Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 27626, 14 pages
doi:10.1155/2007/27626

Research Article
On Relative Homotopy Groups of Modules

C. Joanna Su

Received 10 May 2007; Accepted 16 August 2007

Recommended by Mónica Clapp

In his book "Homotopy Theory and Duality," Peter Hilton described the concepts of relative homotopy theory in module theory. We study in this paper the possibility of parallel concepts of fibration and cofibration in module theory, analogous to the existing theorems in algebraic topology. First, we discover that one can study relative homotopy groups, of modules, from a viewpoint which is closer to that of (absolute) homotopy groups. Then, through the study of various cases, we learn that the classic fibration/cofibration relation does not come automatically. Nonetheless, the ability to see the relative homotopy groups as absolute homotopy groups, in a stronger sense, promises to justify our ultimate search.

1. Introduction

In [1], Peter Hilton developed homotopy theory in module theory, parallel to the existing homotopy theory in topology. However, unlike homotopy theory in topology, there are two types of homotopy theory in module theory, the injective theory and the projective theory. They are dual but not isomorphic. In this paper, we emphasize the injective relative homotopy groups (of modules) and approach the proofs in a way that does not refer to elements of sets, so one can proceed with the dual, in projective relative homotopy theory, without further arguments.

During the search for the analogy between the relative homotopy groups in module theory and those in topology, we realize that the (injective) relative homotopy group, $\overline{\pi}_n(A,\beta)$, $n \geq 1$, for a map $\beta : B_1 \to B_2$ has a structure which is fairly similar to an (injective) absolute homotopy group, namely, $\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$, where $\iota : B_1 \hookrightarrow CB_1$ is the inclusion of B_1 into an injective container CB_1 that induces a short exact sequence:

$$B_1 \xhookrightarrow{\{\iota,\beta\}} CB_1 \oplus B_2 \twoheadrightarrow \operatorname{coker}\{\iota,\beta\}. \tag{1.1}$$

Thereafter, we analyze the phenomena related $\overline{\pi}_n(A,\beta)$ and $\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$ through cases. As expected, the two are not always isomorphic; nevertheless, the fact that all relative homotopy groups are isomorphic to certain "*strong (absolute) homotopy groups*" gives rise to the possibility of developing parallel concepts of fibration and cofibration in projective and injective homotopy theories, respectively, in module theory, corresponding to the existing fibration/cofibration relation in algebraic topology.

2. Relative homotopy groups—from a different viewpoint

In the injective relative homotopy theory of modules, for a given Λ-module homomorphism $\beta : B_1 \to B_2$ and a given Λ-module A, one computes the nth relative homotopy group, $\overline{\pi}_n(A,\beta)$, $n \geq 1$, through the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\
\downarrow & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} & & \downarrow \\
\ker\beta & \rightarrowtail & B_1 & \xrightarrow{\beta} & B_2 & \twoheadrightarrow & \operatorname{coker}\beta
\end{array} \tag{2.1}$$

where ι_0 is the inclusion map which embeds A into an injective container CA, and ϵ_1 is the quotient map to ΣA, called the suspension of A, as the quotient. We say that the map $(\rho,\sigma) : \iota_{n-1} \to \beta$ is i-nullhomotopic, denoted $(\rho,\sigma) \simeq_i 0$, if it can be extended to an injective container of ι_{n-1}, and that $\overline{\pi}_n(A,\beta) = \operatorname{Hom}(\iota_{n-1},\beta)/\operatorname{Hom}_0(\iota_{n-1},\beta)$, where $\operatorname{Hom}(\iota_{n-1},\beta)$ is the abelian group of maps of ι_{n-1} to β, and $\operatorname{Hom}_0(\iota_{n-1},\beta)$ the subgroup consisting of i-nullhomotopic maps.

The computation of such diagrams, as (2.1), is rather challenging at times, especially during the search for suitable definitions of fibration and cofibration in module theory, analogous to those in topology. Therefore we examine the diagram, of relative homotopy groups, from another viewpoint: First assuming that the map $\beta : B_1 \to B_2$ is monomorphic so (2.1) is essentially

$$\begin{array}{ccccc}
\Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\
{\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} \\
B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\kappa} & \operatorname{coker}\beta.
\end{array} \tag{2.2}$$

In (2.2), each pair of maps $(\rho,\sigma) : \iota_{n-1} \to \beta$ induces a map $\sigma' : \Sigma^n A \to \operatorname{coker}\beta$. We define $\operatorname{RHom}_\Lambda(\Sigma^n A,\operatorname{coker}\beta)$ to be the subgroup of $\operatorname{Hom}_\Lambda(\Sigma^n A,\operatorname{coker}\beta)$ consisting of such induced maps; it gives the relative homotopy group $\overline{\pi}_n(A,\beta)$ an alternative aspect.

