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Research Article

Spectral Theory from the Second-Order q-Difference Operator

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Spectral theory from the second-order q-difference operator Δ_q is developed. We give an integral representation of its inverse, and the resolvent operator is obtained. As application, we give an analogue of the Poincare inequality. We introduce the Zeta function for the operator Δ_q and we formulate some of its properties. In the end, we obtain the spectral measure.

1. Basic definitions

Consider $0 < q < 1$. In what follows, the standard conventional notations from [1] will be used

$$\mathbb{R}_q = \{\mp q^n,\ n \in \mathbb{Z}\}, \qquad \mathbb{R}_q^+ = \{q^n,\ n \in \mathbb{Z}\},$$
$$(a,q)_0 = 1, \qquad (a,q)_n = \prod_{i=0}^{n-1}(1 - aq^i), \tag{1.1}$$
$$[n]_q = \frac{1-q^n}{1-q}.$$

The q-schift operator is

$$\Lambda_q f(x) = f(qx). \tag{1.2}$$

Next, we introduce two concepts of q-analysis: the q-derivative and the q-integral. The q-derivative (see [2]) of a function f is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \tag{1.3}$$

and the second-order q-difference operator is

$$\Delta_q f(x) = \left[\frac{1-q}{q}\right]^2 \Lambda_q^{-1} D_q^2 f(x) = \frac{1}{x^2}\left[f(q^{-1}x) - \frac{1+q}{q} f(x) + \frac{1}{q} f(qx)\right]. \tag{1.4}$$

The product rule for the q-derivative is

$$D_q(fg)(x) = D_q f(x) g(x) + \Lambda_q f(x) D_q g(x). \tag{1.5}$$

Jackson's q-integral (see [3]) in the interval $[a,b]$ is defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} q^n [b f(bq^n) - a f(aq^n)]. \tag{1.6}$$

Also the rule of q-integration by parts is given by

$$\int_a^b D_q f(x) g(x) d_q x = [f(b)g(b) - f(a)g(a)] - \int_a^b \Lambda_q f(x) D_q g(x) d_q x. \tag{1.7}$$

The Hahn-Exton q-Bessel function of order $\alpha > -1$ (see [4–6]) is defined by

$$J_\alpha^{(3)}(x,q) = \frac{(q^{\alpha+1},q)_\infty}{(q,q)_\infty} x^\alpha {}_1\phi_1(0, q^{\alpha+1}, q; qx^2). \tag{1.8}$$

The q-trigonometric functions (see [7]) are defined on $\mathbb{C}$ by

$$\begin{aligned}
\cos(x,q^2) &= {}_1\phi_1(0,q;q^2;q^2x^2) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q,q)_{2n}} x^{2n} = \frac{(q^2,q^2)_\infty}{(q,q^2)_\infty} \sqrt{x} J_{-1/2}^{(3)}(x,q^2), \\
\sin(x,q^2) &= (1-q)^{-1} x {}_1\phi_1(0,q^3,q^2;q^2x^2) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q,q)_{2n+1}} x^{2n+1} = \frac{(q^2,q^2)_\infty}{(q,q^2)_\infty} \sqrt{x} J_{1/2}^{(3)}(x,q^2).
\end{aligned} \tag{1.9}$$

Both functions $\cos(x,q^2)$ and $\sin(x,q^2)$ are analytic. In [7], it is proved that

$$\Delta_q f(x) = \begin{cases} -\lambda^2 f(x), & \text{if } f(x) = \cos(\lambda x, q^2) \\ -\dfrac{1}{q}\lambda^2 f(x), & \text{if } f(x) = \sin(\lambda x, q^2). \end{cases} \tag{1.10}$$

Let $\mathcal{L}_{q,2}$ be the space of all real-valued functions defined on

$$[0,1]_q = \{q^n,\ n = 0,1,\dots\}, \tag{1.11}$$

such that

$$\|f\|_{\mathcal{L}_{q,2}} = \left(\int_0^1 |f(x)|^2 d_q x\right)^{1/2} < \infty. \tag{1.12}$$

