TRANSFERS FOR RAMIFIED COVERING MAPS IN HOMOLOGY AND COHOMOLOGY

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Making use of a modified version, due to McCord, of the Dold-Thom construction of ordinary homology, we give a simple topological definition of a transfer for ramified covering maps in homology with arbitrary coefficients. The transfer is induced by a suitable map between topological groups. We also define a new cohomology transfer which is dual to the homology transfer. This duality allows us to show that our homology transfer coincides with the one given by L. Smith. With our definition of the homology transfer we can give simpler proofs of the properties of the known transfer and of some new ones. Our transfers can also be defined in Karoubi's approach to homology and cohomology. Furthermore, we show that one can define mixed transfers from other homology or cohomology theories to the ordinary ones.

1. Introduction

Let X be a pointed topological space (of the same homotopy type of a CW-complex). By a theorem of Moore, the free topological abelian group $F(X,\mathbb{Z})$ generated by the points of X (where the base point is zero in this group) is determined up to homotopy equivalence by its homotopy groups. So, it is natural to associate to X the homotopy groups $\pi_q(F(X,\mathbb{Z}))$. By the Dold-Thom theorem these groups are isomorphic to the singular homology groups $\widetilde{H}_q(X;\mathbb{Z})$. In this paper, we use this approach to homology in order to define a transfer for ramified covering maps. Furthermore, one can also use the same free topological groups to define cohomology, since $F(\mathbb{S}^q,\mathbb{Z})$ is an Eilenberg-Mac Lane space of type $K(\mathbb{Z},q)$. We use this model of Eilenberg-Mac Lane spaces to define a transfer for ramified covering maps also in singular cohomology.

This approach to homology and cohomology has proved to be extremely useful in other branches of mathematics. Karoubi [11] used the groups $F(\mathbb{B}^q,\mathbb{Z})$, where $\mathbb{B}^q$ is the closed q-ball, to construct a chain complex and a cochain complex. This complex is called the complex of noncommutative topological forms and is related to the Connes

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noncommutative geometry. We show that the transfers for ramified covering maps can also be defined in this setting.

On the other hand, let X be a complex projective variety. Lawson [12] (see also [13]) considered a topological abelian group $Z_p(X)$ generated by the subvarieties of dimension p, and he defined the Lawson homology of X by taking $L_pH_q(X) = \pi_{q-2p}(Z_p(X))$, so that $L_0H_q(X) = \pi_q(F(X,\mathbb{Z}))$. Furthermore, the construction of Eilenberg-Mac Lane spaces using $F(\mathbb{S}^q,\mathbb{Z})$ was generalized by Voevodsky [20] in order to have an analog of the Eilenberg-Mac Lane spectrum in algebraic geometry. This allowed him to define motivic cohomology.

The class of finite ramified covering maps on which we work was defined by Smith [18], who constructed for them a transfer in singular homology. Later on, in [6] Dold gave an alternative construction and characterized ramified covering maps as maps between orbit spaces of the action of a finite group and a subgroup, and giving a modified definition of the transfer. Both definitions are algebraic in nature. These transfers have the usual property that when composed with the homomorphism induced by the projection of the ramified covering map, they yield multiplication by the multiplicity of the covering map in the homology of the base space.

There have been previous definitions of both the homology and the cohomology transfers for maps between orbit spaces of certain actions of a finite group and a subgroup, see Bredon [4] (and also tom Dieck [19]). These definitions depend on the equivariant structure of the spaces involved.

In this paper we use the McCord [16] version of the Dold-Thom construction of ordinary homology mentioned above, to produce a topological transfer for general ramified covering maps. Namely, we define a transfer that is a continuous map between the topological groups associated to the total and to the base space of the ramified covering map. This codifies in a sense the fact that a transfer can be seen as a multivalued map. We also define a cohomology transfer using models of Eilenberg-Mac Lane spaces that have the structure of topological abelian groups. We apply either transfer to give some results about the homology or cohomology of orbit maps of the action of a group and a subgroup of finite index. The definitions of the new transfers are rather simple. This fact simplifies their computation (see (7.1), for instance).

Friedlander and Mazur [7] constructed a trace map in the context of simplicial sets and chain complexes, which is similar to our homology transfer. They use their trace maps to study the homology of algebraic varieties. On the other hand, Cohen [5] defined a stable version of the transfer of an n-fold ramified covering map, provided that one inverts $n!$.

In this paper, we will work entirely in the category of weak Hausdorff k-spaces, which we will call *compactly generated spaces* (see [15]). Moreover, unless otherwise stated, we will also assume that all spaces have the same homotopy type of CW-complexes.

2. McCord's topological groups

In this section, we recall briefly the spaces $B(G,X)$ introduced by McCord. We find it convenient to use $F(X,L)$ as an alternative notation. Details can be seen in [16] or [1, 6.3.20ff].

Let L be an abelian group and let X be a pointed topological space with base point $* \in X$. $F(X,L)$ is the abelian group of all functions $u : X \to L$ such that $u(*) = 0$ and $u(x) = 0$ for all but a finite number of elements $x \in X$. If these elements are $x_1,\ldots,x_n$ and the values of u at each of them are $l_1,\ldots,l_n$, respectively, it is sometimes convenient to write u as $\sum_{i=1}^{n} l_i x_i = l_1 x_1 + \cdots + l_n x_n$. In particular, for any $x \in X$, $x \neq *$, one may see lx as the element in $F(X,L)$ whose value at x is l and whose value elsewhere is 0 ($l* = 0$). Taking $L = R$ to be a commutative ring with 1 and $x \in X$, $x \neq *$ then $x \in F(X,R)$ can be interpreted as the function whose value at x is 1 and whose value elsewhere is 0. This defines a canonical inclusion $X \hookrightarrow F(X,R)$. In this case, the elements $x \in F(X,R)$ generate $F(X,R)$ freely as an R-module.

The abelian group $F(X,L)$ has a topology that turns it into a pointed space with base point $0 \in F(X,L)$. It is in fact a topological abelian group (in the category of compactly generated spaces). Consider the natural filtration of closed subspaces

$$F_0(X,L) \subset F_1(X,L) \subset \cdots \subset F(X,L), \tag{2.1}$$

where $F_n(X,L)$ consists of those functions u that are nonzero on at most n points in X. The topology can then be defined as follows. For each n, take the surjection $(L \times X)^n \to F_n(X,L)$ given by mapping $(l_1,x_1,\ldots,l_n,x_n)$ to $\sum_{i=1}^{n} l_i x_i$. Here $(L \times X)^n$ is the product of n copies of $L \times X$, furnished with the compactly generated product topology, and $F_n(X,L)$ is given the corresponding quotient topology. Then provide $F(X,L)$ with the weak topology (of the union).

Given a pointed map $\varphi : X \to Y$ and a homomorphism $\alpha : L \to M$, one has a unique pointed map $F(\varphi,\alpha) : F(X,L) \to F(Y,M)$ given by

$$F(\varphi,\alpha)\left(\sum_{i=1}^{n} l_i x_i\right) = \sum_{i=1}^{n} \alpha(l_i)\varphi(x_i). \tag{2.2}$$

In other words, $F(\varphi,\alpha)(u)$ is the function whose values at $y \in Y$ are 0 unless $y = \varphi(x)$ and $u(x) \neq 0$; in this case, $F(\varphi,\alpha)(u)(y) = \sum_{\varphi(x)=y} \alpha(u(x))$. An easy way of writing this is $F(\varphi,\alpha)(u) = \sum_{x\in X} \alpha(u(x))\varphi(x)$. This definition turns F into a covariant bifunctor from the category $\mathcal{T}\mathrm{op}_* \times \mathcal{A}\mathrm{b}$ of pairs consisting of a pointed topological space and a topological abelian group to the category $\mathcal{T}\mathrm{opab}$ of topological abelian groups.

We will denote $F(\varphi,1_L)$ simply by φ_* and $F(\mathrm{id}_X,\alpha)$ by α_*.

The fundamental property of the McCord topological groups is the following.

Property 2.1. If (X,A) is a pointed triangulable pair, then the quotient map $p : X \to X/A$ induces a locally trivial principal $F(A,L)$-bundle, $p_* : F(X,L) \to F(X/A,L)$ (see [16] or [14]).

There is a natural H-isomorphism, that is, a pointed homotopy equivalence which is also a morphism of H-groups:

$$F(Y,L) \longrightarrow \Omega F(\Sigma Y,L), \tag{2.3}$$

where Ω means the loop space and Σ the (reduced) suspension given by $\Sigma Y = \mathbb{S}^1 \wedge Y$. This H-isomorphism yields a group isomorphism

$$\sigma : [X, F(Y,L)]_* \longrightarrow [X, \Omega F(\Sigma Y, L)]_* \cong [\Sigma X, F(\Sigma Y, L)]_*, \tag{2.4}$$

where $[-,-]_*$ denotes pointed homotopy classes. We call this the *suspension isomorphism* (see [16, 10.4]).

Hence $F(\mathbb{S}^q, L) \xrightarrow{\simeq} \Omega F(\mathbb{S}^{q+1}, L)$, and since $F(\mathbb{S}^0, L) = L$ (see below), we have that the space $F(\mathbb{S}^q, L)$ is an Eilenberg-Mac Lane space of type (L,q) that has the structure of a topological abelian group.

Hence we have a long exact sequence for the homotopy groups $\pi_q(F(A,L))$, $\pi_q(F(X, L))$, and $\pi_q(F(X/A, L))$ for a pair (X,A) of the same homotopy type of a CW-pair. By this and the previous comment, we have that the groups

$$\widetilde{H}_q(X;L) = \pi_q(F(X,L)), \qquad \widetilde{H}^q(X;L) = [X, F(\mathbb{S}^q, L)]_* \tag{2.5}$$

define ordinary (reduced) homology and cohomology theories with coefficients in L.

Remark 2.2. Observe that the unpointed homotopy classes $[X, F(\mathbb{S}^q, L)]$ yield the unreduced cohomology groups $H^q(X;L)$.