Theorem 2.1. *Suppose given a monomorphism $\beta : B_1 \rightarrowtail B_2$. For each A, consider the diagram*

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xhookrightarrow{\iota_n} & C\Sigma^n A \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} & & \\ B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\kappa} & \operatorname{coker}\beta & & \end{array} \tag{2.3}$$

where $\iota_0 : A \hookrightarrow CA$ is the inclusion of A into an injective container CA, ϵ_1 the quotient map with ΣA, called the suspension of A, as the quotient, and κ the expected quotient map. Then,

$$\overline{\pi}_n(A,\beta) \cong \operatorname{RHom}_\Lambda\left(\Sigma^n A, \operatorname{coker}\beta\right) / \kappa_* \iota_n^* \operatorname{Hom}_\Lambda\left(C\Sigma^n A, B_2\right), \tag{2.4}$$

where

$$\begin{aligned} &\operatorname{RHom}_\Lambda\left(\Sigma^n A, \operatorname{coker}\beta\right) \\ &\quad = \left\{ \begin{array}{l} \sigma' \in \operatorname{Hom}_\Lambda\left(\Sigma^n A, \operatorname{coker}\beta\right) \mid \sigma' \ \textit{ is the induced map of} \\ \textit{a commutative square} \quad \begin{array}{ccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} \\ B_1 & \xrightarrow{\beta} & B_2 \end{array} \end{array} \right\}. \end{aligned} \tag{2.5}$$

To prepare for the proof of Theorem 2.1, we first state a couple of existing propositions.

Proposition 2.2 ([2]). *In $\operatorname{Hom}(\iota_{n-1},\beta)$, when β is monomorphic, $(\rho,\sigma) \simeq_i 0$ if and only if $\sigma = \beta\theta + \chi\iota_n\epsilon_n$ for some $\theta : C\Sigma^{n-1}A \to B_1$ and $\chi : C\Sigma^n A \to B_2$;*

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xhookrightarrow{\iota_n} & C\Sigma^n A \\ \downarrow{\scriptstyle\rho} & \swarrow{\scriptstyle\theta} & \downarrow{\scriptstyle\sigma} & \swarrow{\scriptstyle\chi} & \downarrow{\scriptstyle\sigma'} & & \\ B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\kappa} & \operatorname{coker}\beta & & \end{array} \tag{2.6}$$

Proposition 2.3 [1]. *In the commutative diagram of short exact sequences:*

$$\begin{array}{ccccc} A & \xrightarrow{\mu} & B & \xrightarrow{\epsilon} & C \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\gamma} \\ A' & \xrightarrow{\mu'} & B' & \xrightarrow{\epsilon'} & C' \end{array} \tag{2.7}$$

α factors through μ if and only if γ factors through ϵ'.

Proof of Theorem 2.1. We define

$$\phi : \overline{\pi}_n(A,\beta) \to \operatorname{RHom}_\Lambda\left(\Sigma^n A, \operatorname{coker}\beta\right) / \kappa_* \iota_n^* \operatorname{Hom}_\Lambda\left(C\Sigma^n A, B_2\right) \tag{2.8}$$

by $\phi([(\rho,\sigma)]) = [\sigma']$ and show that ϕ is an isomorphism; first, suppose given a $[(\rho,\sigma)] \in \overline{\pi}_n(A,\beta)$ and assume that $(\rho,\sigma) \simeq_i 0$. By Proposition 2.2, $\sigma = \beta\theta + \chi\iota_n\epsilon_n$ for some $\theta : C\Sigma^{n-1}A \to B_1$ and $\chi : C\Sigma^n A \to B_2$. Thus, $\sigma'\epsilon_n = \kappa\sigma = \kappa(\beta\theta + \chi\iota_n\epsilon_n) = \kappa\beta\theta + \kappa\chi\iota_n\epsilon_n = \kappa\chi\iota_n\epsilon_n$, so $\sigma' = \kappa\chi\iota_n$, due to the fact that ϵ_n is surjective. Hence, $\sigma' \in \kappa_*\iota_n^* \operatorname{Hom}_\Lambda(C\Sigma^n A, B_2)$ and ϕ is well defined.