Then, $\mathcal{L}_{q,2}$ is a separable Hilbert space with the inner product

$$\langle f,g\rangle = \int_0^1 f(x)g(x)d_q x. \tag{1.13}$$

In the following, we denote by $\mathcal{D}$ the subspace of $\mathcal{L}_{q,2}$ defined by

$$\mathcal{D} = \left\{f \in \mathcal{L}_{q,2},\ \Delta_q f \in \mathcal{L}_{q,2},\ \lim_{n\to\infty} f(q^n) = f(1) = 0\right\}. \tag{1.14}$$

2. Eigenfunctions of Δ_q in $\mathcal{D}$

THEOREM 2.1. $(\Delta_q,\mathcal{D})$ *has an infinite sequence of nonzero real eigenvalues*

$$\{\eta_n = -\lambda_n^2\}_{n\in\mathbb{N}^*}, \tag{2.1}$$

where $0 < \lambda_1 < \lambda_2 < \cdots$ *are the positive zeros of the following function:*

$$x \longmapsto \sin(\sqrt{q}x, q^2). \tag{2.2}$$

The corresponding set of eigenfunctions is

$$\{\sin(\sqrt{q}\lambda_n x, q^2)\}_{n\in\mathbb{N}^*}. \tag{2.3}$$

Proof. For $f,g \in \mathcal{D}$, using the q-integration by parts we write

$$\begin{aligned}\int_0^1 \Lambda_q^{-1} D_q^2 f(x)g(x)d_q x &= q^2 \int_0^1 D_q^2 \Lambda_q^{-1} f(x)g(x)d_q x\\ &= -q^2 \int_0^1 \Lambda_q D_q \Lambda_q^{-1} f(x) D_q g(x) d_q x\\ &= -q \int_0^1 D_q f(x) D_q g(x) d_q x,\end{aligned} \tag{2.4}$$

hence

$$\int_0^1 \Delta_q f(x)g(x)d_q x = \int_0^1 f(x)\Delta_q g(x)d_q x. \tag{2.5}$$

In particular, we can write

$$\int_0^1 \Delta_q f(x)f(x)d_q x = -\frac{(1-q)^2}{q}\int_0^1 [D_q f(x)]^2 d_q x. \tag{2.6}$$

Let σ_p be the sequence of eigenvalues of $(\Delta_q,\mathcal{D})$. Then

$$\sigma_p \subset \mathbb{R}_-. \tag{2.7}$$

Let $f \in \mathcal{D}$ be a function satisfying the q-differential equation

$$\Delta_q f(x) = -\lambda^2 f(x), \quad \forall x \in \mathbb{R}_q^+. \tag{2.8}$$

Using (1.4), we write

$$q^{-2n}\left[f(q^{n-1}) - \frac{(1+q)}{q} f(q^n) + f(q^{n+1})\right] = -\lambda^2 f(q^n), \quad \forall n \in \mathbb{Z}. \tag{2.9}$$

Therefore the set of solution of (2.8) is a vector space over $\mathbb{R}$ of dimension 2. From (1.10), it follows that f can be written in the form

$$f(x) = a\sin(\sqrt{q}\lambda x, q^2) + b\cos(\lambda x, q^2), \quad a,b \in \mathbb{R}. \tag{2.10}$$

If f satisfies

$$\lim_{n\to\infty} f(q^n) = f(1) = 0, \tag{2.11}$$

then $f(x) = a\sin(\sqrt{q}\lambda x, q^2)$ and $\sin(\sqrt{q}\lambda, q^2) = 0$. In [5], it is proved that The Hahn-Exton q-Bessel function of order $\alpha > -1$ has a countably infinite number of positive simple zeros. This finishes the proof. □

3. The inverse of Δ_q in the space $\mathcal{D}$

Given $x = q^s \in \mathbb{R}_q^+$, we define

$$[x,1]_q = \{q^n,\ n = 0\cdots s\}, \qquad [0,x]_q = \{q^n,\ n = s\cdots\}. \tag{3.1}$$

