# STRUCTURE OF RINGS WITH CERTAIN CONDITIONS ON ZERO DIVISORS

HAZAR ABU-KHUZAM AND ADIL YAQUB

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Let $R$ be a ring such that every zero divisor $x$ is expressible as a sum of a nilpotent element and a potent element of $R : x = a + b$, where $a$ is nilpotent, $b$ is potent, and $ab = ba$. We call such a ring a $D^*$-ring. We give the structure of periodic $D^*$-ring, weakly periodic $D^*$-ring, Artinian $D^*$-ring, semiperfect $D^*$-ring, and other classes of $D^*$-ring.



## 1. Introduction

Throughout this paper, $R$ is an associative ring; and $N$, $C$, $C(R)$, and $J$ denote, respectively, the set of nilpotent elements, the center, the commutator ideal, and the Jacobson radical. An element $x$ of $R$ is called *potent* if $x^n = x$ for some positive integer $n = n(x) > 1$. A ring $R$ is called *periodic* if for every $x$ in $R$, $x^m = x^n$ for some distinct positive integers $m = m(x)$, $n = n(x)$. A ring $R$ is called *weakly periodic* if every element of $R$ is expressible as a sum of a nilpotent element and a potent element of $R : R = N + P$, where $P$ is the set of potent elements of $R$. A ring $R$ such that every zero divisor is nilpotent is called a *D-ring*. The structure of certain classes of $D$-rings was studied in [1]. Following [7], $R$ is called *normal* if all of its idempotents are in $C$. A ring $R$ is called a $D^*$*-ring*, if every zero divisor $x$ in $R$ can be written as $x = a + b$, where $a \in N$, $b \in P$, and $ab = ba$. Clearly every $D$-ring is a $D^*$-ring. In particular every nil ring is a $D^*$-ring, and every domain is a $D^*$-ring. A Boolean ring is a $D^*$-ring but not a $D$-ring. Our objective is to study the structure of certain classes of $D^*$-ring.

## 2. Main results

We start by stating the following known lemmas: Lemmas 2.1 and 2.2 were proved in [5], Lemmas 2.3 and 2.4 were proved in [4].

Lemma 2.1. *Let $R$ be a weakly periodic ring. Then the Jacobson radical $J$ of $R$ is nil. If, furthermore, $xR \subseteq N$ for all $x \in N$, then $N = J$ and $R$ is periodic.*

Lemma 2.2. *If $R$ is a weakly periodic division ring, then $R$ is a field.*

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LEMMA 2.3. *Let $R$ be a periodic ring and $x$ any element of $R$. Then*
(a) *some power of $x$ is idempotent;*
(b) *there exists an integer $n > 1$ such that $x - x^n \in N$.*

LEMMA 2.4. *Let $R$ be a periodic ring and let $\sigma : R \to S$ be a homomorphism of $R$ onto a ring $S$. Then the nilpotents of $S$ coincide with $\sigma(N)$, where $N$ is the set of nilpotents of $R$.*

*Definition 2.5.* A ring is said to be a *D-ring* if every zero divisor is nilpotent. A ring $R$ is called a *$D^*$-ring* if every zero divisor $x$ in $R$ can be written as $x = a + b$, where $a \in N$, $b \in P$, and $ab = ba$.

THEOREM 2.6. *A ring $R$ is a $D^*$-ring if and only if every zero divisor of $R$ is periodic.*

*Proof.* Assume $R$ is a $D^*$-ring and let $x$ be any zero divisor. Then

$$x = a + b, \quad a \in N,\ b \in P,\ ab = ba. \tag{2.1}$$

So, $(x - a) = b = b^n = (x - a)^n$. This implies, since $x$ commutes with $a$, that $(x - a) = (x - a)^n = x^n +$ sum of pairwise commuting nilpotent elements.

Hence

$$x - x^n \in N \quad \text{for every zero divisor } x. \tag{2.2}$$

Since each such $x$ is included in a subring of zero divisors, which is periodic by Chacron's theorem, $x$ is periodic.

