1. \tag{2.8}$$

Thus we obtain

$$\gamma(s) = \frac{s^2}{2}\mathbf{n} + s\mathbf{u} + \mathbf{v}, \tag{2.9}$$

where $\mathbf{v}$ is a constant vector. Hence $\gamma$ is congruent to

$$(bs^2, bs^2, s), \quad b \neq 0. \tag{2.10}$$

Next, assume that $k$ is a nonzero constant, then $\gamma$ is given by

$$\gamma(s) = \frac{1}{k^2}e^{ks}\mathbf{n} + s\mathbf{u} + \mathbf{v}. \tag{2.11}$$

Here the constant vectors $\mathbf{n}$ and $\mathbf{u}$ satisfy (2.8). Hence $\gamma$ is congruent to

$$\left(\frac{a}{k^2}e^{ks}, \frac{a}{k^2}e^{ks}, s\right), \quad a \neq 0. \tag{2.12}$$

*Example 2.3.* Let us determine spacelike curves with $1/k = s + c$, where $c$ is a constant. Then $\gamma$ is given by

$$\gamma(s) = \left(\frac{s^3}{3} + \frac{cs^2}{2}\right)\mathbf{n} + s\mathbf{u} + \mathbf{v}, \tag{2.13}$$

where the constant vectors $\mathbf{n}$ and $\mathbf{u}$ satisfy (2.8). Thus $\gamma$ is congruent to the curve

$$(as^3 + bs^2, as^3 + bs^2, s), \quad a \neq 0. \tag{2.14}$$

## 3. Biharmonic curves

We start this section with recalling the notion of biharmonicity.

Let $\gamma$ be a spacelike curve in $\mathbb{E}^3_1$ parametrized by arclength defined on an open interval $I$. We denote by $\gamma^* T\mathbb{E}^3_1$ the vector bundle over $I$ obtained by pulling back the tangent bundle $T\mathbb{E}^3_1$:

$$\gamma^* T\mathbb{E}^3_1 = \bigcup_{s \in I} T_{\gamma(s)}\mathbb{E}^3_1. \tag{3.1}$$

The *Laplace operator* $\Delta$ acting on the space $\Gamma(\gamma^* T\mathbb{E}^3_1)$ of all smooth vector fields along $\gamma$ is given by

$$\Delta = -\nabla_{\gamma'}\nabla_{\gamma'}. \tag{3.2}$$

A spacelike curve $\gamma$ is said to be *biharmonic* if $\Delta\mathbb{H} = 0$, where $\mathbb{H}$ is the mean curvature vector field of $\gamma$.

Chen and Ishikawa obtained the following result.

THEOREM 3.1 [1]. *Let $\gamma(s)$ be a spacelike curve parametrized by arclength with null acceleration vector field. Then $\gamma$ is biharmonic if and only if $\gamma$ is congruent to*

$$(as^3 + bs^2, as^3 + bs^2, s), \quad a^2 + b^2 \neq 0. \tag{3.3}$$

Now we give a geometric characterization of biharmonic spacelike curve with null principal normal. Let $\gamma(s)$ be a spacelike curve parametrized by arclength with null acceleration vector field. Then the mean curvature vector field $\mathbb{H}$ is given by

$$\mathbb{H} = \nabla_{\gamma'}\gamma' = \mathbf{f}_2. \tag{3.4}$$

Thus we obtain

$$\Delta\mathbb{H} = -(k' + k^2)\mathbf{f}_2. \tag{3.5}$$

Hence $\gamma$ is biharmonic if and only if $k' + k^2 = 0$. Hence the curvature function $k$ is given by $k = 0$ or $1/k(s) = s + c$, where $c$ is a constant.

Proposition 3.2. *A spacelike curve $\gamma(s)$ parametrized by arclength parameter $s$ with null principal normal vector field is biharmonic if and only if its curvature function is given by $k = 0$ or $1/k = s + c$ for some constant $c$. Hence such curves are congruent to the curve (3.3). The former case $(k = 0)$ corresponds to the case $a = 0$ (2.10) and the latter case $(1/k = s + c)$ to $a \neq 0$ (2.14), respectively.*

## References

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