We introduce the operator

$$u_k : \mathcal{L}_{q,2} \longrightarrow u_k(\mathcal{L}_{q,2}) \tag{3.2}$$

defined by

$$u_k(f)(x) = \frac{q}{(1-q)^2}\int_0^1 k(x,y) f(y) d_q y, \tag{3.3}$$

where

$$k(x,y) = \begin{cases} x(y-1) & \text{if } y \in [x,1]_q \\ y(x-1) & \text{if } y \in [0,x]_q. \end{cases} \tag{3.4}$$

Theorem 3.1. *The operator u_k is the inverse of Δ_q in the space $\mathcal{D}$.*

Proof. For $f \in \mathcal{L}_{q,2}$, we write

$$u_k(f)(x) = \frac{q}{(1-q)^2}\left[(x-1)\int_0^x yf(y)d_qy + x\int_x^1 (y-1)f(y)d_qy\right]. \tag{3.5}$$

Then

$$u_k(f) \in \mathcal{L}_{q,2}, \quad \lim_{n\to\infty} u_k(f)(q^n) = u_k(f)(1) = 0. \tag{3.6}$$

On the other hand, using (1.5) we write

$$\begin{aligned} D_qu_k(f)(x) &= \frac{q}{(1-q)^2}\left[(x-1)xf(x)+\int_0^{qx} yf(y)d_qy - x(x-1)f(x) + \int_{qx}^1 (y-1)f(y)d_qy\right] \\ &= \frac{q}{(1-q)^2}\left[\int_0^{qx} yf(y)d_qy + \int_{qx}^1 (y-1)f(y)d_qy\right], \\ D_q^2u_k(f)(x) &= \frac{q^2}{(1-q)^2}[qxf(qx)-(qx-1)f(qx)] = \frac{q^2}{(1-q)^2}f(qx), \end{aligned} \tag{3.7}$$

which shows that

$$\Delta_q \circ u_k(f)(x) = f(x). \tag{3.8}$$

We conclude that

$$u_k(f) \in \mathcal{D}, \quad \Delta_q \circ u_k = \mathrm{id}_{\mathcal{L}_{q,2}}. \tag{3.9}$$

Similarly, we can prove that

$$u_k \circ \Delta_q f(x) = f(x), \quad \forall f \in \mathcal{D}. \tag{3.10}$$

Indeed

$$u_k \circ \Delta_q(f)(x) = \frac{q}{(1-q)^2}\int_0^1 k(x,y)\Delta_q f(y)d_qy, \tag{3.11}$$

with

$$k(x,0) = k(x,1) = 0, \tag{3.12}$$

and we obtain

$$\begin{aligned} &\frac{q}{(1-q)^2}\int_0^1 k(x,y)\Delta_q f(y)d_qy \\ &\quad = \frac{q}{(1-q)^2}\int_0^1 \Delta_q k(x,y)f(y)d_qy = \frac{q}{1-q}\sum_{n=0}^{\infty} q^n\Delta_q k(x,q^n)f(q^n). \end{aligned} \tag{3.13}$$

Next, we have

$$\Delta_q k(x,y) = \frac{1}{y^2}\left[k(x,q^{-1}y) - \frac{1+q}{q}k(x,y) + \frac{1}{q}k(x,qy)\right], \tag{3.14}$$

which implies

$$\begin{aligned}\Delta_q k(x,x) &= \frac{1}{x^2}\left[k(x,q^{-1}x) - \frac{1+q}{q}k(x,x) + \frac{1}{q}k(x,qx)\right] \\ &= \frac{1}{x^2}\left[x(q^{-1}x-1) - \frac{1+q}{q}x(x-1) + \frac{1}{q}qx(x-1)\right] = \frac{1-q}{qx}.\end{aligned} \tag{3.15}$$

Now we will prove that

$$\Delta_q k(x,y) = 0 \quad \text{if } x \neq y. \tag{3.16}$$

For $y \in [0,x]_q$, we have

$$\Delta_q k(x,y) = \frac{1}{y^2}\left[(x-1)q^{-1}y - \frac{1+q}{q}(x-1)y + \frac{1}{q}(x-1)qy\right] = 0, \tag{3.17}$$

and if $y \in [x,1]_q$,

$$\Delta_q k(x,y) = \frac{1}{y^2}\left[(y-1)q^{-1}x - \frac{1+q}{q}(y-1)x + \frac{1}{q}(y-1)qx\right] = 0. \tag{3.18}$$

