Research Article
Solution of Cauchy-Type Singular Integral Equations of the First Kind with Zeros of Jacobi Polynomials as Interpolation Nodes

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Of concern in this paper is the numerical solution of Cauchy-type singular integral equations of the first kind at a discrete set of points. A quadrature rule based on Lagrangian interpolation, with the zeros of Jacobi polynomials as nodes, is developed to solve these equations. The problem is reduced to a system of linear algebraic equations. A theoretical convergence result for the approximation is provided. A few numerical results are given to illustrate and validate the power of the method developed. Our method is more accurate than some earlier methods developed to tackle this problem.

1. Introduction

Given the functions $K(x,t)$ and $f(x)$, we consider the problem of finding a function $y(x)$, $-1 < x < 1$, such that

$$\frac{b}{\pi} \int_{-1}^{1} \frac{y(t)}{t-x} dt + \int_{-1}^{1} K(x,t)y(t)dt = f(x), \tag{1.1}$$

which is the one-dimensional, real, Cauchy-type singular integral equation (SIE) of the first kind, defined on the finite interval $[-1,1]$. There is no loss of generality here since any finite interval $[c,d]$ can be transformed into $[-1,1]$ by the linear transformation $t = (1/2)[c + d + (d-c)s]$. In (1.1), b is a real constant, $y(t)$ is the unknown function which we seek to find, $K(x,t)$ is a known well-behaved kernel, which is continuous on the square $S = \{(x,t) : x,t \in (-1,1) \times (-1,1)\}$ and satisfies the condition $\iint_{-1}^{1} K^2(x,t)dx\,dt < \infty$, and $f(x)$ is a regular known function.

These equations (see [1]) arise most naturally and directly from boundary value problems for special nonsmooth boundaries such as cuts, cracks, foils, slits, strips; and these boundaries possess sharp edges where singularities can be expected. In some applications, the strengths of these singularities may be output quantities of some interest. The most dominant areas where these equations are encountered are in aerodynamics and plane elasticity. Often, they represent the equations of cracks existing in an infinite, isotropic elastic medium [2], which are serious engineering problems. In the last three decades or more, several authors have studied the numerical solution of (1.1) and among the methods used are the direct quadrature method [2], the spline approximation method [3], and the trigonometric polynomial interpolation method [4].

In this paper, we present an efficient quadrature rule which is based on Lagrange interpolation and Gauss-Jacobi quadrature, with the zeros of the Jacobi polynomial $P_n^{\alpha,\beta}(t)$ of degree n adopted as our interpolation nodes. Both the nodes and the weights depend, of course, on α and β. It is well known [1, 5, 6] that the unknown function $y(t)$ in (1.1) possesses singularities at $t = \pm 1$ and is possibly unbounded at these points, and therefore it is appropriate to express it [6] as $y(t) = w(t)\phi(t)$, where $\phi(t)$ is a regular function and $w(t)$ is the weight function, $w(t) = (1-t)^\alpha(1+t)^\beta$, $\alpha, \beta > -1$. For this reason, we will assume that the solution $y(t)$ has the following boundary behavior in $[-1,1]$, namely, (i) $y(t)$ is unbounded at both ends of the interval; (ii) $y(t)$ is bounded at both ends; (iii) $y(t)$ is bounded at either of the ends. Each case is a consequence of the choices made for α and β in the weight function $w(t)$. A quadrature rule for each case (except the last one) will be developed as a special case of our more general quadrature rule developed in Section 2.

The paper is organized as follows. In Section 2, we develop an algorithm based on Lagrange interpolation and Gauss-Jacobi quadrature for the numerical solution of (1.1). In Sections 3 and 4, we develop rules for the first two cases ((i) & (ii)) mentioned previously. Four numerical examples are given in these two sections to validate our method. We shelved the third case (iii), as our method can easily be adapted to it. A theoretical convergence of our method is proved in Section 5.