Lemma 2.3. *The map $\varepsilon : F(X,L) \to L$ given by $\sum_{i=1}^m l_i x_i \mapsto \sum_{i=1}^m l_i$ is well defined and continuous. In particular, $\varepsilon : F(\mathbb{S}^0, L) \to L$ is an isomorphism.*

Proof. This follows easily from the fact that the restriction $\varepsilon_n : F_n(X,L) \to L$ of ε is continuous, since its composite with the identification $(X \times L)^n \to F_n(X,L)$ is obviously continuous. □

Another useful property of the functor F is that one has a well-defined continuous pairing,

$$F(X,L) \times F(Y,M) \longrightarrow F(X \wedge Y, L \otimes M), \tag{2.6}$$

given by

$$\left(\sum_i l_i x_i, \sum_j m_j y_j \right) \longmapsto \sum_{i,j} (l_i \otimes m_j)(x_i \wedge y_j) \tag{2.7}$$

(see [16, 11.6]). If, in particular, $L = M = R$ is a commutative ring with 1, with $m : R \otimes R \to R$ as the ring multiplication, then composing (2.6) with m_*, we obtain another pairing,

$$F(X,R) \times F(Y,R) \longrightarrow F(X \wedge Y, R). \tag{2.8}$$

Using (2.8), one obtains products in homology and cohomology. We will be interested in the following.

Proposition 2.4. *One has cap-products*

$$H^q(X;R) \otimes H_k(X;R) \xrightarrow{\frown} H_{k-q}(X;R), \tag{2.9}$$

if X is 0*-connected and $q \le k$, and*

$$H^q(X;R) \otimes H_k(X;R) \xrightarrow{\frown} H^{q-k}(X;R) \tag{2.10}$$

if $k \le q$. In particular, if $k = q$ one has a Kronecker product

$$H^q(X;R) \otimes H_q(X;R) \xrightarrow{\langle -,- \rangle} R. \tag{2.11}$$

Proof. Taking smash-products and the pairing (2.6), we have

$$\begin{tikzcd}
{[X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^k, F(X^+, R)]_*} \arrow[r] \arrow[dr, dashed, "\kappa"'] & {[X^+ \wedge \mathbb{S}^k, F(\mathbb{S}^q, R) \wedge F(X^+, R)]_*} \arrow[d] \\
& {[\Sigma^k X^+, F(\Sigma^q X^+, R)]_*}
\end{tikzcd} \tag{2.12}$$

If $q \le k$, using σ^{-q} of the suspension property, we desuspend q times. Composing κ with the homomorphism

$$[\Sigma^{k-q} X^+, F(X^+, R)]_* \longrightarrow [\mathbb{S}^{k-q}, F(X^+, R)]_* \tag{2.13}$$

induced by the pointed inclusion $\mathbb{S}^0 \to X^+$ that sends -1 to some point x_{-1} in the path-connected space X, we obtain the homology $\frown$-product

$$\frown : [X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^k, F(X^+, R)]_* \longrightarrow [\mathbb{S}^{k-q}, F(X^+, R)]_*. \tag{2.14}$$

On the other hand, if $k \le q$, using σ^{-k}, we desuspend k times. And then, composing κ with the homomorphism

$$[X^+, F(\Sigma^{q-k} X^+, R)]_* \longrightarrow [X^+, F(\mathbb{S}^{q-k}, R)]_* \tag{2.15}$$

induced by the obvious map $X^+ \to \mathbb{S}^0$, we obtain the cohomology $\frown$-product

$$\frown : [X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^k, F(X^+, R)]_* \longrightarrow [X^+, F(\mathbb{S}^{q-k}, R)]_*. \tag{2.16}$$

In order to obtain the Kronecker product $\langle -, - \rangle$, we take $q = k$ and consider the composite

$$[X^+, F(\mathbb{S}^q, R)]_* \times [\mathbb{S}^q, F(X^+, R)]_* \xrightarrow{\frown} [X^+, F(\mathbb{S}^0, R)]_* \longrightarrow [\mathbb{S}^0, F(\mathbb{S}^0, R)]_* = R, \tag{2.17}$$

where the last arrow is induced by the pointed inclusion $\mathbb{S}^0 \to X^+$, and the equality follows from the bijection $\varepsilon : F(\mathbb{S}^0, R) \to R$ given in Lemma 2.3. $\square$

3. Ramified covering maps

To give the definition of a ramified covering map, we will need the concept of *nth symmetric product* of a compactly generated space Y. It is defined by $\mathrm{SP}^n Y = Y^n/\Sigma_n$, where Y^n has the compactly generated topology and the symmetric group Σ_n acts by permuting the coordinates. We denote its elements by $\langle y_1, y_2, \dots, y_n \rangle$.

Definition 3.1. An *n-fold ramified covering map* is a continuous map $p : E \to X$ together with a *multiplicity function* $\mu : E \to \mathbb{N}$ such that the following hold:

(i) the fibers $p^{-1}(x)$ are finite (discrete), $x \in X$;

(ii) for each $x \in X$, $\sum_{e \in p^{-1}(x)} \mu(e) = n$;

(iii) the map $\varphi_p : X \to \mathrm{SP}^n E$ given by

$$\varphi_p(x) = \langle \underbrace{e_1, \dots, e_1}_{\mu(e_1)}, \dots, \underbrace{e_m, \dots, e_m}_{\mu(e_m)} \rangle, \tag{3.1}$$

where $p^{-1}(x) = \{e_1, \dots, e_m\}$, is continuous.

In the original definition of Smith (see [18]), $\mathrm{SP}^n E$ is taken with the usual topology. Since in this paper we are assuming that all spaces are compactly generated, we have to check that the original definition is equivalent to the one given above. To that end and for the time being, we denote by E^n the product of n copies of E with the usual product topology. Recall that there is a functor k from the category of weak Hausdorff spaces to the category of compactly generated spaces such that the identity $k(X) \to X$ is continuous.

Remark 3.2. If E is a weak Hausdorff space, then one can show that E^n with the usual product topology is also weak Hausdorff. Recall that in a weak Hausdorff space X a set $C \subset X$ is closed in $k(X)$ if and only if $C \cap K$ is closed for every compact Hausdorff subspace $K \subset X$ (see [15]).

Lemma 3.3. *Let E be a compactly generated space. Then, $k(E^n/\Sigma_n) = k(E^n)/\Sigma_n$.*

Proof. Let $p : E^n \to E^n/\Sigma_n$ be the quotient map, which is proper, since Σ_n is a finite group. Consider the following commutative diagram of continuous maps:

$$\begin{tikzcd} k(E^n) \arrow[r] \arrow[d, "p"'] & E^n \arrow[d, "p"] \\ k(E^n)/\Sigma_n \arrow[r, "\iota"'] & E^n/\Sigma_n \end{tikzcd} \tag{3.2}$$

Note that the quotient space $k(E^n)/\Sigma_n$ is again compactly generated (see [15]). Consider the continuous map $k(\iota) : k(k(E^n)/\Sigma_n) \to k(E^n/\Sigma_n)$ and note that $k(k(E^n)/\Sigma_n) = k(E^n)/\Sigma_n$; therefore, any closed set in $k(E^n/\Sigma_n)$ is closed in $k(E^n)/\Sigma_n$.

Now take a closed set $C \subset k(E^n)/\Sigma_n$. To see that C is closed in $k(E^n/\Sigma_n)$, consider $C \cap L$, where $L \subset E^n/\Sigma_n$ is compact Hausdorff. Since $p^{-1}(C \cap L) = p^{-1}C \cap p^{-1}L$ and $p^{-1}L$ is compact Hausdorff in E^n, and by assumption $p^{-1}(C \cap L)$ is closed in E^n, then $C \cap L$ is closed in E^n/Σ_n. Thus C is closed in $k(E^n/\Sigma_n)$. □

Proposition 3.4. *Let X and E be compactly generated spaces. A map $\varphi : X \to \mathrm{SP}^n E$ is continuous with respect to the usual topology on $\mathrm{SP}^n E$ if and only if it is continuous with respect to the compactly generated topology on $\mathrm{SP}^n E$.*

Proof. By the previous lemma, we can consider the following commutative diagram:

$$\begin{tikzcd} k(X) \arrow[r, "k(\varphi)"] \arrow[d, equal] & k(\mathrm{SP}^n E) \arrow[r, equal] \arrow[d] & k(E^n)/\Sigma_n \\ X \arrow[r, "\varphi"'] & \mathrm{SP}^n E & \end{tikzcd} \tag{3.3}$$

Therefore, φ is continuous if and only if $k(\varphi)$ is continuous. □

Remark 3.5. Given an n-fold ramified covering map $p : E \to X$ with multiplicity function μ, one can construct an n-fold ramified covering map $p^+ : E^+ \to X^+$, where $Y^+ = Y \sqcup \{*\}$ for any space Y, p^+ extends p by defining $p^+(*) = *$, and the multiplicity function μ^+ extends μ by setting $\mu^+(*) = n$. More generally, given a (closed) subspace $A \subset X$, one can construct an n-fold ramified covering map $p' : E' \to X/A$, where $E' = E/p^{-1}A$, p' is the map between quotients, and the multiplicity function μ' coincides with μ off $p^{-1}A$ and is extended by setting $\mu'(*) = n$ if $*$ is the base point onto which $p^{-1}A$ collapses.

On the other hand, given a map $f : Y \to X$, one can construct the *induced n-fold ramified covering map* $f^*(p) : f^*(E) \to Y$ by taking the pullback $f^*(E) = \{(y,e) \in Y \times E \mid f(y) = p(e)\}$ and $f^*(p) = \mathrm{proj}_Y$. The *induced multiplicity function* $f^*(\mu) : f^*(E) \to \mathbb{N}$ is given by $f^*(\mu)(y,e) = \mu(e)$. Denote by $\widetilde{f} : f^*(E) \to E$ the projection proj_E.