Suppose, conversely, that each zero divisor is periodic. Then by the proof of [4, Lemma 1], $R$ is a $D^*$-ring. □

THEOREM 2.7. *If $R$ is any normal $D^*$-ring, then either $R$ is periodic or $R$ is a $D$-ring. Moreover, $aR \subseteq N$ for each $a \in N$.*

*Proof.* If $R$ is a normal $D^*$-ring which is not a $D$-ring, then $R$ has a central idempotent zero divisor $e$. Then $R = eR \oplus A(e)$, where $eR$ and $A(e)$ both consist of zero divisors of $R$, hence (in view of Theorem 2.6) are periodic. Therefore $R$ is periodic.

Now consider $a \in N$ and $x \in R$. Since $ax$ is a zero divisor, hence a periodic element, $(ax)^j = e$ is a central idempotent for some $j$. Thus $(ax)^{j+1} = (ax)^j ax = a^2 y$ for some $y \in R$. Repeating this argument, one can show that for each positive integer $k$, there exists $m$ such that $(ax)^m = a^{2^k} w$ for some $w \in R$. Therefore $aR \subseteq N$. □

COROLLARY 2.8. *Let $R$ be a $D^*$-ring which is not a $D$-ring. If $N \subseteq C$, then $R$ is commutative.*

*Proof.* Since $N \subseteq C$, $R$ is normal. Therefore commutativity follows from Theorem 2.7 and a theorem of Herstein. □

Now, we prove the following result for $D^*$-rings.

THEOREM 2.9. *Let $R$ be a normal $D^*$-ring.*
(i) *If $R$ is weakly periodic, then $N$ is an ideal of $R$, $R$ is periodic, and $R$ is a subdirect sum of nil rings and/or local rings $R_i$. Furthermore, if $N_i$ is the set of nilpotents of the local ring $R_i$, then $R_i/N_i$ is a periodic field.*

(ii) *If $R$ is Artinian, then $N$ is an ideal and $R/N$ is a finite direct product of division rings.*

*Proof.* (i) Using Theorem 2.7, we have

$$aR \subseteq N \quad \text{for every } a \in N. \tag{2.3}$$

This implies, using Lemma 2.1, that $N = J$ is an ideal of $R$, and $R$ is periodic.
As is well-known, we have

$$R \cong \text{ a subdirect sum of subdirectly irreducible rings } R_i. \tag{2.4}$$

Let $\sigma : R \to R_i$ be the natural homomorphism of $R$ onto $R_i$. Since $R$ is periodic, $R_i$ is periodic and by Lemma 2.4,

$$N_i = \text{ the set of nilpotents of } R_i = \sigma(N) \text{ is an ideal of } R_i. \tag{2.5}$$

We now distinguish two cases.

*Case 1* $1 \notin R_i$. Let $x_i \in R_i$, and let $\sigma : x \to x_i$. Then by Lemma 2.3, $x^k$ is a central idempotent of $R$, and hence $x_i^k$ is a central idempotent in the subdirectly irreducible ring $R_i$, for some positive integer $k$. Hence $x_i^k = 0$ ($1 \notin R_i$). Thus $R_i = N_i$ is a nil ring.

*Case 2* $1 \in R_i$. The above argument in Case 1 shows that $x_i^k$ is a central idempotent in the subdirectly irreducible ring $R_i$. Hence $x_i^k = 0$ or $x_i^k = 1$ for all $x_i \in R_i$. So, $R_i$ is a local ring and for every $x_i + N_i \in R_i/N_i$,

$$x_i + N_i = N_i \quad \text{or} \quad (x_i + N_i)^k = 1 + N_i. \tag{2.6}$$

So $R_i/N_i$ is a periodic division ring, and hence by Lemma 2.2, $R_i/N_i$ is a periodic field.