2. Construction of the rule

Let $\{t_1, t_2, \ldots, t_n\}$ be the sequence of distinct points in $[-1,1]$. Given these distinct points, such that the values of some function $\rho(t)$ are defined and known at these points, it is known [7] that there exists a unique polynomial $h_{n-1}(t)$ of degree $n-1$ such that,

$$h_{n-1}(t_k) = \rho(t_k), \quad k = 1, 2, \ldots, n. \tag{2.1}$$

This interpolating polynomial $h_{n-1}(t)$, written in Lagrangian form, is

$$\sum_{k=1}^{n} \rho(t_k)\ell_k(t), \tag{2.2}$$

where

$$\ell_k(t) = \frac{W_n(t)}{(t - t_k) W_n'(t_k)},$$

$$W_n(t) = \prod_{k=1}^{n} (t - t_k),$$

$$\ell_i(t_k) = \delta_{ik} = \begin{cases} 0, & i \neq k, \\ 1, & i = k, \end{cases}$$

(2.3)

and the error [7] following this approximation is $(W_n(t)/n!)\rho^{(n)}(\zeta)$, $\zeta \in (-1,1) \cap (t_1, t_n)$.

Our approximate rule will be based on the preceding form of interpolation, which is Lagrangian.

We will assume, from here and the rest of the paper, that $\{t_j\}_{j=1}^{n}$ are the zeros of the Jacobi polynomials $P_n^{\alpha,\beta}(t)$, which are classical orthogonal polynomials defined on $[-1,1]$ with the weight function $w(t) = (1-t)^\alpha (1+t)^\beta$, $\alpha, \beta > -1$; some special cases of these polynomials are the Chebyshev polynomials (first and second kind), Legendre polynomials, and the Gegenbauer polynomials.

Suppose as mentioned earlier that we set $y(t) = w(t)\phi(t)$ and interpolate to $\phi(t)$ at the set of points t_j, $j = 1, \ldots, n$, to give

$$\phi(t) \approx \sum_{j=1}^{n} \phi(t_j)\ell_j(t),$$

(2.4)

then, on substituting this approximation in the first integral of (1.1), we have

$$\int_{-1}^{1} \frac{w(t)\phi(t)}{t - x} dt \approx \sum_{j=1}^{n} \phi(t_j) \int_{-1}^{1} \frac{w(t) P_n^{\alpha,\beta}(t)}{(t-x)(t-t_j) P_n'^{\alpha,\beta}(t_j)} dt.$$

(2.5)

Using Christoffel-Darboux identity [8],

$$-\gamma_n h_n \sum_{k=0}^{n-1} \frac{1}{h_k} \frac{P_k^{\alpha,\beta}(t) P_k^{\alpha,\beta}(t_j)}{P_{n+1}^{\alpha,\beta}(t_j)} = \frac{P_n^{\alpha,\beta}(t)}{(t - t_j)},$$

(2.6)

where $h_n = \langle P_n^{\alpha,\beta}(t), P_n^{\alpha,\beta}(t) \rangle$ is a positive normalization constant defined by

$$h_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta)\Gamma(n+\alpha+\beta+1)},$$

$$\gamma_n = \frac{c_{n+1}}{c_n},$$

(2.7)

where c_n is the coefficient of t^n in $P_n^{\alpha,\beta}(t)$. The pairs (h_r, c_r) have been tabulated [8] for some orthogonal polynomials. From (2.5), on using (2.6), (2.7),

$$\int_{-1}^{1} \frac{w(t)\phi(t)}{t-x} dx \approx -\gamma_n h_n \sum_{j=1}^{n} \frac{\phi(t_j)}{P_{n+1}^{\alpha,\beta}(t_j) P_n'^{\alpha,\beta}(t_j)} \sum_{k=0}^{n-1} \frac{1}{h_k} P_k^{\alpha,\beta}(t_j) v_k(x), \qquad (2.8)$$

where $v_k(x)$ are functions of the second kind, which are defined in this case by

$$v_k(x) = \int_{-1}^{1} \frac{w(t) P_k^{\alpha,\beta}(t)}{t-x} dt. \qquad (2.9)$$