Examples 3.6. Typical examples of ramified covering maps are the following:

(1) standard covering maps with finitely many leaves;

(2) orbit maps $E/H \to E/G$ for actions of a finite group G on a space E and $H \subset G$. They can be considered as $[G : H]$-fold ramified covering maps. In fact, Dold [6] proves that all ramified covering maps are of this form for $G = \Sigma_n$ and $H = \Sigma_{n-1}$ (see Proposition 7.2).

(3) *Branched covering maps* on manifolds, namely, open maps $p : M^d \to N^d$, where M^d and N^d are orientable closed topological manifolds of dimension d, p has finite fibers and its degree is n. Indeed, Berstein and Edmonds [3] prove that p is of the form $E/H \to E/G$, with $[G : H] = n$, so that by (2), p is in fact an n-fold ramified covering map. An interesting special case of this is given by Montesinos [17] and Hilden [9], who show that for any closed orientable 3-manifold M^3, there is a branched covering map $p : M^3 \to \mathbb{S}^3$ of degree 3.

(4) It will be of particular interest to consider the following example. Let B be a space and let $\pi_B : B^n \times_{\Sigma_n} \overline{n} \to \mathrm{SP}^n B$, where $\overline{n} = \{1, 2, \dots, n\}$ and $\times_{\Sigma_n}$ represents the twisted product, be given by $\pi_B\langle b_1, b_2, \dots, b_n; i\rangle = \langle b_1, b_2, \dots, b_n\rangle$. Then π_B is an n-fold ramified covering map with multiplicity function $\mu_B : B^n \times_{\Sigma_n} \overline{n} \to \mathbb{N}$ given by $\mu_B\langle b_1, b_2, \dots, b_n; i\rangle = \#\{j \mid b_j = b_i\}$ (see [18]).

4. The homology transfer

We will now define the homology transfer. In this section we will assume that all spaces and maps are pointed.

Definition 4.1. Let $p : E \to X$ be an n-fold ramified covering map with multiplicity function μ. Define the *pretransfer*

$$t_p : F(X,L) \longrightarrow F(E,L) \quad \text{by } t_p(u) = \widetilde{u}, \tag{4.1}$$

where $\widetilde{u}(e) = \mu(e)u(p(e))$. In other words, if $u = \sum_{i=1}^{n} l_i x_i \in F(X,L)$, then

$$t_p(u) = \sum_{\substack{p(e)=x_i \\ i=1,\dots,n}} \mu(e) l_i e. \tag{4.2}$$

Remark 4.2. The pretransfer $t_p : F(X,L) \to F(E,L)$ is clearly a homomorphism of topological groups and it is thus convenient to see what it does to generators. Namely, if lx is the function in $F(X,L)$ such that it is zero everywhere, with the exception of x, where its value is l, then it is a generator and the pretransfer satisfies

$$t_p(lx)(e) = \mu(e)lx(p(e)) = \begin{cases} \mu(e)l & \text{if } p(e) = x, \text{ that is, if } e \in p^{-1}(x), \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Hence, the only points where $t_p(lx)$ is nonzero are the elements of $p^{-1}(x) = \{e_1, e_2, \dots, e_r\}$, that is,

$$t_p(lx)(e_1) = \mu(e_1)l, \qquad t_p(lx)(e_2) = \mu(e_2)l, \dots, t_p(lx)(e_r) = \mu(e_r)l, \tag{4.4}$$

and thus

$$t_p(lx) = \mu(e_1)le_1 + \mu(e_2)le_2 + \cdots + \mu(e_r)le_r. \tag{4.5}$$

We will prove below that t_p is continuous. Hence, on homotopy groups, the map t_p induces the *homolgy transfer*

$$\tau_p : \widetilde{H}_q(X;L) \longrightarrow \widetilde{H}_q(E;L). \tag{4.6}$$

We have the following.

Proposition 4.3. *Let $p : E \to X$ be an n-fold ramified covering map with multiplicity function $\mu : E \to \mathbb{N}$. Then the pretransfer $t_p : F(X,L) \to F(E,L)$ is continuous.*

Proof. Since $F(X,L)$ has the topology of the union of the closed subspaces

$$\cdots \subset F_r(X,L) \subset F_{r+1}(X,L) \subset \cdots \subset F(X,L), \tag{4.7}$$

t_p is continuous if and only if the restriction $t_p|_{F_r(X,L)}$ is continuous for each $r \in \mathbb{N}$. We have a quotient map $q_r : (L \times X)^r \twoheadrightarrow F_r(X,L)$ for each r. Define $\delta : L \times X \to F_n(E,L)$ by

$\delta(l,x) = t_p q_1(l,x) = t_p(lx)$, and $\alpha : L \times X \to (L \times E)^n/\Sigma_n$ by

$$\alpha(l,x) = \langle \underbrace{(l,e_1),\ldots,(l,e_1)}_{\mu(e_1)},\ldots,\underbrace{(l,e_m),\ldots,(l,e_m)}_{\mu(e_m)} \rangle, \tag{4.8}$$

where $p^{-1}(x) = \{e_1,\ldots,e_m\}$. For each $l \in L$, let $i_l : X \to L \times X$ be given by $i_l(x) = (l,x)$, and let $j_l : E^n/\Sigma_n \to (L \times E)^n/\Sigma_n$ be given by $j_l\langle e_1,\ldots,e_n\rangle = \langle (l,e_1),\ldots,(l,e_n)\rangle$. Then $\alpha \circ i_l = j_l \circ \varphi_p$, where $\varphi_p : X \to \mathrm{SP}^n E$. Since j_l and φ_p are continuous and L is discrete, α is continuous.

The quotient map q_n factors through the quotient map $q'_n : (L \times E)^n \twoheadrightarrow (L \times E)^n/\Sigma_n$, yielding the following commutative diagram:

$$\begin{tikzcd}
(L \times E)^n \arrow[r, two heads, "q'_n"] \arrow[d, "q_n"'] & (L \times E)^n/\Sigma_n \arrow[dl, "\rho_n"] \\
F_n(E,L) &
\end{tikzcd} \tag{4.9}$$

where ρ_n is also a quotient map.

Now, δ makes the following diagram commute:

$$\begin{tikzcd}
& (L \times E)^n/\Sigma_n \arrow[d, "\rho_n"] \\
L \times X \arrow[ur, "\alpha"] \arrow[r, "\delta"'] & F_n(E,L)
\end{tikzcd} \tag{4.10}$$

therefore, δ is continuous.

In order to show that $t_p|_{F_r(X,L)}$ is continuous, consider the diagram

$$\begin{tikzcd}
(L \times X)^r \arrow[r, "\delta^r"] \arrow[d, "q_r"'] & F_n(E,L) \times \cdots \times F_n(E,L) \arrow[d, "\mathrm{sum}"] \\
F_r(X,L) \arrow[r, "t_p|_{F_r(X,L)}"'] & F(E,L)
\end{tikzcd} \tag{4.11}$$

where sum is given by the operation in $F(E,L)$, which is continuous. Since δ is continuous and q_r is a quotient map, $t_p|_{F_r(X,L)}$ is continuous. $\square$

Corollary 4.4. *Let $p : E \to X$ be an n-fold ramified covering map with multiplicity function $\mu : E \to \mathbb{N}$. Then there is a homology transfer $\tau_p : \widetilde{H}_q(X;L) \to \widetilde{H}_q(E;L)$.*

Remark 4.5. Besides the transfer τ_p defined above, for every integer k there is another homology transfer ${}_k\tau$ given by $({}_k\tau)_p(\xi) = k \cdot \tau_p(\xi)$, $\xi \in H_q(X;L)$. This transfer, in turn, is determined by the pretransfer $({}_kt)_p : F(X,L) \to F(E,L)$ given by $({}_kt)_p(u) = k \cdot t_p(u)$, $u \in F(X,L)$.

Example 4.6. For the ramified covering map $\pi_B : B^n \times_{\Sigma_n} \overline{n} \to \mathrm{SP}^n B$ of Examples 3.6, the homology transfer is given as follows. We first compute

$$t_{\pi_B} : F(\mathrm{SP}^n B, L) \longrightarrow F(B^n \times_{\Sigma_n} \overline{n}, L) \tag{4.12}$$

on the generators. Set

$$b = (\underbrace{b_1,\dots,b_1}_{i_1},\underbrace{b_2,\dots,b_2}_{i_2},\dots,\underbrace{b_r,\dots,b_r}_{i_r}) \in B^n, \tag{4.13}$$

where $i_1 + i_2 + \cdots + i_r = n$. Then

$$\pi_B^{-1}\langle b\rangle = \{\langle b, i_1\rangle, \langle b, i_1 + i_2,\rangle, \dots, \langle b, n\rangle\}. \tag{4.14}$$

Therefore,

$$\begin{aligned}
t_{\pi_B}(l\langle x\rangle) &= \mu\langle b, i_1\rangle l\langle b, i_1\rangle + \mu\langle b, i_1 + i_2\rangle l\langle b, i_1 + i_2\rangle \\
&\quad + \cdots + \mu\langle b, i_1 + i_2 + \cdots + i_r\rangle l\langle b, i_1 + i_2 + \cdots + i_r\rangle \\
&= i_1 l\langle b, i_1\rangle + i_2 l\langle b, i_1 + i_2\rangle + \cdots + i_r l\langle b, i_1 + i_2 + \cdots i_r\rangle \\
&= \underbrace{l\langle b, i_1\rangle + l\langle b, i_1\rangle + \cdots + l\langle b, i_1\rangle}_{i_1} \\
&\quad + \underbrace{l\langle b, i_1 + i_2\rangle + l\langle b, i_1 + i_2\rangle + \cdots + l\langle b, i_1 + i_2\rangle}_{i_2} \\
&\quad + \cdots + \underbrace{l\langle b, n\rangle + l\langle b, n\rangle + \cdots + l\langle b, n\rangle}_{i_r} \\
&= l\langle b, 1\rangle + \cdots + l\langle b, i_1\rangle + l\langle b, i_1 + 1\rangle + \cdots + l\langle b, i_1 + i_2\rangle \\
&\quad + \cdots + l\langle b, i_1 + i_2 + \cdots + i_{r-1} + 1\rangle + \cdots + l\langle b, n\rangle \\
&= l\langle b, 1\rangle + l\langle b, 2\rangle + \cdots + l\langle b, n\rangle,
\end{aligned} \tag{4.15}$$

hence,

$$t_{\pi_B}(l\langle b_1, \dots, b_n\rangle) = l\langle b_1, \dots, b_n; 1\rangle + \cdots + l\langle b_1, \dots, b_n; n\rangle. \tag{4.16}$$

Thus, in general, if $\beta = \sum_{i=1}^{k} l_i\langle b_1^i, \dots, b_n^i\rangle$, then

$$t_{\pi_B}(\beta) = \sum_{(i,j)=(1,1)}^{(k,n)} l_i\langle b_1^i, \dots, b_n^i; j\rangle, \tag{4.17}$$

since by varying j from 1 to n, the fiber elements over $\langle b_1^i, \dots, b_n^i\rangle$, namely $\langle b_1^i, \dots, b_n^i; j\rangle$, are repeated $\mu_B\langle b_1^i, \dots, b_n^i; j\rangle$ times.