Therefore

$$\frac{q}{(1-q)^2}\int_0^1 k(x,y)\Delta_q f(y)\,d_q y = f(x). \tag{3.19}$$

This finishes the proof. □

Corollary 3.2. *The sequence*

$$\{\sin(\sqrt{q}\lambda_n x, q^2)\}_{n\in\mathbb{N}}, \tag{3.20}$$

is an orthogonal basis of $\mathcal{L}_{q,2}$.

Proof. Since u_k is a Hilbert-Schmidt operator, then u_k is normal and compact because $\mathcal{L}_{q,2}$ is separable. The eigenfunctions of u_k are the elements of the sequence

$$\{\sin(\sqrt{q}\lambda_n x, q^2)\}_{n\in\mathbb{N}}, \tag{3.21}$$

associated with corresponding eigenvalues

$$\eta_n = -\frac{1}{\lambda_n^2}, \tag{3.22}$$

and they form an orthogonal basis of $\mathcal{L}_{q,2}$. □

4. Resolvent operator and Green kernel

We introduce the q-hyperbolic sine and the q-hyperbolic cosine function as follows:

$$\sinh(x,q^2) = -i\sin(ix,q^2), \qquad \cosh(x,q^2) = \cos(ix,q^2). \tag{4.1}$$

For $z \in \mathbb{C}/\{\sigma_p\}$, we have the following result.

Theorem 4.1. *The q-Sturm-Liouville problem*

$$\Delta_q U(x) = zU(x) - f(x), \quad U \in \mathcal{D}, \tag{4.2}$$

has a unique solution in the form

$$U(x) = (z-\Delta_q)^{-1} f(x) = \int_0^1 G_{q,z}(x,y)\Lambda_q f(y)d_q y, \tag{4.3}$$

where $G_{q,z}$ is the Green kernel defined by

$$G_{q,z}(x,y) = -\frac{q^2}{(1-q)}\frac{1}{\sqrt{qz}\sinh(\sqrt{qz},q^2)}\begin{cases} U_1(x)U_2(qy), & y \in [x,1]_q \\ U_1(qy)U_2(x), & y \in [0,x]_q \end{cases} \tag{4.4}$$

and U_1 and U_2 are defined by

$$\begin{aligned} U_1(x) &= \sinh(\sqrt{qz}x,q^2), \\ U_2(x) &= \cosh(\sqrt{z}x,q^2)\sinh(\sqrt{qz},q^2) - \sinh(\sqrt{qz}x,q^2)\cosh(\sqrt{z},q^2). \end{aligned} \tag{4.5}$$

Proof. We will solve this q-problem using the q-analogue of the method of variation of constants. We write U in the following form:

$$U(x) = U_1(x)V_1(x) + U_2(x)V_2(x). \tag{4.6}$$

Note that U_1 and U_2 form a fundamental solution set of the q-difference equation

$$\Delta_q U(x) = zU(x). \tag{4.7}$$

Using (1.5), we write

$$D_q U(x) = D_q U_1(x)V_1(x) + \Lambda_q U_1(x) D_q V_1(x) + D_q U_2(x) V_2(x) + \Lambda_q U_2(x) D_q V_2(x). \tag{4.8}$$

From the first condition

$$\Lambda_q U_1(x) D_q V_1(x) + \Lambda_q U_2(x) D_q V_2(x) = 0, \tag{4.9}$$

we get

$$D_q U(x) = D_q U_1(x) V_1(x) + D_q U_2(x) V_2(x). \tag{4.10}$$

Therefore

$$D_q^2U(x) = D_qV_1(x)D_qU_1(x) + \Lambda_qV_1(x)D_q^2U_1(x) + D_qV_2(x)D_qU_2(x) + \Lambda_qV_2(x)D_q^2U_2(x). \tag{4.11}$$