For the special cases of the Jacobi polynomials, $v_k(x)$ are usually known in a closed form, (see [8, page 785]). However, where this is not readily available, v_k may be evaluated from the recurrence relations satisfied by $P_k^{\alpha,\beta}(t)$. Since $P_k^{\alpha,\beta}(t)$ satisfies the recurrence relation of the form

$$P_{k+1}^{\alpha,\beta}(t) = (A + Bt) P_k^{\alpha,\beta}(t) - C P_{k-1}^{\alpha,\beta}(t), \qquad (2.10)$$

where

$$A = A(\alpha,\beta,k), \qquad B = B(\alpha,\beta,k), \qquad C = C(\alpha,\beta,k) \qquad (2.11)$$

then, it is easy to show that v_k satisfies the recursion equation

$$v_{k+1} = (A + Bx) v_k - C v_{k-1} + B\Upsilon, \quad k = 1,2,\dots, \qquad (2.12)$$

with starting values

$$v_0 = \int_{-1}^{1} \frac{(1-t)^{\alpha}(1+t)^{\beta}}{t-x} dt,$$

$$v_1 = \psi\Upsilon + \left(\psi x + \frac{1}{2}(\alpha - \beta)\right) v_0, \quad \psi = 1 + \frac{1}{2}(\alpha + \beta), \qquad (2.13)$$

where

$$\Upsilon = 2^{\alpha+\beta+1} \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+2)}. \qquad (2.14)$$

From experiment, the recursion equation (2.12) is most often stable in the increasing direction of k for all $\alpha,\beta > -1$.

Suppose that we apply the Gauss-Jacobi quadrature rule to the second integral of (1.1) as follows:

$$\int_{-1}^{1} w(t)K(x,t)\phi(t)dt \approx \sum_{j=1}^{n} w_j K(x,t_j)\phi(t_j).$$

(2.15)

Then substituting (2.8) and (2.15) in (1.1) and collocating at the points x_i (Nyström's method), we have

$$-\frac{b}{\pi}\gamma_n h_n \sum_{j=1}^{n} \frac{\phi(t_j)}{P_{n+1}^{\alpha,\beta}(t_j)P_n'^{\alpha,\beta}(t_j)} \sum_{k=0}^{n-1} \frac{1}{h_k} P_k^{\alpha,\beta}(t_j) v_k(x_i) + \sum_{j=1}^{n} w_j K(x_i,t_j)\phi(t_j) = f(x_i),$$

$$i = 1,\dots,n-1.$$

(2.16)

We let

$$S_{n-1}(i,j) = \sum_{k=0}^{n-1} \frac{1}{h_k} P_k^{\alpha,\beta}(t_j) v_k(x_i),$$

$$\eta_n = -\frac{b}{\pi}\gamma_n h_n,$$

$$R_j = \frac{\eta_n}{P_{n+1}^{\alpha,\beta}(t_j)P_n'^{\alpha,\beta}(t_j)},$$

$$\phi(t_j) = \phi_j, \qquad f(x_i) = f_i, \qquad K(x_i,t_j) = K_{i,j}.$$

(2.17)

Then we have the rule

$$\sum_{j=1}^{n} \left(R_j S_{n-1}(i,j) + w_j K_{i,j}\right)\phi_j = f_i, \quad i = 1,\dots,n-1.$$

(2.18)

Let us assume an additional condition equation of the form

$$\sum_{j=1}^{n} \phi(t_j) = 0.$$

(2.19)

This assumption is logical since (1.1) is a special case of Cauchy-type singular integral equations of the second kind in which (2.19) may be required for a unique solution when some index $k = -(\alpha+\beta) = 1$.

