Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 27368, 10 pages
doi:10.1155/2007/27368

*Research Article*

# Periodic Solutions of Evolution *m*-Laplacian Equations with a Nonlinear Convection Term

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Received 24 April 2006; Revised 23 October 2006; Accepted 29 November 2006

Recommended by Vladimir Mityushev

We show the existence and gradient estimates of periodic solutions in the case of $0 \leq \alpha < m+1$ to the evolution $m$-Laplacian equations of form $u_t - \operatorname{div}\{|\nabla \mathrm{u}|^m \nabla u\} + \mathbf{b}(u) \cdot \nabla u = f(t)u^\alpha + h(x,t)$, in $\Omega \times \mathbb{R}^1$ with the Dirichlet boundary value condition.



## 1. Introduction and main results

In this paper, we are concerned with the existence and gradient estimates for periodic solutions of the evolution $m$-Laplacian equations with a nonlinear convection term and with the Dirichlet boundary value condition

$$\begin{gathered} u_t - \operatorname{div}\{|\nabla u|^m \nabla u\} + \mathbf{b}(u) \cdot \nabla u = f(t)u^\alpha + h(x,t), \quad \text{in } \Omega \times \mathbb{R}^1, \\ u(x,t) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^1, \\ u(x,t+\omega) = u(x,t), \quad \text{in } \Omega \times \mathbb{R}^1, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\omega > 0$, $m > 1$, and $\mathbf{b}(u)$ is a nonlinear vector field such that $|\mathbf{b}(u)| \leq k|u|^\beta$, with some $k > 0$, $0 \leq \beta \leq m-1$. $f(t)$ and $h(x,t)$ are $\omega$-periodic (in $t$) functions.

Equation (1.1) is a class of degenerate parabolic equations and appears to be relevant in the theory of non-Newtonian fluids perturbed by nonlinear terms and forced by rather irregular period in time excitations, see [1, 2] for instance. The term $\mathbf{b}(u) \cdot \nabla u$ describes an effect of convection with a velocity field $\mathbf{b}(u)$.

In the last two decades, periodic parabolic equations have been the subject of extensive study (see [3–11]). In Particular, Nakao [7] considered the following equation:

$$u_t - \Delta\beta(u) + B(x,t,u) = f(x,t), \tag{1.2}$$

where $B$ and $f$ are periodic in $t$ with common period $\omega > 0$, $\beta(u)$ satisfies $\beta'(u) > 0$ except for $u = 0$ and $B(x,t,u)u \geq -b_0|u|$ with some constant $b_0 \geq 0$. The existence and $L^\infty$ estimates of periodic solutions were established.

When $\mathbf{b}(u) = \mathbf{0}$ and $f(t)u^\alpha$ replaced by $g(x,u)$ with $g(x,u)u \leq k_0|u|^{\beta+1} + k_1|u|$, $0 \leq \beta \leq m+1$, Nakao and Ohara [8] obtained the existence and $\|\nabla u(t)\|_\infty$ estimate of periodic solutions of (1.1).

For $\mathbf{b}(u) = \mathbf{0}$ and $h(x,t) = 0$, applying the topological degree theory, Wang et al. [9] discussed the existence of periodic solutions of (1.1) in the case of strongly nonlinear sources $(m+1 < \alpha < m+1+(m+2)/N)$.

The object of this paper is to prove the existence of periodic solutions in the case of $0 \leq \alpha < m+1$ and to derive an estimates of $\|\nabla u(t)\|_\infty$ for the problem (1.1). For the proof of our result, we employ Moser's technique as in [12] and make some devices as in [8] to obtain the existence of periodic solutions. Leray-Shauder fixed point theorem instead of approximate method used in [8] is applied to prove the existence of periodic solutions. To derive estimates of $\nabla u(t)$, we must treat the terms $\mathbf{b}(u) \cdot \nabla u$ and $f(t)u^\alpha$ at the same time very carefully. To our best knowledge, this result is not found in others work.

Let $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ denote $L^p = L^p(\Omega)$ and $W^{m,p} = W^{m,p}(\Omega)$ norms, respectively, $1 \leq p \leq \infty$.