From the second condition

$$D_qV_1(x)D_qU_1(x) + D_qV_2(x)D_qU_2(x) = -\frac{q^2}{(1-q)^2}\Lambda_qf(x), \tag{4.12}$$

we obtain

$$D_q^2U(x) = \Lambda_qV_1(x)D_q^2U_1(x) + \Lambda_qV_2(x)D_q^2U_2(x) - \frac{q^2}{(1-q)^2}\Lambda_qf(x). \tag{4.13}$$

Conditions (4.9) and (4.12) form a linear system

$$\begin{pmatrix} \Lambda_qU_1(x) & \Lambda_qU_2(x) \\ D_qU_1(x) & D_qU_2(x) \end{pmatrix}\begin{pmatrix} D_qV_1(x) \\ D_qV_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\dfrac{q^2}{(1-q)^2}\Lambda_qf(x) \end{pmatrix}. \tag{4.14}$$

The solution of this system is

$$\begin{aligned} D_qV_1(x) &= \frac{q^2}{(1-q)^2w(x)}\Lambda_qU_2(x)\Lambda_qf(x), \\ D_qV_2(x) &= -\frac{q^2}{(1-q)^2w(x)}\Lambda_qU_1(x)\Lambda_qf(x), \end{aligned} \tag{4.15}$$

where

$$w(x) = \det\begin{pmatrix} \Lambda_qU_1(x) & \Lambda_qU_2(x) \\ D_qU_1(x) & D_qU_2(x) \end{pmatrix} = \Lambda_qU_1(x)D_qU_2(x) - \Lambda_qU_2(x)D_qU_1(x) \tag{4.16}$$

is the q-Wronskian of the q-Sturm-Liouville problem.

Now since

$$\Lambda_q^{-1}w(x) = U_1(x)\Lambda_q^{-1}D_qU_2(x) - U_2(x)\Lambda_q^{-1}D_qU_1(x), \tag{4.17}$$

using (1.5), and the fact that U_1 and U_2 are fundamental solution of the q-difference equation $\Delta_qU(x) = zU(x)$, we obtain

$$\begin{aligned} D_q\Lambda_q^{-1}w(x) &= D_q\Lambda_q^{-1}D_qU_2(x)U_1(x) + D_qU_2(x)D_qU_1(x) \\ &\quad - D_q\Lambda_q^{-1}D_qU_1(x)U_2(x) - D_qU_1(x)D_qU_2(x) \\ &= \frac{q}{(1-q)^2}\left[\Delta_qU_2(x)U_1(x) - \Delta_qU_1(x)U_2(x)\right] = 0. \end{aligned} \tag{4.18}$$

Therefore

$$w(q^n) = \text{constant}, \quad \forall n \in \mathbb{Z}. \tag{4.19}$$

Finally, for $x \in \mathbb{R}_q^+$ we get

$$w(x) = w(0) = U_1(0)D_qU_2(0) - U_2(0)D_qU_1(0). \tag{4.20}$$

Therefore the functions V_1 and V_2 satisfy

$$\begin{aligned} D_qV_1(x) &= \frac{q^2}{(1-q)^2w(0)}\Lambda_qU_2(x)\Lambda_qf(x), \\ D_qV_2(x) &= -\frac{q^2}{(1-q)^2w(0)}\Lambda_qU_1(x)\Lambda_qf(x), \end{aligned} \tag{4.21}$$

which gives

$$\begin{aligned} V_1(x) &= -\frac{q^2}{(1-q)^2w(0)}\int_x^1 \Lambda_qU_2(y)\Lambda_qf(y)d_qy, \\ V_2(x) &= -\frac{q^2}{(1-q)^2w(0)}\int_0^x \Lambda_qU_1(y)\Lambda_qf(y)d_qy. \end{aligned} \tag{4.22}$$

The condition $U \in \mathfrak{D}$ requires

$$U_1(0) = U_2(1) = 0, \tag{4.23}$$

which implies

$$w(0) = -U_2(0)D_qU_1(0). \tag{4.24}$$

Using the fact that

$$D_q \sin(x,q^2) = \frac{1}{1-q}\cos(x,q^2), \tag{4.25}$$

we obtain

$$w(0) = \frac{1}{1-q}\sqrt{qz}\sinh(\sqrt{qz},q^2). \tag{4.26}$$

This completes the proof. □

5. Poincare inequality

Here, we give a q-analogue of the Poincare inequality.