Equations (2.18) and (2.19) constitute an $n \times n$ system of linear equations in ϕ_j and in matrix form may be expressed as

$$A\mathbf{\Phi} = \mathbf{F}, \tag{2.20}$$

where

$$\mathbf{\Phi} = [\phi_1, \phi_2, \ldots, \phi_n]^T,$$
$$\mathbf{F} = [f_1, f_2, \ldots, f_{n-1}, 0]^T, \tag{2.21}$$

$$A = \begin{pmatrix} (R_1 S_{n-1}(1,1) + w_1 K_{11}) & (R_2 S_{n-1}(1,2) + w_2 K_{12}) & \ldots & (R_n S_{n-1}(1,n) + w_n K_{1n}) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ (R_1 S_{n-1}(n,1) + w_1 K_{n1}) & (R_2 S_{n-1}(n,2) + w_n K_{n2}) & \ldots & (R_n S_{n-1}(n,n) + w_n K_{nn}) \end{pmatrix}. \tag{2.22}$$

The linear algebraic system (2.20) gives an approximation of the solution of (1.1), (2.19) at a discrete set of points t_j, $j = 1, 2, \ldots, n$. For all the numerical experiments considered below, we found the coefficient matrix A in (2.20) to be invertible and nearly diagonally dominant in each case of n. The determinant of the matrix grew with increasing n and the growth was $\propto n^2$. There is a rule of the thumb which suggests that a matrix is ill-conditioned if its determinant is small compared to the entries in the matrix. Therefore, our matrix is well conditioned. Nevertheless, we used Gaussian elimination with partial pivoting while solving the linear systems.

3. Solution unbounded at both endpoints of interval

For this case, we set the solution to be of the form

$$y(t) = (1 - t^2)^{-1/2} \phi(t), \quad -1 \leq t \leq 1, \tag{3.1}$$

and therefore we have considered $\alpha, \beta = -1/2$, and as a sequel, $P_n^{-1/2, -1/2}(t) = T_n(t)$, which is the Chebyshev polynomial of the first kind degree n.

Then, it follows immediately in this case that

$$\theta_j = \frac{(2j-1)}{2n} \pi = \cos^{-1}(t_j), \quad j = 1, \ldots, n,$$
$$t_j = \cos\theta_j, \quad T_n(t_j) = 0. \tag{3.2}$$

Suppose that we choose the discrete points $\{x_i\}_{i=1}^{n-1}$ as the zeros of $U_{n-1}(x)$, which is the Chebyshev polynomial of the second-kind degree $n-1$, then we will have

$$x_i = \cos\left(\frac{\pi i}{n}\right), \quad U_{n-1}(x_i) = 0, \quad i = 1, \ldots, n-1,$$

$$h_n = \begin{cases} \pi & \text{if } n = 0, \\ \dfrac{\pi}{2} & \text{if } n \neq 0, \end{cases}$$

$$\gamma_n = 2,$$

$$T_{n+1}(t_j) = \cos\left((n+1)\theta_j\right),$$

$$T_n'(t_j) = \frac{n \sin(n\theta_j)}{\sin(\theta_j)},$$

$$w_j = \frac{\pi}{n},$$

$$v_k(t_i) = \pi U_{k-1}(t_i) = \frac{\pi \sin(k\theta_i)}{\sin(\theta_i)}, \tag{3.3}$$

$$T_k(t_j) = \cos(k\theta_j),$$

$$\eta_n = -b, \quad n \geq 1,$$

$$R_j = \frac{\eta_n \sin(\theta_j)}{n \sin(n\theta_j) \cos\left((n+1)\theta_j\right)},$$

$$\begin{aligned} S_{n-1}(i, j) &= \sum_{k=0}^{n-1} \frac{\cos(k\theta_j) \sin(k\theta_i)}{\sin(\theta_i)} \\ &= \frac{\sin(n/2)(\theta_i - \theta_j) \sin\left((n-1)/2\right)(\theta_i - \theta_j)}{\cos(1/2)(\theta_i + \theta_j) - \cos(1/2)(3\theta_i - \theta_j)} \\ &\quad + \frac{\sin(n/2)(\theta_i + \theta_j) \sin\left((n-1)/2\right)(\theta_i + \theta_j)}{\cos(1/2)(\theta_i - \theta_j) - \cos(1/2)(3\theta_i + \theta_j)}. \end{aligned}$$