Due to the degeneracy of the equations considered, problem (1.1) has no classical solutions in general, and thus we consider its weak solutions in the following sense.

*Definition 1.1.* Assume that $h(x,t) \in E = C_\omega(\overline{Q})$, the set of all functions in $C(\overline{\Omega} \times \mathbb{R}^1)$ which are periodic in $t$ with period $\omega$, where $Q = \Omega \times (0,\omega)$. A function $u$ is said to be a periodic solution of problem (1.1) if

$$u \in L^{m+2}(0,\omega; W_0^{1,m+2}(\Omega)) \cap C_\omega(\overline{Q}), \tag{1.3}$$

and $u$ satisfies

$$\iint_Q \{-u\,\varphi_t + |\nabla u|^m \nabla u \cdot \nabla\varphi - \mathbf{B}(u) \cdot \nabla\varphi - f(t)u^\alpha\varphi - h(x,t)\varphi\}\,dx\,dt = 0 \tag{1.4}$$

for any $\varphi \in C_0^1(0,\omega; C_0^1(\Omega))$ with $\varphi(x,0) = \varphi(x,\omega)$, where $\mathbf{B}(u) = \int_0^u \mathbf{b}(s)ds$ is set.

We assume

(H1) $\mathbf{b}(u) = (b_1(u), b_2(u), \ldots, b_N(u))$ is an $\mathbb{R}^N$-valued function on $\mathbb{R}^1$, satisfying

$$|\mathbf{b}(u)| \leq k|u|^\beta \tag{1.5}$$

for some $0 \leq \beta < m-1$ and $k > 0$, or $\beta = m-1$, and $k > 0$ is sufficiently small.

(H2) $h(x,t) \in C_\omega(\overline{Q}) \cap L^\infty(0,\omega; W_0^{1,\infty}(\Omega))$, $h(x,t) > 0$ for $(x,t) \in \Omega \times \mathbb{R}^1$ and we set $M_0 = \sup_t \|h(t)\|_\infty$, $M_1 = \sup_t \|\nabla h(t)\|_\infty$.

(H3) $f(t) \in L^\infty(0,\omega)$ is periodic in $t$ with period $\omega$. We also assume that $0 \leq \alpha < m+1$.

(H4) $\partial\Omega$ is of $C^2$ class and the mean curvature $H(x)$ at $x \in \partial\Omega$ is nonpositive with respect to the outward normal.

*Remark 1.2.* (H4) is satisfied in particular if $\Omega$ is convex. Without (H4), we cannot control the boundary integral which appears in the estimation of $\|\nabla u(t)\|_\infty$.

Our main results of this paper read as follows.

Theorem 1.3. *Under the assumptions (H1)–(H3), $N > 1$, problem (1.1) admits at least one solution $u(t)$, which satisfies*

$$u(t) \in L^\infty(0,\omega; W_0^{1,m+2}(\Omega)) \cap C_\omega(\overline{Q}), \quad u_t \in L^2(Q). \tag{1.6}$$

Theorem 1.4. *Under the assumptions (H1)–(H4), the solution $u(t)$ of problem (1.1) further belongs to $L^\infty(0,\omega; W_0^{1,\infty}(\Omega))$, and satisfies*

$$\sup_t \|\nabla u(t)\|_\infty \le C_1 < \infty, \tag{1.7}$$

*where $C_1$ is a constant, depending on $M_0$, $M_1$, and $\alpha$.*

For the proof of theorems, we use the following lemmas.