THEOREM 5.1. *Given $f \in \mathfrak{D}$, then*

$$\int_0^1 [f(x)]^2 d_qx \le \frac{(1-q)^2}{q\lambda_1^2}\int_0^1 [D_qf(x)]^2 d_qx. \tag{5.1}$$

Proof. For $f \in \mathfrak{D}$, we write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{k_n} a_n \sin\left(\sqrt{q}\lambda_n x, q^2\right), \quad \forall x \in [0,1]_q, \tag{5.2}$$

where

$$a_n = \int_0^1 f(x) \sin\left(\sqrt{q}\lambda_n x, q^2\right) d_q x, \qquad k_n = \left\| \sin\left(\sqrt{q}\lambda_n x, q^2\right) \right\|_{\mathcal{L}_{q,2}}^2. \tag{5.3}$$

Therefore

$$\Delta_q f(x) = -\sum_{n=1}^{\infty} \frac{1}{k_n} a_n \lambda_n^2 \sin\left(\sqrt{q}\lambda_n x, q^2\right), \quad x \in [0,1]_q. \tag{5.4}$$

This implies

$$\int_0^1 \Delta_q f(x) f(x) d_q x = -\sum_{n=1}^{\infty} a_n^2 \lambda_n^2, \qquad \int_0^1 [f(x)]^2 d_q x = \sum_{n=1}^{\infty} a_n^2. \tag{5.5}$$

Using that

$$\begin{aligned}\int_0^1 \Lambda_q^{-1} D_q^2 f(x) f(x) d_q x &= q \int_0^1 D_q^2 f(x) \Lambda_q^{-1} f(x) d_q x \\ &= q[D_q f(1) f(1) - D_q f(0) f(0)] - q \int_0^1 [D_q f(x)]^2 d_q x \\ &= -q \int_0^1 [D_q f(x)]^2 d_q x,\end{aligned} \tag{5.6}$$

we obtain

$$\begin{aligned}\int_0^1 [D_q f(x)]^2 d_q x &= -\frac{q}{(1-q)^2} \int_0^1 \Delta_q f(x) f(x) d_q x \\ &= \frac{q}{(1-q)^2} \sum_{n=1}^{\infty} a_n^2 \lambda_n^2 = \frac{q\lambda_1^2}{(1-q)^2} \sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_1}\right)^2 a_n^2.\end{aligned} \tag{5.7}$$

From the inequality

$$\frac{\lambda_n}{\lambda_1} \geq 1, \quad \text{for every } n \geq 1, \tag{5.8}$$

we conclude that

$$\int_0^1 [f(x)]^2 d_q x \leq \frac{(1-q)^2}{q\lambda_1^2} \int_0^1 [D_q f(x)]^2 d_q x. \tag{5.9}$$

This completes the proof. □

As an application, consider the function

$$f(x) = x(x-1). \tag{5.10}$$

After simple calculations, we obtain

$$\left[\frac{\lambda_1}{1-q}\right]^2 \le \frac{1/[3]_q}{1/[5]_q - 2/[4]_q + 1/[3]_q}. \tag{5.11}$$

6. Zeta function for the operator Δ_q

Theorem 6.1. (1) *If $q^3 < (1-q^2)^2$, then the Zeta function for the operator Δ_q*

$$\zeta_q(s) = \sum_{p=1}^{\infty} \left(\frac{\lambda_1}{\lambda_p}\right)^s \tag{6.1}$$

is analytic in the region $\{s \in \mathbb{C},\ \Re(s) > 0\}$.