(ii) Suppose $R$ is Artinian. Using (2.3), $aR$ is a nil right ideal for every $a \in N$. So, $N \subseteq J$. But $J \subseteq N$ since $R$ is Artinian. Therefore $N = J$ is an ideal of $R$ and $R/N = R/J$ is semisimple Artinian. This implies that $R/N$ is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a complete $t_i \times t_i$ matrix ring over a division ring $D_i$. Since $R$ is Artinian, the idempotents of $R/J$ lift to idempotents in $R$ [2], and hence the idempotents of $R/J$ are central. If $t_j > 1$, then $E_{11} \in R_j$, and $(0,\dots,0,E_{11},0,\dots,0)$ is an idempotent element of $R/J$ which is not central in $R/J$. This is a contradiction. So $t_i = 1$ for every $i$. Therefore each $R_i$ is a division ring and $R/N$ is isomorphic to a finite direct product of division rings. □

The next result deals with a special kind of $D^*$-rings.

Theorem 2.10. *Let $R$ be a ring such that every zero divisor $x$ can be written uniquely as $x = a + e$, where $a \in N$ and $e$ is idempotent.*

(i) *If $R$ is weakly periodic, then $N$ is an ideal of $R$, and $R/N$ is isomorphic to a subdirect sum of fields.*

(ii) *If $R$ is Artinian, then $N$ is an ideal and $R/N$ is a finite direct product of division rings.*

*Proof.* Let $e^2 = e \in R$, $x \in R$, and let $f = e + ex - exe$. Then $f^2 = f$ and hence $(ef - e)f = 0$. So if $f$ is not a zero divisor, then $ef - e = 0$. So $ef = e$, and thus $f = e$, which implies that $ex = exe$. The net result is $ex - exe = 0$ if $f$ is not a zero divisor. Next, suppose $f$ is a zero divisor. Then since

$$f = (ex - exe) + e; \quad ex - exe \in N,\ e \text{ idempotent}; \\ f = 0 + f, \tag{2.7}$$

it follows from uniqueness that $ex - exe = 0$, and hence $ex = exe$ in all cases. Similarly $xe = exe$, and thus

$$\text{all idempotents of } R \text{ are central, and hence } R \text{ is a normal } D^*\text{-ring.} \tag{2.8}$$

(i) Using (2.8), $R$ satisfies all the hypotheses of Theorem 2.9(i), and hence $N$ is an ideal, and $R$ is periodic. Using Lemma 2.2, for each $x \in R$, there exists an integer $k > 1$, such that $x - x^k \in N$, and hence

$$(x + N)^k = (x + N), \quad k = k(x) > 1. \tag{2.9}$$

By a well-known theorem of Jacobson [6], (2.9) implies that $R/N$ is a subdirect sum of fields.

(ii) If $R$ is Artinian, then using (2.8), $R$ satisfies the hypotheses of Theorem 2.9(ii). Therefore $N$ is an ideal and $R/N$ is a finite direct product of division rings. □

Theorem 2.11. *Let $R$ be a semiprime $D^*$-ring with $N$ commutative. Then $R$ is either a domain or a $J$-ring.*

*Proof.* As in the proof of [3, Theorem 1] we can show that if $a^k = 0$, then $(ar)^k = 0$ for all $r \in R$. Therefore, by Levitzki's theorem, $N = \{0\}$. Assume $R$ is not a domain, and let $a$ be any nonzero divisor of zero. Then $a$ is potent and $aR$ consists of zero divisors, hence is a $J$-ring containing $a$. Therefore $[ax, a] = 0$ for all $x \in R$, hence $(ax)^n = a^n x^n$ for all $x \in R$, and all $n \geq 2$. For $x$ not a zero divisor, choose $n > 1$ such that $a^n = a$ and $(ax)^n = ax$. Then $a^n x^n = ax$, so $a(x^n - x) = 0$ and $x^n - x$ is a zero divisor, hence is periodic. It follows by Chacron's theorem that $R$ is a periodic ring; and since $N = \{0\}$, $R$ is a $J$-ring. □

*Example 2.12.* Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad 0, 1 \in GF(2). \tag{2.10}$$

Then $R$ is a normal weakly periodic $D^*$-ring with commuting nilpotents. $R$ is not semiprime since the set of nilpotent elements $N$ is a nonzero nilpotent ideal. This example shows that we cannot drop the hypothesis "$R$ is semiprime" in Theorem 2.11.