Using these equations in (2.18) reduces (2.18) to the approximate rule

$$\sum_{j=1}^{n} \left[R_j S_{n-1}(i, j) + \frac{\pi}{n} K_{ij} \right] \phi_j = f(\cos\theta_i), \quad i = 1, \ldots, n-1. \tag{3.4}$$

To validate (3.4) in conjunction with (2.19), we present below the results of two numerical experiments.

(a) Consider [9] the equation

$$\frac{1}{\pi} \int_{-1}^{1} \frac{y(t)}{t - x} dt = \frac{2}{\pi} \left[1 + x^2 (1 - x^2)^{-1/2} \log \left| \frac{(1 - x^2)^{1/2} - x + 1}{(1 - x^2)^{1/2} + x - 1} \right| \right] \tag{3.5}$$

with the exact solution $y(x) = x|x|(1 - x^2)^{-1/2}$.

TABLE 3.1

	Our method
n	$\|y - y_n\|_\infty$
3	2.18×10^{-2}
7	3.8×10^{-3}
9	9.88×10^{-4}

TABLE 3.2

	Method [9]
n	$\|\phi - \phi_n\|_\infty$
4	5.33×10^{-2}
8	1.26×10^{-2}
16	3.07×10^{-3}

TABLE 3.3

	Our method
n	$\|y - y_n\|_\infty$
4	2.1×10^{-6}
6	1.2×10^{-10}
8	1.27×10^{-10}

Comparing this integral with (1.1), it can be noted that $b = 1$ and $Ky = 0$. Using (2.19) and (3.4) and noting the changes in this equation, we obtain the results shown in Table 3.1 and which are more accurate than those in Table 3.2.

(b) Consider the integral equation [3]

$$\frac{1}{\pi} \int_{-1}^{1} \frac{y(t)}{t-x} dt + \frac{1}{\pi} \int_{-1}^{1} \sin(t-x)y(t)dt = J_1(1)\cos(x) + 1, \quad -1 < x < 1, \tag{3.6}$$

where J_1 is the Bessel function of the first kind of order 1. The exact solution is $y(t) = t(1 - t^2)^{-1/2}$. Applying the rule (3.4) with (2.19) leads to the results shown in Table 3.3, and because the results $\|g - g_n\|_\infty$ are not given in [3], we cannot make a direct comparison but we believe that our results will not be less accurate.

4. Solution is bounded at both ends of interval

For this case, we require that $y(-1) = y(1) = 0$, and so require that the solution be of the form

$$y(t) = (1 - t^2)^{1/2} \phi(t), \quad -1 \le t \le 1. \tag{4.1}$$

Substituting (4.1) into (1.1) gives

$$\frac{b}{\pi} \int_{-1}^{1} \frac{(1-t^2)^{1/2}\phi(t)}{t-x} dt + \int_{-1}^{1} (1-t^2)^{1/2} K(x,t)\phi(t) dt = f(x). \tag{4.2}$$

Let t_j, $j = 1,\dots,n$, be the zeros of $U_n(t)$. By interpolating to $\phi(t)$ at $\{t_j\}_{j=1}^{n}$, we have

$$\phi(t) \approx \sum_{j=1}^{n} \ell_j(t)\phi(t_j), \quad \ell_j(t) = \frac{U_n(t)}{(t-t_j)U_n'(t_j)}. \tag{4.3}$$