(2) *For every $n \in \mathbb{N}^*$,*

$$\zeta_q(2n) = \left(-\frac{q\lambda_1^2}{(1-q)^2}\right)^n \int_0^1 \cdots \int_0^1 k(x_1,x_2)\cdots k(x_{n-1},x_n)k(x_n,x_1)\,d_qx_1\cdots d_qx_n. \tag{6.2}$$

Proof. In [8], it is proved that if

$$q^{2\alpha+2} < \left(1-q^2\right)^2, \tag{6.3}$$

then the positive roots $w_k^{(\alpha)}(q^2)$ of the Hahn-Exton q-Bessel function $J_\alpha^{(3)}(x,q)$ satisfy

$$\lim_{k\to\infty} q^k w_k^{(\alpha)}(q^2) = 1. \tag{6.4}$$

Since

$$\sin(x,q^2) = \frac{(q^2,q^2)_\infty}{(q,q^2)_\infty}\sqrt{x}J_{1/2}^{(3)}(x,q^2), \tag{6.5}$$

we have

$$\lambda_k = \frac{w_n^{(1/2)}(q^2)}{\sqrt{q}} \sim \frac{1}{\sqrt{q}}q^{-k}, \tag{6.6}$$

which leads to the first result.

To prove the second result we use the Mercer theorem for the operator u_k

$$\frac{q}{(1-q^2)}k(x,y) = -\sum_{p=1}^{\infty}\frac{1}{k_p}\frac{1}{\lambda_p^2}\sin(\sqrt{q}\lambda_p x,q^2)\sin(\sqrt{q}\lambda_p y,q^2), \tag{6.7}$$

and the orthogonality relations

$$\frac{1}{k_p}\int_0^1 \sin\left(\sqrt{q}\lambda_p x, q^2\right)\sin\left(\sqrt{q}\lambda_m x, q^2\right) d_q x = \delta_{pm}. \tag{6.8}$$

This completes the proof. □

Example 6.2.

$$\begin{aligned}\zeta_q(2) &= -\frac{q\lambda_1^2}{(1-q)^2}\int_0^1 k(x,x)d_q x = \frac{q\lambda_1^2}{(1-q)^2}\left[\frac{1}{[2]_q}-\frac{1}{[3]_q}\right],\\ \zeta_q(4) &= \int_0^1\cdots\int_0^1 k(x,y)^2 d_q y\, d_q x = \frac{q^2\lambda_1^4}{(1-q)^4}\frac{1}{[3]_q}\left[\frac{1}{[5]_q}-\frac{2}{[4]_q}+\frac{1}{[3]_q}\right].\end{aligned} \tag{6.9}$$

In [7], it is proved that $\sin((1-q)x, q^2) \to \sin(x)$, as $q \to 1^-$. This implies

$$\lim_{q\to 1^-}\frac{\lambda_n}{1-q} = n\pi. \tag{6.10}$$

Thus, when $q \to 1^-$, we obtain the well-known identities for the Euler Zeta function

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}. \tag{6.11}$$

7. Spectral resolution

We can now find the spectral measure of Δ_q from the resolvent, see the Stieltjes-Perron inversion formula [9].

Theorem 7.1. *Let $\eta_n = -\lambda_n^2$ be a point from the spectrum of Δ_q. If*

$$u \in \mathcal{D}, \qquad v \in \mathcal{L}_{q,2}, \tag{7.1}$$

then the spectral measure E of $\{\eta_n\}$ is given by

$$\langle E\{\eta_n\}u, v\rangle = \frac{2}{(1-q)}\frac{\cosh\left(\sqrt{\eta_n}, q^2\right)}{(d/dx)\sinh\left(x, q^2\right)\big|_{x=\sqrt{q\eta_n}}}\left\langle u, \sin\left(\sqrt{q}\lambda_n x, q^2\right)\right\rangle \cdot \left\langle v, \sin\left(\sqrt{q}\lambda_n x, q^2\right)\right\rangle. \tag{7.2}$$

Proof. In order to calculate $E\{\eta_n\}$, we choose the interval (a,b) so that it contains only $\eta_n = -\lambda_n^2$ as a point from the spectrum. Then

$$\begin{aligned}\langle E\{\eta_n\}u, v\rangle &= \langle E(a,b)u, v\rangle\\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{2i\pi}\int_a^b \left[\left\langle \left(s-i\varepsilon-\Delta_q\right)^{-1}u, v\right\rangle - \left\langle \left(s+i\varepsilon-\Delta_q\right)^{-1}u, v\right\rangle\right]ds\\ &= \frac{1}{2i\pi}\int_{\{\eta_n\}}\left\langle \left(s-\Delta_q\right)^{-1}u, v\right\rangle ds.\end{aligned} \tag{7.3}$$