Remark 4.7. Given an n-fold ramified covering map $p: E \to X$ with multiplicity function $\mu: E \to \mathbb{N}$, and a (closed) subspace $A \subset X$, we have the *restricted ramified covering map* $p_A: E_A \to A$, $E_A = p^{-1}A$, and the *quotient ramified covering map* $p': E' = E/E_A \to X/A$, as described in Remark 3.5. The following diagram obviously commutes:

$$\begin{tikzcd} E_A \arrow[r, hook] \arrow[d, "p_A"'] & E \arrow[r, two heads] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ A \arrow[r, hook] & X \arrow[r, two heads] & X/A \end{tikzcd} \tag{4.18}$$

Thus the above diagram yields

$$\begin{tikzcd} F(A,L) \arrow[r] \arrow[d, "t_A"'] & F(X,L) \arrow[r] \arrow[d, "t"'] & F(X/A,L) \arrow[d, "t'"] \\ F(E_A,L) \arrow[r] & F(E,L) \arrow[r] & F(E',L) \end{tikzcd} \tag{4.19}$$

where the horizontal arrows are obvious and t_A, t, and t' are the corresponding pretransfers. Therefore, using t', we have a *relative homology transfer* $\tau_p: H_n(X,A;L) \to H_n(E,E_A;L)$, and by the commutativity of the diagram, also this transfer maps the long exact sequences of (X,A) into the long exact sequence of (E,E_A), provided that the inclusion $A \hookrightarrow X$ is a closed cofibration (in general it is also true by constructing an adequate ramified covering map over $X \cup CA$).

The following theorems establish the fundamental properties of the transfer.

Theorem 4.8. *Let $p: E \to X$ be an n-fold ramified covering map. Then the composite*

$$p_* \circ \tau_p: \widetilde{H}_q(X;L) \longrightarrow \widetilde{H}_q(X;L) \tag{4.20}$$

is multiplication by n.

The *proof* follows immediately from the following proposition.

Proposition 4.9. *If $p: E \to X$ is an n-fold ramified covering map, then the composite*

$$F(X,L) \xrightarrow{t_p} F(E,L) \xrightarrow{p_*} F(X,L) \tag{4.21}$$

is multiplication by n.

Proof. If $u = \sum_{i=1}^r l_i x_i \in F(X,L)$, then $p_* t_p(u) = p_* t_p(\sum_{i=1}^r l_i x_i) = \sum_{p(e)=x_i, i=1,\ldots,r} \mu(e) l_i x_i = \sum_{i=1}^r l_i x_i \sum_{p(e)=x_i} \mu(e) = n \sum_{i=1}^r l_i x_i = n \cdot u$. □

The invariance under pullbacks is given by the following.

Theorem 4.10. *Let $p : E \to X$ be an n-fold ramified covering map and assume that $f : Y \to X$ is continuous. Then the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{H}_q(Y;L) & \xrightarrow{\tau_{f^*(p)}} & \widetilde{H}_q(f^*(E);L) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \widetilde{f}_*} \\ \widetilde{H}_q(X;L) & \xrightarrow[\tau_p]{} & \widetilde{H}_q(E;L) \end{array} \tag{4.22}$$

where $f^*(p) : f^*(E) \to Y$ is the n-fold ramified covering map induced by $p : E \to X$ over f.

As for the previous theorem, the *proof* follows immediately from the next proposition.

Proposition 4.11. *If $p : E \to X$ is an n-fold ramified covering map and $f : Y \to X$ is continuous, then the following diagram commutes:*

$$\begin{array}{ccc} F(Y,L) & \xrightarrow{t_{f^*(p)}} & F(f^*(E),L) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \widetilde{f}_*} \\ F(X,L) & \xrightarrow[t_p]{} & F(E,L) \end{array} \tag{4.23}$$

Proof. Let $v = \sum_{i=1}^{r} l_i y_i \in F(Y,L)$. Then $t_{f^*(p)}(v) \in F(f^*(E),L)$ is such that

$$\begin{aligned} \widetilde{f}_*\left(t_{f^*(p)}(v)\right) &= \widetilde{f}_*\left(\sum_{\substack{f^*(p)(y,e)=y_i \\ i=1,\dots,r}} f^*(\mu)(y,e)l_i(y,e)\right) \\ &= \sum_{\substack{f^*(p)(y,e)=y_i \\ i=1,\dots,r}} \mu(e)l_i\widetilde{f}(y,e) = \sum_{\substack{p(e)=f(y_i) \\ i=1,\dots,r}} \mu(e)l_i e = t_p\left(f_*(v)\right). \end{aligned} \tag{4.24}$$

$\square$

Take maps $f_0, f_1 : Y \to X$ and let $p : E \to X$ be an n-fold ramified covering map. We have the induced covering maps over Y as follows:

$$\begin{array}{ccc} f_0^*(E) & \xrightarrow{\widetilde{f}_0} & E \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f_0]{} & X \end{array} \qquad \begin{array}{ccc} f_1^*(E) & \xrightarrow{\widetilde{f}_1} & E \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f_1]{} & X \end{array} \tag{4.25}$$

Another property of the transfer is the following homotopy invariance.

Theorem 4.12. *If $f_0, f_1 : Y \to X$ are homotopic and $p : E \to X$ is an n-fold ramified covering map, then*

$$\widetilde{f}_{0*} \circ \tau_{p_0} = \widetilde{f}_{1*} \circ \tau_{p_1} : \widetilde{H}_q(Y;L) \longrightarrow \widetilde{H}_q(E;L). \tag{4.26}$$

The *proof* is an immediate consequence of the next proposition.

PROPOSITION 4.13. *If $f_0 \simeq f_1 : Y \to X$ and $p : E \to X$ is an n-fold ramified covering map, then*

$$\widetilde{f}_{0*} \circ t_{p_0} \simeq \widetilde{f}_{1*} \circ t_{p_1} : F(Y,L) \longrightarrow F(E,L). \tag{4.27}$$

Proof. If $H : Y \times I \to X$ is a homotopy from f_0 to f_1, then $\widehat{H} : F(Y,L) \times I \to F(X,L)$ given by $\widehat{H}(v,t) = \sum_{y\in Y} v(y)H(y,t)$ is a (continuous) homotopy from f_{0*} to f_{1*}. Thus, applying Proposition 4.11, $\widetilde{f}_{0*} \circ t_{p_0} = t_p \circ f_{0*} \simeq t_p \circ f_{1*} = \widetilde{f}_{1*} \circ t_{p_1}$. □

One further property of the homology transfer that is useful is given by the following proposition.

PROPOSITION 4.14. *Let $f : B \to C$ be continuous and consider the commutative diagram*

$$\begin{tikzcd} B^n \times_{\Sigma_n} \overline{n} \arrow[r, "f^n \times_{\Sigma_n} 1_{\overline{n}}"] \arrow[d, "\pi_B"'] & C^n \times_{\Sigma_n} \overline{n} \arrow[d, "\pi_C"] \\ \mathrm{SP}^n B \arrow[r, "\mathrm{SP}^n f"'] & \mathrm{SP}^n C \end{tikzcd} \tag{4.28}$$

Then the following diagram commutes:

$$\begin{tikzcd} F(B^n \times_{\Sigma_n} \overline{n}, L) \arrow[r, "(f^n \times_{\Sigma_n} 1_{\overline{n}})_*"] & F(C^n \times_{\Sigma_n} \overline{n}, L) \\ F(\mathrm{SP}^n B, L) \arrow[r, "(\mathrm{SP}^n f)_*"'] \arrow[u, "t_{\pi_B}"] & F(\mathrm{SP}^n C, L) \arrow[u, "t_{\pi_C}"'] \end{tikzcd} \tag{4.29}$$

Proof. Using the description of the transfers given in Example 4.6, one can easily verify that the diagram commutes. □

In Proposition 4.9, we computed the composite $p_* \circ t_p$. The opposite composite $t_p \circ p_*$ is also interesting. An immediate computation yields the following.

PROPOSITION 4.15. *Let $p : E \to X$ by an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$F(E,L) \xrightarrow{p_*} F(X,L) \xrightarrow{t_p} F(E,L) \tag{4.30}$$

is given by

$$t_p p_*(v)(e) = \mu(e) \sum_{p(e')=p(e)} v(e'), \tag{4.31}$$

for any $v \in F(E,L)$.

In the case of an action of a finite group G on E and $X = E/G$, we have the following consequence.

Corollary 4.16. *For $v \in F(E,L)$, one has $t_p p_*(v)(e) = \sum_{g\in G} v(ge)$. Therefore, the composite*

$$F(E,L) \xrightarrow{p_*} F(E/G,L) \xrightarrow{t_p} F(E,L) \tag{4.32}$$

is given by $t_p p_*(v) = \sum_{g\in G} g_*(v)$.

Proof. Just observe that the element ge is repeated in the sum $\mu(e) = |G_e|$ times. □

The two previous results yield the following in homology.