Substituting (4.3) and applying the Gauss-Chebyshev quadrature rule to the second integral,

$$\frac{b}{\pi} \sum_{j=1}^{n} \phi(t_j) \int_{-1}^{1} \frac{(1-t^2)^{1/2} U_n(t)}{(t-x)U_n'(t_j)(t-t_j)} dt + \sum_{j=1}^{n} W_j K(x,t_j)\phi(t_j) = f(x). \tag{4.4}$$

Using Christoffel-Darboux identity [8] and applying some algebraic manipulations, we have

$$\sum_{j=1}^{n} \gamma R_j \phi(t_j) \sum_{k=0}^{n-1} \frac{\sin((k+1)z_j)}{\sin(z_j)} T_{k+1}(x) + \sum_{j=1}^{n} W_j K(x,t_j)\phi(t_j) = f(x), \tag{4.5}$$

where (see [8])

$$W_j = \frac{\pi}{n+1} \sin^2\left(\frac{j\pi}{n+1}\right),$$
$$t_j = \cos\left(\frac{j\pi}{n+1}\right). \tag{4.6}$$

We let

$$z_j = \cos^{-1}(t_j) = \frac{j\pi}{n+1},$$
$$\delta_j = U_{n+1}(t_j) = \frac{\sin((n+2)z_j)}{\sin(z_j)},$$
$$\lambda_j = U_n'(t_j) = \frac{\sin((n+1)z_j)\cos(z_j)}{\sin^3(z_j)} - \frac{(n+1)\cos((n+1)z_j)}{\sin^2(z_j)}, \tag{4.7}$$
$$R_j = (\delta_j \lambda_j)^{-1},$$
$$\gamma = (2\pi)\left(\frac{b}{\pi}\right).$$

By choosing the collocation points $x_i = \cos(i - 1/2)(\pi/(n+1))$, $i = 1,\dots,n$, we further reduce (4.5) into the linear algebraic system

$$\sum_{j=1}^{n}\left[\gamma R_j \sum_{k=0}^{n-1}\frac{\sin\left((k+1)z_j\right)\cos\left((k+1)\cos^{-1}x_i\right)}{\sin\left(z_j\right)} + W_j K\left(x_i, t_j\right)\right]\phi(t_j) = f\left(x_i\right), \quad i = 1,\dots,n$$

(4.8)

which we may write in matrix form as

$$\left(A_n + B_n\right)\Phi_n = \mathbf{F}_n,$$

(4.9)

where

$$\begin{aligned}
\left(A_n\right)_{i,j} &= \gamma R_j \sum_{k=0}^{n-1}\frac{\sin\left((k+1)z_j\right)\cos\left((k+1)\cos^{-1}x_i\right)}{\sin\left(z_j\right)}, \quad i,j = 1,\dots,n, \\
\left(B_n\right)_{i,j} &= W_j K\left(x_i, t_j\right), \quad i,j = 1,\dots,n, \\
\Phi_n &= \left[\phi(t_1), \phi(t_2),\dots,\phi(t_n)\right]^T, \\
\mathbf{F}_n &= \left[f(x_1), f(x_2),\dots,f(x_n)\right].
\end{aligned}$$

(4.10)

As previously mentioned in Section 2, the matrix $D_n = A_n + B_n$ of (4.9) is invertible and stable with increasing n.

For a numerical experiment, we consider the simple equation

(a)

$$\int_{-1}^{1}\frac{y(t)}{t-x}dt + \int_{-1}^{1}y(t)x^2 dt = \pi\left(\frac{1}{2}x + \frac{1}{8}x^2 - x^3\right)$$

(4.11)

which has the exact solution $y(t) = \sqrt{(1-t^2)}t^2$. Using (4.9), the following results are obtained. With $n = 3$, the maximum absolute error obtained is $\|y - y_n\|_\infty = 0.1665 \times 10^{-15}$, which is correct to the machine accuracy. This is expected as both ϕ and $K(x,t)$ have been approximated exactly by our method.