Now observe that

$$\langle (s-\Delta_q)^{-1}u, v\rangle = \int_0^1 \cdots \int_0^1 G_s(x,y)\Lambda_q u(y)v(x)d_q y\, d_q x. \tag{7.4}$$

If $z=\eta_n$, then $\sinh(\sqrt{qz},q^2)=0$, which implies that the fundamental solutions U_1 and U_2 of the q-difference equation

$$\Delta_q U = zU \tag{7.5}$$

are proportional as follows:

$$U_2(x) = -\cosh\left(\sqrt{\eta_n},q^2\right)U_1(x), \quad \forall x\in[0,1]_q. \tag{7.6}$$

Therefore

$$\begin{aligned}\langle E\{\eta_n\}u,v\rangle &= \int_0^1\cdots\int_0^1 [\mathrm{Res}_{s=\eta_n}G_s(x,y)]\Lambda_q u(y)v(x)d_q y\, d_q x\\ &= \frac{q^2}{(1-q)}\cosh\left(\sqrt{\eta_n},q^2\right)\left[\mathrm{Res}_{s=\eta_n}\frac{1}{\sqrt{qs}\sinh\left(\sqrt{qs},q^2\right)}\right]\\ &\quad\times\langle\Lambda_q u,\sinh\left(\sqrt{q\eta_n}qx,q^2\right)\rangle\cdot\langle v,\sinh\left(\sqrt{q\eta_n}x,q^2\right)\rangle.\end{aligned} \tag{7.7}$$

If $f(1)=0$, then

$$\int_0^1 \Lambda_q f(x)d_q x = \frac{1}{q}\int_0^1 f(x)d_q x, \tag{7.8}$$

which implies

$$\langle\Lambda_q u,\sinh\left(\sqrt{q\eta_n}qx,q^2\right)\rangle = \frac{1}{q}\langle u,\sinh\left(\sqrt{q\eta_n}x,q^2\right)\rangle. \tag{7.9}$$

Finally we have

$$\langle E\{\eta_n\}u,v\rangle = \frac{2}{(1-q)}\frac{\cosh\left(\sqrt{\eta_n},q^2\right)}{(d/dx)\sinh\left(x,q^2\right)\big|_{x=\sqrt{q\eta_n}}}\langle u,\sin\left(\sqrt{q}\lambda_n x,q^2\right)\rangle\cdot\langle v,\sin\left(\sqrt{q}\lambda_n x,q^2\right)\rangle. \tag{7.10}$$

This completes the proof. □

Corollary 7.2.

$$k_n = \left\|\sin\left(\sqrt{q}\lambda_n x,q^2\right)\right\|^2_{\mathcal{L}_{q,2}} = \frac{(1-q)}{2}\frac{(d/dx)\sinh\left(x,q^2\right)\big|_{x=\sqrt{q\eta_n}}}{\cosh\left(\sqrt{\eta_n},q^2\right)}. \tag{7.11}$$

Proof. For

$$u=v=\sin\left(\sqrt{q}\lambda_n x,q^2\right), \tag{7.12}$$

the result follows immediately from the following equality:

$$E\{\eta_n\}\sin\left(\sqrt{q}\lambda_n x, q^2\right) = \sin\left(\sqrt{q}\lambda_n x, q^2\right). \tag{7.13}$$

This finishes the proof. □

Remark 7.3. In [5], it is proved that

$$\int_0^1 x\left[J_\alpha^{(3)}\left(aqx, q^2\right)\right]^2 d_q x = -\frac{1-q}{2} q^{\alpha-1} J_{\alpha+1}^{(3)}\left(aq, q^2\right)\frac{d}{dx} J_\alpha^{(3)}\left(x, q^2\right)\Big|_{x=a}, \tag{7.14}$$

where $a \neq 0$ is a real zero of $J_\alpha^{(3)}(x, q^2)$. This formula can be employed to evaluate k_n by another method.

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