Theorem 4.17. *Let $p : E \to X$ by an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$\widetilde{H}_q(E;L) \xrightarrow{p_*} \widetilde{H}_q(X;L) \xrightarrow{\tau_p} \widetilde{H}_q(E;L) \tag{4.33}$$

is given by $\tau_p p_*(y) = y'$, where $y' = [v'] \in \pi_q(F(E,L))$ and

$$v'(s)(e) = \mu(e) \sum_{p(e')=p(e)} v(s)(e'), \tag{4.34}$$

where $y = [v] \in \pi_q(F(E,L))$ and $s \in \mathbb{S}^q$.

Corollary 4.18. *For an action of a finite group G on E and $X = E/G$, one has that the composite*

$$\widetilde{H}_q(E;L) \xrightarrow{p_*} \widetilde{H}_q(E/G;L) \xrightarrow{\tau_p} \widetilde{H}_q(E;L) \tag{4.35}$$

is given by $\tau_p p_*(y) = \sum_{g\in G} g_*(y)$.

Remark 4.19. Considering an action of G on E and a subgroup $H \subset G$, one has different ramified covering maps as depicted in

$$\begin{tikzcd} & E \arrow[dl, "q_H"'] \arrow[dr, "q_G"] & \\ E/H \arrow[rr, "q_G^H"'] & & E/G \end{tikzcd} \tag{4.36}$$

One may easily compute several combinations of the maps induced by these covering maps and their transfers.

Another interesting property of the transfer is the relationship given by computing the transfer of the composition of two ramified covering maps. Before giving it we need the following.

Definition 4.20. Let $p : Y \to X$ be an n-fold ramified covering map, with multiplicity function $\mu : Y \to \mathbb{N}$, and let $q : Z \to Y$ be an m-fold ramified covering map, with multiplicity function $\nu : Z \to \mathbb{N}$. Then, the composite $p \circ q : Z \to X$ is an mn-fold ramified covering map, with multiplicity function $\xi : Z \to \mathbb{N}$ given by $\xi(z) = \nu(z)\mu(q(z))$. In order to verify that this composite is indeed an mn-fold ramified covering map, consider the *wreath product* $\Sigma_n \wr \Sigma_m$, defined as the semidirect product of Σ_n and $(\Sigma_m)^n$, where Σ_n acts on $(\Sigma_m)^n$ by permuting the n factors. There exists an action $(Z^m \times \cdots \times Z^m) \times \Sigma_n \wr \Sigma_m \to Z^m \times \cdots \times Z^m$ given by $(\zeta_1, \dots, \zeta_n) \cdot (\sigma, \tau_1, \dots, \tau_n) = (\zeta_{\sigma(1)} \cdot \tau_1, \dots, \zeta_{\sigma(n)} \cdot \tau_n)$, where $\zeta_i \in Z^m$. Then we have the following diagram, where all maps are open:

$$\begin{tikzcd}
Z^m \times \cdots \times Z^m \arrow[r, "q \times \cdots \times q"] \arrow[d, "\pi"'] & Z^m/\Sigma_m \times \cdots \times Z^m/\Sigma_m \arrow[d, "\pi'"] \\
(Z^m)^n/\Sigma_n \wr \Sigma_m \arrow[r, dashed] & \mathrm{SP}^n(\mathrm{SP}^m Z)
\end{tikzcd} \tag{4.37}$$

One may easily show that π is compatible with $\pi' \circ (q \times \cdots \times q)$. Therefore, there is a homeomorphism $Z^{mn}/\Sigma_n \wr \Sigma_m \approx \mathrm{SP}^n(\mathrm{SP}^m Z)$, and hence one has a canonical quotient map $\rho : \mathrm{SP}^n(\mathrm{SP}^m Z) \to \mathrm{SP}^{mn} Z$. Then one can easily verify that $\varphi_{p\circ q} = \rho \circ \mathrm{SP}^n(\varphi_q) \circ \varphi_p : X \to \mathrm{SP}^n(\mathrm{SP}^m Z) \xrightarrow{\rho} \mathrm{SP}^{mn} Z$. Thus $\varphi_{p\circ q}$ is continuous.

The homology transfer behaves well with respect to composite ramified covering maps.

Theorem 4.21. *Let $p : Y \to X$ and $q : Z \to Y$ be ramified covering maps. Then the following hold:*

$$\begin{gathered}
t_{p\circ q} = t_q \circ t_p : F(X;L) \xrightarrow{t_p} F(Y;L) \xrightarrow{t_q} F(Z;L); \\
\tau_{p\circ q} = \tau_q \circ \tau_p : H_k(X;L) \xrightarrow{\tau_p} H_k(Y;L) \xrightarrow{\tau_q} H_k(Z;L).
\end{gathered} \tag{4.38}$$

Proof. As before, the second formula follows from the first. Take $u \in F(X;L)$, $v \in F(Y;L)$, $w \in F(Z;L)$, then $v = t_p(u)$ if $v(y) = \mu(y)u(p(y))$, and $w = t_q(v)$ if $w(z) = \nu(z)v(q(z))$. Hence $(t_q t_p(u))(z) = \nu(z)v(q(z)) = \nu(z)\mu(q(z))u(pq(z)) = \xi(z)u((p\circ q)(z)) = t_{p\circ q}(u)(z)$. □

Corollary 4.22. *Given an n-fold ramified covering map $p : E \to X$ with multiplicity function μ and an integer m, there is an mn-fold ramified covering map $p_m : E \to X$ such that $p_m = p$ and $\mu_m(e) = m\mu(e)$, $e \in E$. Then $t_{p_m} = mt_p : F(X;L) \to F(E;L)$ and $\tau_{p_m} = m\tau_p : \widetilde{H}_k(X;L) \to \widetilde{H}_k(E;L)$.*

Proof. Consider the m-fold ramified covering map $q : E \to E$ such that $q = \mathrm{id}_E$ and $\nu(e) = m$ for all $e \in E$. Hence $p_m = p \circ q$. Then apply Theorem 4.21. □

Remark 4.23. The mn-fold covering map p_m obtained from p is a sort of spurious ramified covering map, since the multiplicity of p is artificially multiplied by m. It is interesting to remark that the previous result shows that the transfer of this new ramified covering map p_m is just the corresponding multiple of the transfer of the original ramified covering map p. Thus on this sort of artificial ramified covering maps, the transfer remains essentially unchanged.

5. The cohomology transfer

In this section, we define the cohomology transfer and prove some of its properties.

Definition 5.1. Let $p : E \to X$ be an n-fold ramified covering map with multiplicity function μ. Define its *cohomology transfer*

$$\tau^p : H^q(E;L) = \big[E, F(\mathbb{S}^q, L)\big] \longrightarrow \big[X, F(\mathbb{S}^q, L)\big] = H^q(X;L) \tag{5.1}$$

by $\tau^p([\widetilde{\alpha}]) = [\alpha]$, where $\alpha(x) = \sum_{p(e)=x} \mu(e)\widetilde{\alpha}(e)$, $x \in X$. To see that the map α is continuous and that its homotopy class depends only on the homotopy class of $\widetilde{\alpha}$, observe that α is given by the composite

$$\alpha : X \xrightarrow{\varphi_p} \mathrm{SP}^n E \xrightarrow{\mathrm{SP}^n \widetilde{\alpha}} \mathrm{SP}^n F(\mathbb{S}^q, L) \longrightarrow F(\mathbb{S}^q, L), \tag{5.2}$$

where the last map is given by the group structure on $F(\mathbb{S}^q, L)$, adding the components.

Remark 5.2. We might assume that E and X are paracompact spaces instead of spaces of the same homotopy type of a CW-complex. In this case, the same definition yields a transfer that is a homomorphism between Čech cohomology groups:

$$\tau^p : \check{H}^q(E;L) \longrightarrow \check{H}^q(X;L), \tag{5.3}$$

since in this case homotopical cohomology coincides with Čech cohomology (see [10]).

Note 5.3. In order to define the cohomology transfer, the only property of the Eilenberg-Mac Lane spaces given by $F(\mathbb{S}^q, L)$ required is the fact that they are topological abelian groups in the category of compactly generated spaces.

Similarly to the homology transfer, the cohomology transfer has the following fundamental properties.

Theorem 5.4. *If $p : E \to X$ is an n-fold ramified covering map, then the composite*

$$\tau^p \circ p^* : H^q(X;L) \longrightarrow H^q(X;L) \tag{5.4}$$

is multiplication by n.

Proof. If $[\alpha] \in [X, F(\mathbb{S}^q, L)]$, then $\tau^p p^*(\alpha) = \tau^p(\alpha \circ p) : X \to F(\mathbb{S}^q, L)$, and $\tau^p(\alpha \circ p)(x) = \sum_{p(e)=x} \mu(e)\alpha p(e) = (\sum_{p(e)=x} \mu(e))\alpha(x) = n \cdot \alpha(x)$. Thus, $\tau^p p^*([\alpha]) = n \cdot [\alpha]$. □

Theorem 5.5. *Let $p : E \to X$ be an n-fold ramified covering map and assume that $f : Y \to X$ is continuous. Then the following diagram commutes:*

$$\begin{array}{ccc} H^q(E;L) & \xrightarrow{\tau^p} & H^q(X;L) \\ {\scriptstyle \widetilde{f}^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^q(f^*(E);L) & \xrightarrow[\tau^{f^*(p)}]{} & H^q(Y;L) \end{array} \tag{5.5}$$

where $f^*(p) : f^*(E) \to Y$ is the n-fold ramified covering map induced by $p : E \to X$ over f.

Proof. Let $\widetilde{\alpha} : E \to F(\mathbb{S}^q, L)$ represent an element in $H^q(E;L)$. Then the map

$$y \longmapsto \sum_{f_*(p)(y,e)=y} f^*(\mu)(y,e)\widetilde{f}^*(\widetilde{\alpha})(y,e) = \sum_{p(e)=f(y)} \mu(e)\widetilde{\alpha}(e), \tag{5.6}$$

that represents $\tau^{f^*(p)}\widetilde{f}^*(\widetilde{\alpha})$, clearly represents also $f^*\tau^p([\widetilde{\alpha}]) \in H^q(Y;L)$. □

Similarly to Theorem 4.12, we also have the next property, which easily follows from Theorem 5.5.