Lemma 1.5 [12] (Gagliardo-Nirenberg). *Let $\beta \ge 0$, $N > p \ge 1$, $\beta + 1 \le q$, and $1 \le r \le q \le (\beta+1)Np/(N-p)$, then for $u$ such that $|u|^\beta u \in W^{1,p}(\Omega)$,*

$$\|u\|_q \le C^{1/(\beta+1)} \|u\|_r^{1-\theta} \| \mid u \mid^\beta u \|_{1,p}^{\theta/(\beta+1)} \tag{1.8}$$

*with $\theta = (\beta+1)(r^{-1} - q^{-1})/\{N^{-1} - p^{-1} + (\beta+1)r^{-1}\}$, where $C$ is a constant independent of $q$, $r$, $\beta$, and $\theta$.*

Lemma 1.6 [8]. *Let $y(t) \in C^1(\mathbb{R}^1)$ be a nonnegative $\omega$ periodic function satisfying the differential inequality*

$$y'(t) + Ay^{1+\alpha}(t) \le By(t) + C, \quad t \in \mathbb{R}^1, \tag{1.9}$$

*with some $\alpha > 0$, $A > 0$, $B \ge 0$, and $C \ge 0$. Then*

$$y(t) \le \max\{1, (A^{-1}(B+C))^{1/\alpha}\}. \tag{1.10}$$

The paper is organized as follows. Section 2 is devoted to the proof of the existence of periodic solutions for problem (1.1) by using the Leray-Shauder fixed point theorem, which is different from that adopted in [8, 9]. Subsequently, we present the proof of Theorem 1.4 in Section 3.

## 2. The proof of Theorem 1.3

Our result will be proved by means of parabolic regularization. Namely, we consider the regularized equations

$$u_t - \operatorname{div}\{(|\nabla u|^2 + \varepsilon)^{m/2} \nabla u\} + \mathbf{b}(u) \cdot \nabla u = f(t)u^\alpha + h(x,t), \quad (x,t) \in Q, \tag{2.1}$$

where $\varepsilon > 0$. The desired solution $u(t)$ of problem (1.1) will be obtained as a limit point of the approximate solutions $u_\varepsilon(t)$ of (2.1). To prove the existence of the approximate solutions $u_\varepsilon(t)$, we apply the Leray-Shauder fixed point theorem. For our purpose, we need the following a priori estimate.

Proposition 2.1. *Let $u_0$ be a periodic solution of the equation*

$$u_t - \operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\} + \mathbf{b}(u)\cdot\nabla u = \tau f(t)u^\alpha + \tau h(x,t), \quad (x,t)\in Q, \tag{2.2}$$

*with $\tau\in[0,1]$, and $u_0$ satisfying the Dirichlet boundary value condition of (1.1). Then there exists a constant $C_0>0$ independent of $\tau$ and $\varepsilon$ such that*

$$||u_0(t)||_\infty \le C_0. \tag{2.3}$$

*Proof.* We only consider $N>m+2$, the other case can be treated similarly.

Multiplying (2.2) by $|u|^{p-2}u$ ($p>2$), integrating by parts, and noticing that

$$\begin{aligned}\int_\Omega \mathbf{b}(u)\cdot\nabla u|u|^{p-2}u\,dx &= \int_\Omega\sum_{i=1}^N b_i(u)|u|^{p-2}u\frac{\partial u}{\partial x_i}\,dx\\ &= \sum_{i=1}^N\int_\Omega\left(\int_0^u b_i(s)|s|^{p-2}s\,ds\right)_{x_i}dx\\ &= \sum_{i=1}^N\int_{\partial\Omega}\left(\int_0^u b_i(s)|s|^{p-2}s\,ds\right)\cos(\mathbf{n},x_i)\,ds\\ &= 0,\end{aligned} \tag{2.4}$$

we have

$$\frac{1}{p}\frac{d}{dt}||u(t)||_p^p+\varepsilon(p-1)\left(\frac{m+2}{p+m}\right)^{m+2}||\nabla u^{(p+m)/(m+2)}||_{m+2}^{m+2} \le C\left(f(t)\|u\|_{p+\alpha-1}^{p+\alpha-1}+\|h\|_p\|u\|_p^{p-1}\right). \tag{2.5}$$