In Theorem 2.14 below, we study the structure of a special kind of $D^*$-rings, the class of rings in which every zero divisor is potent. Recall that a ring is semiperfect [2] if and only if $R/J$ is semisimple (Artinian) and idempotents lift modulo $J$. We need the following lemma.

LEMMA 2.13. *Let $R$ be a ring in which every zero divisor is potent. Then $N = \{0\}$ and $R$ is normal. Moreover, If $R$ is not a domain, then $J = \{0\}$.*

*Proof.* If $a \in N$, then $a$ is a zero divisor and hence potent by hypothesis. So $a^n = a$ for some positive integer $n$, and since $a \in N$, there exists a positive integer $k$ such that $0 = a^{n^k} = a$. So $N = \{0\}$. Let $e$ be any idempotent element of $R$ and $x$ is any element of $R$. Then $ex - exe \in N$, and hence $ex - exe = 0$. Similarly, $xe = exe$. So $ex = xe$ and $R$ is normal.

Let $x$ be a nonzero divisor of zero. Then $xJ$ consists of zero divisors, which are potent. Therefore $xJ = \{0\}$. But then $J$ consists of zero divisors, hence potent elements, and therefore $J = \{0\}$. □

THEOREM 2.14. *Let $R$ be a ring such that every zero divisor is potent.*

(i) *If $R$ is weakly periodic, then every element of $R$ is potent and $R$ is a subdirect sum of fields.*
(ii) *If $R$ is prime, then $R$ is a domain.*
(iii) *If $R$ is Artinian, then $R$ is a finite direct product of division rings.*
(iv) *If $R$ is semiperfect, then $R/J$ is a finite direct product of division rings.*

*Proof.* (i) Since $R$ is weakly periodic, every element $x \in R$ can be written as

$$x = a + b, \quad \text{where } a \in N,\ b \text{ is potent.} \tag{2.11}$$

But $N = \{0\}$ (Lemma 2.13), so every $x \in R$ is potent and hence $R$ is isomorphic to a subdirect sum of fields by a well-known theorem of Jacobson.

(ii) Suppose $R$ is a prime, then $R$ is a prime ring with $N = \{0\}$, and hence $R$ is a domain.

(iii) Let $R$ be an Artinian ring such that every zero divisor is potent. Since $N = \{0\}$ (Lemma 2.13) and $R$ is Artinian, $J = N = \{0\}$. So $R$ is semisimple Artinian and hence it is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a complete $t_i \times t_i$ matrix ring over a division ring $D_i$. If $t_j > 1$, then $E_{11} \in R_j$, and $(0,\dots,0,E_{11},0,\dots,0)$ is an idempotent element of $R$ which is not central in $R$ contradicting Lemma 2.13. So $t_i = 1$ for every $i$. Therefore each $R_i$ is a division ring and $R$ is isomorphic to a finite direct product of division rings.

(iv) Let $R$ be a semiperfect ring such that every zero divisor is potent. Then $R/J$ is semisimple Artinian and hence it is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a complete $t_i \times t_i$ matrix ring over a division ring $D_i$. Since $R$ is semiperfect, the idempotents of $R/J$ lift to idempotents in $R$, and hence the argument of part (iii) above implies that each $R_i$ is a division ring and $R/J$ is isomorphic to a finite direct product of division rings. □

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Hazar Abu-Khuzam: Department of Mathematics, American University of Beirut,
Beirut 1107 2020, Lebanon
*E-mail address*: hazar@aub.edu.lb

Adil Yaqub: Department of Mathematics, University of California, Santa Barbara,
CA 93106-3080, USA
*E-mail address*: yaqub@math.ucsb.edu

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