Theorem 5.6. *If $f_0, f_1 : Y \to X$ are homotopic and $p : E \to X$ is an n-fold ramified covering map, then*

$$\tau^{p_0} \circ \widetilde{f}_0^* = \tau^{p_1} \circ \widetilde{f}_1^* : H^q(E;L) \longrightarrow H^q(Y;L). \tag{5.7}$$

In Theorem 5.4 we computed the composite $\tau^p \circ p^*$. The opposite composite $p^* \circ \tau^p$ is also interesting. As it was the case for the homology transfer, an immediate computation yields the following results for the cohomology transfer.

Proposition 5.7. *Let $p : E \to X$ be an n-fold ramified covering map with multiplicity function μ. Then the composite*

$$H^q(E;L) \xrightarrow{\tau^p} H^q(X;L) \xrightarrow{p^*} H^q(E;L) \tag{5.8}$$

is given as follows. Take $[\widetilde{\alpha}] \in H^q(E;L) = [E, F(\mathbb{S}^q, L)]$, then $p^*\tau^p[\widetilde{\alpha}]$ is represented by the map $\widetilde{\alpha}' : E \to F(\mathbb{S}^q, L)$ given by

$$\widetilde{\alpha}'(e) = \sum_{p(e')=p(e)} \mu(e')\widetilde{\alpha}(e'). \tag{5.9}$$

In the case of an action of a finite group G on E and $X = E/G$, we have the following consequence.

Corollary 5.8. *If $\xi \in H^q(E;L)$, then*

$$p^*\tau^p(\xi) = \sum_{g\in G} g^*(\xi) \in H^q(E;L). \tag{5.10}$$

Proof. Just observe that in the sum the element $g^*(\xi)$ is repeated $\mu(e) = |G_e|$ times. □

Generalizations and further properties of the cohomology transfer are studied in [2].

6. Transfers in other theories

Karoubi [11] has defined the cohomology of a pointed space X as the cohomology of the de Rham complex of noncommutative topological forms on X. The de Rham complex is defined as follows. Let L be any abelian group and consider the group of continuous maps

$$\Omega^q(X;L) = \operatorname{Map}\left(X, F\left(\mathbb{B}^{q+1}, L\right)\right), \tag{6.1}$$

where $\mathbb{B}^{q+1}$ is the $(q+1)$-dimensional ball. The coboundary homomorphism

$$\delta : \Omega^{q-1}(X;L) \longrightarrow \Omega^q(X;L) \tag{6.2}$$

is given by sending $f : X \to F(\mathbb{B}^q, L)$ to the composite

$$X \xrightarrow{f} F\left(\mathbb{B}^q, F\right) \xrightarrow{\kappa_*} F\left(\mathbb{B}^{q+1}, L\right), \tag{6.3}$$

where $\kappa : \mathbb{B}^q \to \mathbb{S}^q \hookrightarrow \mathbb{B}^{q+1}$; here the first map collapses the boundary of the ball to a point while the second includes the sphere as the boundary of the next ball.

Given an n-fold ramified covering map $p : E \to X$, define

$$\rho^p : \Omega^q(E;L) \longrightarrow \Omega^q(X;L) \tag{6.4}$$

by $f \mapsto m_{\mathbb{B}^{q+1}} \circ \mathrm{SP}^n f \circ \varphi_p$, namely, by the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi_p"] \arrow[drr, dashed, "\rho^p(f)"'] & \mathrm{SP}^n E \arrow[r, "\mathrm{SP}^n f"] & \mathrm{SP}^n F\left(\mathbb{B}^{q+1}, L\right) \arrow[d, "m_{\mathbb{B}^{q+1}}"] \\ & & F\left(\mathbb{B}^{q+1}, L\right) \end{tikzcd} \tag{6.5}$$

where φ_p is the structure map of p and $m_{\mathbb{B}^{q+1}}$ stands for the group operation in $F(\mathbb{B}^{q+1}, L)$.

One easily proves that this is a cochain homomorphism and thus it defines a *transfer* in the cohomology of Karoubi's cochain complex

$$\rho^p : H^q\left(\Omega^*(E;L)\right) \longrightarrow H^q\left(\Omega^*(X;L)\right). \tag{6.6}$$

Karoubi [11] has also defined (reduced) homology as follows. Take the group of continuous pointed maps

$$\widetilde{\Omega}_q(X;L) = \operatorname{Map}_*\left(\mathbb{B}^q, F(X, L)\right), \tag{6.7}$$

where the boundary homomorphism

$$\partial : \widetilde{\Omega}_{q+1}(X;L) \longrightarrow \widetilde{\Omega}_q(X;L) \tag{6.8}$$

is given by $f \mapsto f \circ \kappa$, κ as above. For a ramified covering map $p : E \to X$, using the pretransfer of Definition 4.1, $t_p : F(X,L) \to F(E,L)$, we define $\rho_p : \widetilde{\Omega}_q(X;L) \to \widetilde{\Omega}_q(E;L)$ by

$f \mapsto t_p \circ f$. This map is easily seen to commute with the boundary homomorphism, so that one has a *transfer* in the homology of Karoubi's chain complex

$$\rho_p : H_q(\widetilde{\Omega}_*(X;L)) \longrightarrow H_q(\widetilde{\Omega}_*(E;L)). \tag{6.9}$$

These transfers are equivalent to the ones defined above; namely, one has the following.

Theorem 6.1. *Let $p : E \to X$ be a ramified covering map.*

(a) *There is a natural isomorphism $[-,F(\mathbb{S}^q,L)] \to H^q(\Omega^*(-;L))$ such that the following is a commutative diagram:*

$$\begin{tikzcd}
{[E,F(\mathbb{S}^q,L)]} \arrow[r, "\cong"] \arrow[d, "\tau^p"'] & H^q(\Omega^*(E;L)) \arrow[d, "\rho^p"] \\
{[X,F(\mathbb{S}^q,L)]} \arrow[r, "\cong"] & H^q(\Omega^*(X;L))
\end{tikzcd} \tag{6.10}$$

(b) *There is a natural isomorphism $H_q(\widetilde{\Omega}_*(-;L)) \to \pi_q(F(-,L))$ such that the following is a commutative diagram:*

$$\begin{tikzcd}
H_q(\widetilde{\Omega}_*(X;L)) \arrow[r, "\cong"] \arrow[d, "\rho_p"'] & \pi_q(F(X,L)) \arrow[d, "t_{p*}=\tau_p"] \\
H_q(\widetilde{\Omega}_*(E;L)) \arrow[r, "\cong"] & \pi_q(F(E,L))
\end{tikzcd} \tag{6.11}$$

Proof. (a) The isomorphism on the top of the diagram is given by mapping the homotopy class $[f]$ to the cohomology class $[i_* \circ f]$, where $i : \mathbb{S}^q \hookrightarrow \mathbb{B}^{q+1}$ is the canonical inclusion. The isomorphism on the bottom is given similarly. It is now a direct confirmation that the diagram commutes.

(b) The isomorphism on the top of the diagram is given by mapping the homology class of the cycle $[f]$ (which is such that $f|_{\partial\mathbb{B}^q}$ is constant) to the homotopy class $[\overline{f}]$, where $\overline{f} : \mathbb{S}^q = \mathbb{B}^q/\partial\mathbb{B}^q \to F(X;L)$ is induced by f. The isomorphism on the bottom is similar and the diagram is clearly commutative. □

More generally, given a natural transformation of $\Phi : h \to H_q(-;L)$, where h is any homotopy functor, and given an n-fold ramified covering map $p : E \to X$, one can define a *transfer*

$$T_p : h(X) \xrightarrow{\Phi_X} H_q(X;L) \xrightarrow{\tau_p} H_q(E;L). \tag{6.12}$$

This transfer has similar properties to those established in Theorem 4.10 and Proposition 4.14. Also we have the following formula:

$$H_q(p)T_p(a) = n\Phi_X(a). \tag{6.13}$$

Examples of such natural transformations are the following:

(1) the Hurewicz homomorphisms

$$\pi_q \longrightarrow H_q(-;Z), \qquad \pi_q^{\mathrm{st}} \longrightarrow H_q(-;Z); \tag{6.14}$$

(2) the ones given by the Thom homomorphisms

$$\mathfrak{N}_q^G(-) \longrightarrow H_q(-;L), \tag{6.15}$$

where $\mathfrak{N}_q^G$ is any bordism theory, for instance, unoriented, oriented, or complex bordism ($G = O, SO, U$); L varies according to the bordism theory.

Similarly, given a natural transformation $\Phi : h \to H^q(-;L)$ of any contravariant homotopy functor h, one has a *transfer*

$$T^p : h(E) \xrightarrow{\Phi_X} H^q(E;L) \xrightarrow{\tau^p} H^q(X;L) \tag{6.16}$$

with similar properties to those of the cohomology transfer like Theorem 5.4 (see [2], where a classification of transfers between representable cofunctors is given). Examples of these natural transformations are the Hurewicz homomorphisms from stable cohomotopy to cohomology or the Thom homomorphisms from cobordism theories to cohomology.

7. Some applications of the transfers

First we start considering a standard n-fold covering map $p : E \to X$. In this case, the pretransfer (and thus also the transfer in homology) has a particularly nice definition. Since the multiplicity function $\mu : E \to \mathbb{N}$ is constant $\mu(e) = 1$, the pretransfer $t_p : F(X,L) \to F(E,L)$ is given by

$$t_p(u)(e) = u(p(e)). \tag{7.1}$$

This fact has a nice consequence.

Theorem 7.1. *Let G be a finite group acting freely on a Hausdorff space E. Then the orbit map $p : E \to E/G$ is a standard covering map, and its pretransfer induces an isomorphism*

$$t_p : F(E/G,L) \xrightarrow{\cong} F(E,L)^G, \tag{7.2}$$

where the second term represents the fixed points under the induced G-action on $F(E,L)$. Consequently, the pretransfer yields an isomorphism

$$\widetilde{H}_q(E/G;L) \xrightarrow{\cong} \pi_q\bigl(F(E,L)^G\bigr), \tag{7.3}$$

for all q.