Finally, we consider the equation

(b)

$$\int_{-1}^{1}\frac{(1-t^2)^{1/2}\phi(t)dt}{t-x} + \frac{1}{\pi}\int_{-1}^{1}(1-t^2)^{1/2}\phi(t)t^3 e^{x^2}dt = \frac{e^{x^2}}{16} - \pi T_4(x)$$

(4.12)

which has the exact solution, $y = (1-t^2)^{1/2}U_3(t)$. Here, T_r and U_m are the Chebyshev polynomials of the first and second kind, respectively. Again, with $n = 4$, the maximum absolute error obtained is $\|y - y_n\|_\infty = .122 \times 10^{-14}$.

5. Convergence analysis

Since $y(t) = w(t)\phi(t)$, then (1.1) becomes

$$\frac{b}{\pi} \int_{-1}^{1} \frac{w(t)\phi}{t-x} dt + \int_{-1}^{1} w(t)K(x,t)\phi(t)dt = f(x). \tag{5.1}$$

We treat this problem as an operator equation on the real-weighted $C[-1,1]$ space with the weight function $w(t) = (1-t)^{\alpha}(1+t)^{\beta}$, $\alpha, \beta > -1$.

Let **L** and **K** be the linear and bounded integral operators defined by

$$\begin{aligned} \mathbf{L} &= \frac{b}{\pi} \int_{-1}^{1} \frac{w(t)}{t-x} dt, \\ \mathbf{K} &= \int_{-1}^{1} w(t)K(x,t)dt. \end{aligned} \tag{5.2}$$

Rewriting (5.1) using the operator notation, we have

$$\mathbf{L}\phi + \mathbf{K}\phi = f(x). \tag{5.3}$$

Let

$$\phi_n = \sum_{j=1}^{n} \ell_j(t)\phi(t_j), \tag{5.4}$$

where $\ell_j(t)$ are the usual Lagrangian interpolation polynomials, and ϕ_n a polynomial of degree $n-1$.

Hence,

$$\mathbf{L}\phi_n = \sum_{j=1}^{n} \phi(t_j)h_j, \tag{5.5}$$

where

$$h_j = \frac{b}{\pi} \int_{-1}^{1} \frac{w(t)\ell_j(t)}{t-x} dt < \infty \tag{5.6}$$

and h_j is calculated analytically and therefore exact.

Applying the Gauss-Jacobi rule to $\mathbf{K}\phi$, we obtain

$$\mathbf{K}_n\phi = \sum_{j=1}^{n} w_j K(x,t_j)\phi(t_j). \tag{5.7}$$

Then,

$$\mathbf{L}\phi_n + \mathbf{K}_n\phi = f_n. \tag{5.8}$$

THEOREM 5.1. *Assume that $f, \phi \in C[-1,1]$ and $K(x,t)$ is bounded in the closed domain $-1 \leq x, t \leq 1$. If $\|\mathbf{L}\|^{-1}$ exists in the uniform norm, then our rule converges uniformly to the true solution.*

Proof. From (5.3) and (5.8), we may write

$$\mathbf{L}(\phi - \phi_n) + (\mathbf{K} - \mathbf{K}_n)\phi = (f - f_n). \tag{5.9}$$

Therefore,

$$\begin{aligned}
\|\mathbf{L}\| \, |\phi - \phi_n| &\leq \|\mathbf{K} - \mathbf{K}_n\| |\phi| + \|f - f_n\|, \\
|\phi - \phi_n| &\leq \|\mathbf{L}\|^{-1}\{\|\mathbf{K} - \mathbf{K}_n\| |\phi| + \|f - f_n\|\}.
\end{aligned} \tag{5.10}$$

By Gauss-Jacobi quadrature rule, $\|\mathbf{K} - \mathbf{K}_n\| \to 0$ as $n \to \infty$, and by collocation, $\|f - f_n\| \to 0$ as $n \to \infty$ and this proves our theorem. $\square$

Acknowledgment

References

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