If $1\le\alpha<m+1$, by Hölder's inequality and Lemma 1.5, we have

$$\begin{aligned}\|u\|_{p+\alpha-1}^{p+\alpha-1} &= \int_\Omega |u|^{\theta_1}|u|^{\theta_2}\,dx \le C\|u\|_p^{\theta_1}\|u\|_q^{\theta_2}\\ &\le C\|u\|_p^{\theta_1}||\nabla u^{(p+m)/(m+2)}||_{m+2}^{\theta_2(m+2)/(p+m)}\\ &\le \frac{\varepsilon}{2M_0}(p-1)\left(\frac{m+2}{p+m}\right)^{m+2}||\nabla u^{(p+m)/(m+2)}||_{m+2}^{m+2}+C\left(\|u\|_p^{\theta_1}\right)^{r/\theta_1}p^\sigma,\end{aligned} \tag{2.6}$$

where we set $q=(p+m)N/(N-m-2)$, $\theta_1=p[q-(p+\alpha-1)]/(q-p)$, $\theta_2=q(\alpha-1)/(q-p)$, $r<p$, which imply $\theta=1$ in Gagliardo-Nirenberg inequality, and $\sigma>0$ is a constant independent of $p$.

If $0<\alpha<1$, by Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}\|u\|_{p+\alpha-1}^{p+\alpha-1} &= \int_\Omega |u|^{\alpha p}|u|^{(1-\alpha)(p-1)}\,dx \le \left(\int_\Omega |u|^p\,dx\right)^\alpha\left(\int_\Omega |u|^{p-1}\,dx\right)^{1-\alpha}\\ &\le \|u\|_p^p + C\|u\|_p^{p-1}.\end{aligned} \tag{2.7}$$

If $\alpha = 0$, then we use Hölder's inequality to obtain

$$\|u\|_{p+\alpha-1}^{p+\alpha-1} \le \left(\int_\Omega |u|^p dx\right)^{(p-1)/p} \left(\int_\Omega dx\right)^{1/p} \le \max\{1, |\Omega|^{1/2}\}\|u\|_p^{p-1}. \tag{2.8}$$

It follows from (2.5)–(2.8) that

$$\frac{d}{dt}\|u(t)\|_p^p + C_1 p^{-m}\|\nabla u^{(p+m)/(m+2)}\|_{m+2}^{m+2} \le C(M_0)\left(p^{\sigma+1}\|u(t)\|_p^p + 1\right), \tag{2.9}$$

where we set

$$\begin{gathered} p_1 = m+2, \qquad p_n = (m+2)p_{n-1} - m, \qquad \alpha_n = (p_n+m)\theta_n^{-1} - p_n (> m), \\ \theta_n = \frac{1 - p_{n-1}p_n^{-1}}{1 + N^{-1} - (m+2)^{-1}} = \frac{N[(m+1)p_n - m]}{p_n[(m+1)N + m + 2]}. \end{gathered} \tag{2.10}$$

By using Lemma 1.5, we have

$$\|u\|_{p_n} \le C\|u\|_{p_{n-1}}^{1-\theta_n}\|\nabla u^{(p_n+m)/(m+2)}\|_{m+2}^{\theta_n(m+2)/(p_n+m)}. \tag{2.11}$$

Set $p = p_n$ in (2.9) and by (2.11), we have

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|_{p_n}^{p_n} + C_1 C^{-(p_n+m)\theta_n^{-1}} p_n^{-m}\|u(t)\|_{p_{n-1}}^{(p_n+m)(\theta_n-1)/\theta_n}\|u(t)\|_{p_n}^{(p_n+m)/\theta_n} \\ &\quad \le C(M_0)\left(p_n^{\sigma+1}\|u(t)\|_{p_n}^{p_n} + 1\right). \end{aligned} \tag{2.12}$$

Therefore,

$$\frac{d}{dt}\|u(t)\|_{p_n} + C_1 C^{-(p_n+m)\theta_n^{-1}} p_n^{-m-1}\|u(t)\|_{p_{n-1}}^{m-\alpha_n}\|u(t)\|_{p_n}^{\alpha_n+1} \le C(M_0)\left(p_n^{\sigma}\|u(t)\|_{p_n} + 1\right). \tag{2.13}$$