Proof. We assume that the projection $p : E \to E/G$ maps the base point to the base point. The pretransfer t_p is a monomorphism. Namely, if $t_p(u) = 0$, then, by (7.1), $u(p(e)) = t_p(u)(e) = 0$ for all $e \in E$. Since p is surjective, $u = 0$.

On the other hand, obviously $t_p(u) \in F(E,L)^G$ for all $u \in F(E/G,L)$. To see that it is an epimorphism, take any $v \in F(E,L)^G$. Then $v(e) = v(eg)$ for all $g \in G$, and thus v determines a well-defined element $u \in F(E/G,L)$ by $u(eG) = v(e)$. Then clearly $t_p(u) = v$.

Finally, note that $p_*(F(E,L)^G) \subset F(E/G,|G|L)$. Dividing by $|G|$ yields a continuous homomorphism $\gamma : F(E/G,|G|L) \to F(E/G,L)$. Thus, the inverse is given by $\gamma \circ (p_*|_{F(E,L)^G})$. □

In what follows, we use the fundamental properties of Theorem 4.8 and Corollary 4.18, and Theorem 5.4 and Corollary 5.8 of both the homology and the cohomology transfers to prove some results about the homology and cohomology of orbit maps between orbit spaces of the action of a topological group G and a subgroup H of finite index. We will require that the (right) action of a topological group G on a space Y, both in the category of compactly generated spaces, satisfies that the set $\{(y, yg) \mid g \in G\} \subset Y \times Y$ is closed. This guarantees that the orbit space Y/G is again compactly generated (see [15]). This will always be the case, when G is a compact group. (There are corresponding results in Čech cohomology for paracompact spaces.)

Before starting we need to recall Dold's definition of an n-fold ramified covering map [6]. It is a finite-to-one map $p : E \to X$ together with a continuous map $\psi_p : X \to \mathrm{SP}^n E$ such that

(i) for every $e \in E$, e appears in the n-tuple $\psi_p(p(e)) = \langle e_1, \ldots, e_n \rangle$,

(ii) $\mathrm{SP}^n(p)\psi_p(x) = \langle x, \ldots, x \rangle \in \mathrm{SP}^n X$.

This definition is equivalent to Smith's (see Definition 3.1), by setting $\varphi_p = \psi_p$ and defining $\mu(e)$ as the number of times that e is repeated in $\psi_p(p(e))$.

Proposition 7.2. *Let G be a topological group acting on a space Y on the right and let $H \subset G$ be a subgroup of finite index n. Then the orbit map $p : Y/H \to Y/G$ is an n-fold ramified covering map.*

Proof. There is a commutative diagram

$$\begin{tikzcd} Y \times G \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \times (G/H) \arrow[r, "\nu"'] & Y/H \end{tikzcd} \tag{7.4}$$

where the top map is the action and the vertical maps are the quotient maps (with the compactly generated topology in the products). Take the adjoint map of ν, $\eta : Y \to \mathrm{Map}(G/H, Y/H)$. The function space $\mathrm{Map}(G/H, Y/H)$ has a right G-action given as follows. For an $f : G/H \to Y/H$, define $(f \cdot g)(g_1 H) = f(g(g_1 H)) = f(g g_1 H)$. The map η is then G-equivariant and thus induces a map

$$\overline{\eta} : Y/G \longrightarrow \mathrm{Map}(G/H, Y/H)/G. \tag{7.5}$$

On the other hand, if we identify G/H with the set $\underline{n} = \{1, \ldots, n\}$, then we have a map

$$\mathrm{Map}(G/H, Y/H)/G \longrightarrow \mathrm{Map}(\underline{n}, Y/H)/\Sigma_n = \mathrm{SP}^n(Y/H). \tag{7.6}$$

Let $\psi_p : Y/G \to \mathrm{SP}^n(Y/H)$ be $\overline{\eta}$ followed by the previous map. Then ψ_p satisfies conditions (i) and (ii) and thus p is an n-fold ramified covering map. □

We apply the results of Proposition 4.9 and Corollary 4.16 that we have for the pretransfer to the n-fold ramified covering map described above to obtain the following proposition.

Proposition 7.3. *Let Y be a space with an action of a topological group G and let $H \subset G$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p_* : F(Y/H,R) \to F(Y/G,R)$ is a split (continuous) epimorphism. Moreover, if G is finite and its order m is invertible in R, then the kernel of p_* is the complement in $F(Y/H,R)$ of the invariant subgroup $F(Y/H,R)^G$ under the induced action of G. Thus in this case,*

$$F(Y/G,R) \cong F(Y/H,R)^G; \tag{7.7}$$

in particular, if G is finite and H is trivial, then $m = n$ and

$$F(Y/G,R) \cong F(Y,R)^G. \tag{7.8}$$

Proof. By Proposition 7.2, $p : Y/H \to Y/G$ is an n-fold ramified covering map. By Proposition 4.9, $p_* \circ t_p : F(Y/H,R) \to F(Y/H,R)$ is multiplication by n, hence it is an isomorphism, and consequently p_* is a split epimorphism. Moreover, if G is finite of order m, by Corollary 4.16, we have that $t_p \circ p_* : F(Y/H,R)^G \to F(Y/H,R)^G$ is multiplication by m. So, if m is invertible in R, then $p_* : F(Y/H,R)^G \to F(Y/G,R)$ is an isomorphism. □

With a similar proof to the one above, but now applying Theorem 4.8 and Corollary 4.18, we obtain the following two well-known results (cf. [18, 2.5], [4, 19]).

Theorem 7.4. *Let Y be a space with an action of a topological group G and let $H \subset G$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p_* : H_q(Y/H;R) \to H_q(Y/G;R)$ is a split epimorphism. Moreover, if G is finite and its order m is invertible in R, then the kernel of p_* is the complement of $H_q(Y/H;R)^G$ in $H_q(Y/H;R)$. Thus in this case*

$$H_q(Y/G;R) \cong H_q(Y/H;R)^G; \tag{7.9}$$

and in particular,

$$H_q(Y/G;R) \cong H_q(Y;R)^G. \tag{7.10}$$

Similarly, using Theorem 5.4 and Corollary 5.8 one has for cohomology the following theorem.

Theorem 7.5. *Let Y be a space with an action of a topological group G and let $H \subset G$ be a subgroup of finite index n. Assume that R is a ring where the integer n is invertible. Then $p^* : H^q(Y/G;R) \to H^q(Y/H;R)$ is a split monomorphism. Moreover, if G is finite and its order m is invertible in R, then the image of p^* is $H^q(X;R)^G$. Thus in this case*

$$H^q(Y/G;R) \cong H^q(Y/H;R)^G; \tag{7.11}$$

and in particular,

$$H^q(Y/G;R) \cong H^q(Y;R)^G. \tag{7.12}$$

Remark 7.6. One may take a paracompact space Y with an action of a topological group G and obtain for Čech cohomology an analogous result, namely, $p^* : \check{H}^q(Y/G;R) \to \check{H}^q(Y/H;R)$ is a split monomorphism, and

$$\check{H}^q(Y/G;R) \cong \check{H}^q(Y/H;R)^G. \tag{7.13}$$

A nice application of the previous ideas is the following generalization of a well-known result of Grothendieck [8] (who considers the case $Y = EG$).

Theorem 7.7. *Let G be a compact Lie group and let G_1 be the component of $1 \in G$. Let R be a ring where $n = [G : G_1]$ is an invertible element. For an action of G on a topological space Y, one has*

$$\begin{aligned} H_q(Y/G;R) &\cong H_q(Y/G_1;R)^{G/G_1}, \\ H^q(Y/G;R) &\cong H^q(Y/G_1;R)^{G/G_1}. \end{aligned} \tag{7.14}$$

Moreover, if Y is paracompact instead of having the same homotopy type of a CW*-complex, then $\check{H}^q(Y/G;R) \cong \check{H}^q(Y/G_1;R)^{G/G_1}$.*

8. Duality between the homology and cohomology transfers

In this section, we prove the existence of a duality between the homology transfer and the cohomology transfer.

Theorem 8.1. *Let $p : E \to X$ be an n-fold ramified covering map with multiplicity function $\mu : E \to \mathbb{N}$ and E path connected, and let $\tau_p : H_q(X;R) \to H_q(E;R)$ and $\tau^p : H^q(E;R) \to H^q(X;R)$ be its homology and cohomolgy transfers. If $\xi \in H_q(X;L)$ and $\tilde{\xi} \in H^q(E;L)$, then*

$$\langle \tau_p(\xi), \tilde{\xi} \rangle_E = \langle \xi, \tau^p(\tilde{\xi}) \rangle_X \in R, \tag{8.1}$$

for the Kronecker products for E and X, respectively, and R a commutative ring with 1 (see (2.11)).