Let $\chi_n \equiv \sup_t \|u(t)\|_{p_n}$, by Lemma 1.6, we obtain

$$\chi_n \le \max\left\{1, \left(C(M_0)C^{(p_n+m)\theta_n^{-1}} p_n^{m+\sigma+1}\chi_{n-1}^{\alpha_n-m}\right)^{1/\alpha_n} \equiv B_n^{1/\alpha_n}\right\}. \tag{2.14}$$

We set without loss of generality that $B_n^{1/\alpha_n} > 1$, which implies $\chi_n \le B_n^{1/\alpha_n}$. It is easy to verify that $\{\chi_n\}$ is bounded (see [7]), and

$$\sup_t \|u(t)\|_\infty \le \overline{\lim_{n\to\infty}}\chi_n \le C(M_0) < \infty. \tag{2.15}$$

□

To prove the convergence of $u_\varepsilon(t)$, we need the following proposition.

PROPOSITION 2.2. *Under the assumptions (H1)–(H3), the solution $u_\varepsilon(t)$ of (2.1) satisfies*

$$\int_0^\omega \|\nabla u\|_{m+2}^{m+2} dt \le C(M_0), \tag{2.16}$$

$$\int_0^\omega \|u_t(t)\|_2^2 dt \le C(M_0), \tag{2.17}$$

*where $C(M_0)$ denotes a constant depending on $M_0$ and independent of $\varepsilon$.*

*Proof.* Multiplying (2.1) by $u$ and integrating, we obtain

$$\begin{aligned}\int_0^\omega\int_\Omega uu_t\,dx\,dt+\int_0^\omega\int_\Omega|\nabla u|^{m+2}dx\,dt+\int_0^\omega\int_\Omega \mathbf{b}(u)\cdot\nabla uu\,dx\,dt\\ =\int_0^\omega\int_\Omega fu^{\alpha+1}dx\,dt+\int_0^\omega\int_\Omega hu\,dx\,dt.\end{aligned}\tag{2.18}$$

By the periodicity, Hölder's inequality, and Poincare's inequality, we have

$$\begin{aligned}\int_0^\omega\|\nabla u(t)\|_{m+2}^{m+2}dt&\le\int_0^\omega\|f(t)\|_{p^*}\|u(t)\|_{m+2}^{\alpha+1}dt+\int_0^\omega\|h(t)\|_{m+2}\|u(t)\|_{m+2}^{m+1}dt\\&\le\left(\int_0^\omega\|f(t)\|_{p^*}^{p^*}dt\right)^{1/p^*}\left(\int_0^\omega\|u(t)\|_{m+2}^{m+2}dt\right)^{(\alpha+1)/(m+2)}\\&\quad+\left(\int_0^\omega\|h(t)\|_{m+2}^{m+2}dt\right)^{1/(m+2)}\left(\int_0^\omega\|u(t)\|_{m+2}^{m+2}dt\right)^{(m+1)/(m+2)}\\&\le C(M_0)\left[\left(\int_0^\omega\|\nabla u(t)\|_{m+2}^{m+2}dt\right)^{(\alpha+1)/(m+2)}\right.\\&\qquad\left.+\left(\int_0^\omega\|\nabla u(t)\|_{m+2}^{m+2}dt\right)^{(m+1)/(m+2)}\right],\end{aligned}\tag{2.19}$$

in which $p^*=(m+2)(m+1-\alpha)^{-1}$. Thus, we have

$$\int_0^\omega\|\nabla u\|_{m+2}^{m+2}dt\le C(M_0)<\infty.\tag{2.20}$$

In order to derive (2.17), we multiply (2.1) by $u_t$ and integrate over $[o,\omega]\times\Omega$,

$$\int_0^\omega\|u_t(t)\|_2^2\,dt+\int_0^\omega\int_\Omega \mathbf{b}(u)\cdot\nabla uu_t\,dx\,dt\le\int_0^\omega\int_\Omega|u_th|\,dx\,dt+\int_0^\omega f(t)dt\int_\Omega u^\alpha u_t\,dx.\tag{2.21}$$