Proof. We have to prove the commutativity of the following diagram:

$$\begin{tikzcd}
{[X^+,F(\mathbb{S}^q,R)]_* \times [\mathbb{S}^q,F(X^+,R)]_*} \arrow[r, "\frown"] & {[X^+,F(\mathbb{S}^0,R)]_*} \arrow[dr] & \\
{[E^+,F(\mathbb{S}^q,R)]_* \times [\mathbb{S}^q,F(X^+,R)]_*} \arrow[u, "\tau^p \times 1"] \arrow[d, "1 \times \tau_p"'] & & R \\
{[E^+,F(\mathbb{S}^q,R)]_* \times [\mathbb{S}^q,F(E^+,R)]_*} \arrow[r, "\frown"'] & {[E^+,F(\mathbb{S}^0,R)]_*} \arrow[ur] &
\end{tikzcd} \tag{8.2}$$

By the naturality of the construction of the pretransfers and the definition of the $\frown$-product (see Proposition 2.4), it is fairly easy to check that this commutativity follows from the commutativity of the following:

$$\begin{tikzcd}
{[X^+,F(X^+,R)]_*} \arrow[r] & {[\mathbb{S}^0,F(X^+,R)]_*} \arrow[dr] & \\
{[E^+,F(X^+,R)]_*} \arrow[u, "\tau^p"] \arrow[d, "\tau_p"'] & & R \\
{[E^+,F(E^+,R)]_*} \arrow[r] & {[\mathbb{S}^0,F(E^+,R)]_*} \arrow[ur] &
\end{tikzcd} \tag{8.3}$$

Let $\delta : E^+ \to F(X^+,R)$ be given by $\delta(e) = \sum_{i=1}^{m(e)} r_i(e)x_i(e)$, $e \in E$. Chasing this element δ along the top of the diagram, one easily verifies that it maps to the element

$$d = \sum_{p(e)=x_{-1}} \mu(e) \sum_{i=1}^{m(e)} r_i(e), \tag{8.4}$$

while chasing it along the bottom of the diagram, it maps to the element

$$d' = \sum_{i=1}^{m(e_{-1})} r_i(e_{-1}) \sum_{p(e_i)=x_i(e_{-1})} \mu(e_i) = n \sum_{i=1}^{m(e_{-1})} r_i(e_{-1}). \tag{8.5}$$

Call $\rho(e) = \sum_{i=1}^{m(e)} r_i(e)$. Since $\rho = \varepsilon \circ \delta$, this defines by Lemma 2.3 a continuous map $\rho : E \to R$, but since E is path connected and R is discrete, ρ is constant with value $r_\delta \in R$. Hence

$$d = \sum_{p(e)=x_{-1}} \mu(e)\rho(e) = n \cdot r_\delta, \qquad d' = n\rho(e_{-1}) = n \cdot r_\delta. \tag{8.6}$$

Thus $d = d'$ and the diagram commutes. $\square$

For simplicity, in what follows we omit to write the coefficient ring R in homology and cohomology. Taking the adjoints of the Kronecker product $\langle -,- \rangle_Y : H^q(Y) \otimes H_q(Y) \to R$ one has induced natural homomorphisms

$$\Phi_Y : H^q(Y) \longrightarrow \operatorname{Hom}(H_q(Y),R), \qquad \Psi_Y : H_q(Y) \longrightarrow \operatorname{Hom}(H^q(Y),R) \tag{8.7}$$

for every space Y, given by $\Phi_Y(y)(\eta) = \langle y,\eta \rangle_Y$ and $\Psi_Y(\eta)(y) = \langle y,\eta \rangle_Y$, $y \in H^q(Y)$, $\eta \in H_q(Y)$.

Corollary 8.2. *The following diagrams commute:*

$$\begin{tikzcd} H^q(E) \arrow[r, "\Phi_E"] \arrow[d, "\tau^p"'] & \operatorname{Hom}(H_q(E),R) \arrow[d, "\operatorname{Hom}(\tau_p,1)"] \\ H^q(X) \arrow[r, "\Phi_X"'] & \operatorname{Hom}(H_q(X),R) \end{tikzcd} \qquad \begin{tikzcd} H_q(X) \arrow[r, "\Psi_X"] \arrow[d, "\tau_p"'] & \operatorname{Hom}(H^q(X),R) \arrow[d, "\operatorname{Hom}(\tau^p,1)"] \\ H_q(E) \arrow[r, "\Psi_E"'] & \operatorname{Hom}(H^q(E),R) \end{tikzcd} \tag{8.8}$$

Remark 8.3. Under suitable conditions, Φ or Ψ are isomorphisms, in whose case one of the transfers determines the other.

9. Comparison with Smith's transfer

In this section, we show that the transfer defined in [18] coincides with ours if we take $\mathbb{Z}$-coefficients. To that end, we first recall Smith's definition of the transfer. It makes use of a result of Moore, that we state below. Recall that the *weak product* $\widetilde{\prod}_{n=1}^{\infty} X_n$ of a family of pointed spaces is the colimit over n of the directed system of spaces

$$X_1 \hookrightarrow X_1 \times X_2 \hookrightarrow X_1 \times X_2 \times X_3 \hookrightarrow \cdots, \tag{9.1}$$

where the inclusions are given by letting the last coordinate be the base point. Moore's result, as it appears in [21], is as follows.

Theorem 9.1 (Moore). *A connected space X of the same homotopy type of a CW-complex is homotopy equivalent to the weak product $\widetilde{\prod}_{n\geq 1} K(\pi_n(X),n)$ of Eilenberg-Mac Lane spaces if and only if the Hurewicz homomorphism $h_n : \pi_n(X) \to \widetilde{H}_n(X;\mathbb{Z})$ is a split monomorphism for all $n \geq 1$.*

Suppose that $\rho_n : \widetilde{H}_n(X) = \widetilde{H}_n(X;\mathbb{Z}) \to \pi_n(X)$ is a left inverse of h_n. The Kronecker product defined in Section 2 determines an epimorphism

$$\widetilde{H}^n(X;\pi_n(X)) \longrightarrow \operatorname{Hom}(\widetilde{H}_n(X),\pi_n(X)). \tag{9.2}$$

Let $[\xi_n] \in \widetilde{H}^n(X;\pi_n(X)) = [X,K(\pi_n(X),n)]_*$ be some preimage of ρ_n. Then the family of pointed maps (ξ_n) defines the homotopy equivalence of the previous theorem.

Corollary 9.2. *If X is a connected topological abelian monoid of the same homotopy type of a* CW*-complex, then there is a homotopy equivalence $X \to \widetilde{\prod}_{n\geq 1} K(\pi_n(X),n)$.*

Proof. Since X is a topological abelian monoid, there is a retraction $r : \mathrm{SP}^{\infty}X \to X$ given by the retractions

$$r_m : \mathrm{SP}^m X \longrightarrow X, \qquad r_m\langle x_1,x_2,\dots,x_m\rangle = x_1 + x_2 + \cdots + x_m. \tag{9.3}$$

Recall, on the other hand, that by the Dold-Thom theorem one has an isomorphism $\pi_n(\mathrm{SP}^\infty X) \cong \widetilde{H}_n(X)$, so that the inclusion $i : X \hookrightarrow \mathrm{SP}^\infty X$ defines the Hurewicz homomorphism (see [1]). Since $r \circ i = \mathrm{id}_X$, the homomorphism $\rho_n = r_* : \widetilde{H}_n(X) = \pi_n(\mathrm{SP}^\infty X) \to \pi_n(X)$ provides a left inverse of the Hurewicz homomorphism h_n. Hence, by Moore's theorem, we obtain the result. $\square$

For any space E, the space $\mathrm{SP}^\infty E$ is a topological abelian monoid. Thus we have the following.

Corollary 9.3. *For a connected space E of the same homotopy type of a* CW*-complex, there is a homotopy equivalence $w_E : \mathrm{SP}^\infty E \to K(\widetilde{H}_*(E)) = \widetilde{\prod}_{n=1}^{\infty} K(\widetilde{H}_n(E), n)$.*

The definition of Smith's transfer is as follows. Given an n-fold ramified covering map $p : E \to X$ with multiplicity function $\mu : E \to \mathbb{N}$, consider the following composite:

$$\widehat{p} : X \xrightarrow{\varphi_p} \mathrm{SP}^n E \longrightarrow \mathrm{SP}^\infty E \xrightarrow{\simeq} K(\widetilde{H}_*(E)). \tag{9.4}$$

This map defines a family of elements $[\widehat{p}] \in \widetilde{H}^*(X; \widetilde{H}_*(E))$. On the other hand, the Kronecker product determines a homomorphism

$$\psi : \widetilde{H}^*(X; \widetilde{H}_*(E)) \longrightarrow \mathrm{Hom}(\widetilde{H}_*(X), \widetilde{H}_*(E)). \tag{9.5}$$

Smith's transfer is the image $p_\sharp : \widetilde{H}_*(X) \to \widetilde{H}_*(E)$ of $[\widehat{p}]$ under the homomorphism ψ.

Theorem 9.4. *Let $p : E \to X$ be an n-fold ramified covering map with multiplicity function $\mu : E \to \mathbb{N}$. Then $p_\sharp = \tau_p : \widetilde{H}_*(X; \mathbb{Z}) \to \widetilde{H}_*(E; \mathbb{Z})$, where τ_p is the transfer in reduced homology.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
{[E, \mathrm{SP}^n E]_*} \arrow[r] \arrow[d, "\tau^p"'] & {[E, \mathrm{SP}^\infty E]_*} \arrow[r, "\cong"] \arrow[d, "\tau^p"'] & \widetilde{H}^*(E, \widetilde{H}_*(E)) \arrow[r] \arrow[d, "\tau^p"'] & \mathrm{Hom}(\widetilde{H}_*(E), \widetilde{H}_*(E)) \arrow[d, "\mathrm{Hom}(\tau_p, 1)"] \\
{[X, \mathrm{SP}^n E]_*} \arrow[r] & {[X, \mathrm{SP}^\infty E]_*} \arrow[r, "\cong"] & \widetilde{H}^*(X, \widetilde{H}_*(E)) \arrow[r] & \mathrm{Hom}(\widetilde{H}_*(X), \widetilde{H}_*(E))
\end{tikzcd} \tag{9.6}$$

The two squares on the left-hand side, where τ^p represents the cohomology transfer, commute obviously. The one on the right-hand side commutes by Corollary 8.2. Take $[i] \in [E, \mathrm{SP}^n E]_*$, where $i : E \hookrightarrow \mathrm{SP}^n E$ is the canonical inclusion. Chasing $[i]$ down and then right on the bottom of the diagram, we obtain $p_\sharp$, while chasing it to the right on the top of the diagram and then down, we obtain τ_p. This is true, because the image of $[i]$ along the top row of the diagram is the identity homomorphism $1 \in \mathrm{Hom}(\widetilde{H}_*(E), \widetilde{H}_*(E))$. This follows from the naturality of the Kronecker product, since by Corollary 9.2, we have an explicit description of the weak homotopy equivalence that defines the isomorphism in the middle arrow. □

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