Hence, by (H1), (2.16), and Young's inequality, we have

$$\int_0^\omega\|u_t(t)\|_2^2\,dt\le C(M_0)<\infty.\tag{2.22}$$

□

*Completion of the proof of Theorem 1.3.* Now we apply the Leray-Schauder fixed point theorem to show the existence of periodic solutions of problem (1.1). To do this, we investigate the following regularized equation:

$$u_t-\operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\}+\mathbf{b}(u)\cdot\nabla u=g(x,t),\quad x\in\Omega,\ t>0,\tag{2.23}$$

where $g\in E=C_\omega(\overline{Q})$. By using Faedo-Galerkin method and Browder fixed pointed theorem, Crema and Boldrini [13] have proved that for any $g\in E$, the regularized problem has a solution $u\in L^\infty(0,\omega;W_0^{1,m+2}(\Omega))$ and $u_t\in L^2(Q)$.

We defined $T: E \to E$ by $Tg = u$, then the map $T$ is continuous and compact.

In fact, by [14, Theorem 1.2 in page 42] and noticing the periodicity of $u$, we arrive at

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \|u\|_\infty \left(|x_1 - x_2| + \|u\|_\infty^{(p-2)/p} |t_1 - t_2|^{1/p}\right)^\beta \tag{2.24}$$

for every pair of points $(x_1,t_1),(x_2,t_2) \in \overline{Q}$, where the positive constants $\gamma$, $\beta$ depend only on $N$, $\varepsilon$, $m$, $\|g\|_\infty$. Ascoli-Arezela theorem implies that $T$ maps any bounded set of $E$ into a compact set of $E$.

Next, suppose that $g_k \to g$ as $k \to \infty$ and denote $u_k = Tg_k$, then there exists a function $u \in E$ such that

$$u_k(x,t) \longrightarrow u(x,t) \quad \text{uniformly in } Q, \tag{2.25}$$

by taking some subsequence if necessary.

Noticing the fact that

$$\int_\Omega \mathbf{b}(u) \cdot \nabla u u \, dx = 0, \tag{2.26}$$

we can prove that $u = Tg$ by using the argument similar to [9].

Let $\Phi(v) = f(t)v_+^\alpha + h(x,t)$, by the conditions (H2)-(H3) and the estimate above, we can see that $T(\tau,\Phi(v))$ is also the complete continuous map for $\tau \in [0,1]$. Proposition 2.1 shows that if $u_0$ is a fixed point of $T(\tau,\Phi(v))$, then

$$\|u_0(t)\|_\infty \le C_0 \tag{2.27}$$

with $C_0 > 0$ is a constant independent of $\tau$, $\varepsilon$. Hence, applying the Leray-Schauder fixed point theorem, we conclude that (2.1) admits a periodic solution $u_\varepsilon$.

Therefore, we can obtain a periodic solution $\{u(t)\}$ of the problem (1.1) as a limit point of $\{u_\varepsilon(t)\}$ (see [8, 12]).

## 3. The proof of Theorem 1.4

In this section, we will derive the estimates of $\|\nabla u(t)\|_\infty$ for an assumed smooth solution of the problem and prove Theorem 1.4.

Proposition 3.1. *Under the assumptions (H1)–(H4), the (smooth) periodic solution $u(t)$ of problem (1.1) satisfies*

$$\sup_t \|\nabla u(t)\|_\infty \le C_1 < \infty, \tag{3.1}$$

*where $C_1$ is a constant only dependent on $M_0$, $M_1$, and $\alpha$.*

*Proof.* Multiplying (2.1) by $-\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}$ $(p>m+2)$, and integrating over $\Omega$, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}||\nabla u(t)||_p^p&+\int_\Omega \operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\}\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\&=\int_\Omega \mathbf{b}(u)\cdot\nabla u\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx-\int_\Omega f(t)u^\alpha\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\&\quad-\int_\Omega h(x,t)\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx.\end{aligned}\tag{3.2}$$

Further, integrating by parts, we obtain (see [12])

$$\begin{aligned}&\int_\Omega \operatorname{div}\{(|\nabla u|^2+\varepsilon)^{m/2}\nabla u\}\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\&\quad\ge\int_\Omega|\nabla u|^{p+m-2}|D^2u|^2dx+\frac{p-2}{4}\int_\Omega|\nabla u|^{p+m-4}|\nabla(|\nabla u|^2)|^2dx\\&\qquad-C(N-1)\int_{\partial\Omega}|\nabla u|^{p+m}H(x)ds.\end{aligned}\tag{3.3}$$

It follows from (H1)–(H3) and Young's inequality that

$$\begin{aligned}&\int_\Omega \mathbf{b}(u)\cdot\nabla u\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\\&\quad\le\int_\Omega|\nabla u|^{p+m-2}|D^2u|^2dx+\int_\Omega p^2|\mathbf{b}(u)|^2|\nabla u|^{p-m}dx\\&\quad\le\int_\Omega|\nabla u|^{p+m-2}|D^2u|^2dx+C_0p^2(1+||\nabla u(t)||_p^p),\end{aligned}\tag{3.4}$$

$$-\int_\Omega f(t)u^\alpha\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx\le C_1(\|\nabla u\|_p^{p-1}+||\nabla u(t)||_p^p),\tag{3.5}$$

$$-\int_\Omega h(x,t)\operatorname{div}\{|\nabla u|^{p-2}\nabla u\}dx=\int_\Omega\nabla h\cdot\nabla u|\nabla u|^{p-2}dx\le CM_1\|\nabla u\|_p^{p-1}.\tag{3.6}$$

We have from (3.2)–(3.6) and (H4) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}||\nabla u(t)||_p^p+\frac{C_1}{p}|||\nabla u|^{(p+m)/2}||_{1,2}^2\\&\quad\le C_1p^2(1+||\nabla u(t)||_p^p)+C_1\|\nabla u\|_p^{p-1}+\frac{C_1}{p}\int_\Omega|\nabla u|^{p+m}dx.\end{aligned}\tag{3.7}$$

For the third term of the right-hand side of (3.7), by Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\|\nabla u\|_{p+m}^{p+m}\le\frac{1}{2}|||\nabla u|^{(p+m)/2}||_{1,2}^2+C\|\nabla u\|_{m+2}^{m+1}\|\nabla u\|_p^{p-1}.\tag{3.8}$$

Therefore, (3.7) can be rewritten as

$$\frac{d}{dt}||\nabla u(t)||_p^p+C_1|||\nabla u|^{(p+m)/2}||_{1,2}^2\le C_1p^3(1+||\nabla u(t)||_p^p)+C_1p\|\nabla u\|_p^{p-1}.\tag{3.9}$$

Then, setting

$$p_1 = m, \quad p_n = 2p_{n-1} - m, \quad \theta_n = 2N\left(1 - p_{n-1}p_n^{-1}\right)(N+2)^{-1}, \quad n = 2,3,\dots, \tag{3.10}$$

we have, by a variant of the Gagliardo-Nirenberg inequality, again

$$\|\nabla u\|_{p_n} \le C^{2/(p_n+m)} \|\nabla u\|_{p_{n-1}}^{1-\theta_n} \left|\left| |\nabla u|^{(p_n+m)/2} \right|\right|_{1,2}^{2\theta_n/(p_n+m)}. \tag{3.11}$$

Therefore, from (3.9) and (3.11) (set $p = p_n$),

$$\begin{aligned} &\frac{d}{dt}\left|\left|\nabla u(t)\right|\right|_{p_n}^{p_n} + C_1 C^{-2/\theta_n} \|\nabla u\|_{p_{n-1}}^{(p_n+m)(\theta_n-1)/\theta_n} \|\nabla u\|_{p_n}^{(p_n+m)/\theta_n} \\ &\quad \le C_1 p_n^3 \left(1 + \left|\left|\nabla u(t)\right|\right|_{p_n}^{p_n}\right) + C_1 p_n \|\nabla u\|_{p_n}^{p_n-1}. \end{aligned} \tag{3.12}$$

Applying Lemma 1.6, by the same argument as in Proposition 2.1, we can obtain (3.1). □

## Acknowledgments

The author wishes to thank the referees for valuable suggestions which improved the presentation of the paper. This work was supported by the Science Research Foundation of NUIST.

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