To prove ϕ monomorphic, suppose given a $[(\rho,\sigma)] \in \overline{\pi}_n(A,\beta)$ and assume that $\phi([(\rho,\sigma)]) = [\sigma'] = 0 \in \operatorname{RHom}_\Lambda(\Sigma^n A, \operatorname{coker}\beta)/\kappa_*\iota_n^* \operatorname{Hom}_\Lambda(C\Sigma^n A, B_2)$. That is, $\sigma' = \kappa\chi\iota_n$ for some $\chi : C\Sigma^n A \to B_2$, which means that σ' factors through the map κ. Then, by an immediate corollary of Proposition 2.3, namely, $\gamma = 0$ if and only if ξ factors through μ', there exists an $\eta : C\Sigma^{n-1}A \to B_1$ such that $\sigma - \chi\iota_n\epsilon_n = \beta\eta$;

$$\begin{array}{ccccc} \Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\ \downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle \eta} & \downarrow{\scriptstyle \sigma-\chi\iota_n\epsilon_n} & & \downarrow{\scriptstyle 0} \\ B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{\kappa} & \operatorname{coker}\beta \end{array} \tag{2.9}$$

Hence, $(\rho,\sigma) \simeq_i 0$ by Proposition 2.2, and thus ϕ is monomorphic.

Finally, the definition of $\operatorname{RHom}_\Lambda(\Sigma^n A, \operatorname{coker}\beta)$ yields that each σ' is induced from a commutative square

$$\begin{array}{ccc} \Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\beta} & B_2 \end{array} \tag{2.10}$$

Thus, ϕ is epimorphic. □

We remark that one can interpret $\operatorname{RHom}_\Lambda(\Sigma^n A, \operatorname{coker}\beta)$ as the "*reversible*" subgroup of $\operatorname{Hom}_\Lambda(\Sigma^n A, \operatorname{coker}\beta)$; suppose given a map $\sigma' \in \operatorname{Hom}_\Lambda(\Sigma^n A, \operatorname{coker}\beta)$, we say that σ' is *reversible* if it can pull back and produce a commutative diagram (2.2). Furthermore, it reveals a connection between the relative homotopy group $\overline{\pi}_n(A,\beta)$ and the (absolute) homotopy group $\overline{\pi}_n(A, \operatorname{coker}\beta)$.

Next, for the general case that $\beta : B_1 \to B_2$ is arbitrary, we exploit the mapping cylinder of β and Theorem 2.5 follows immediately after Proposition 2.4.

Proposition 2.4 [2]. *Suppose given maps $\beta : B_1 \to B_2$ and $\iota : B_1 \hookrightarrow CB_1$, where CB_1 is an injective container of B_1 so that $\{\iota,\beta\} : B_1 \rightarrowtail CB_1 \oplus B_2$ is a monomorphism, then, for arbitrary A, $\overline{\pi}_n(A,\{\iota,\beta\}) \cong \overline{\pi}_n(A,\beta)$ canonically, $n \geq 1$.*

Theorem 2.5. *Suppose given $\beta : B_1 \to B_2$. For each A, consider the diagram*

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xrightarrow{\iota_n} & C\Sigma^n A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma'} & & \\ B_1 & \xrightarrow{\{\iota,\beta\}} & CB_1 \oplus B_2 & \xrightarrow{\kappa} & \operatorname{coker}\{\iota,\beta\} & & \end{array} \tag{2.11}$$

where $\iota_0 : A \hookrightarrow CA$ *is the inclusion of* A *into an injective container* CA, ϵ_1 *is the quotient map with* ΣA, *called the suspension of* A, *as the quotient,* $\iota : B_1 \hookrightarrow CB_1$ *is the inclusion of* B_1 *into an injective container* CB_1, *and* κ *is the expected quotient map. Then,*

$$\overline{\pi}_n(A,\beta) \cong \operatorname{RHom}_\Lambda\left(\Sigma^n A, \operatorname{coker}\{\iota,\beta\}\right)/\kappa_*\iota_n^* \operatorname{Hom}_\Lambda\left(C\Sigma^n A, CB_1 \oplus B_2\right), \tag{2.12}$$

where

$$\begin{aligned} &\operatorname{RHom}_\Lambda\left(\Sigma^n A, \operatorname{coker}\{\iota,\beta\}\right) \\ &= \left\{ \begin{array}{l} \sigma' \in \operatorname{Hom}_\Lambda\left(\Sigma^n A, \operatorname{coker}\{\iota,\beta\}\right) \mid \sigma' \text{ is the induced map} \\ \text{of a commutative square } \begin{array}{ccc} \Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A \\ \rho\downarrow & & \downarrow\sigma \\ B_1 & \xrightarrow{\{\iota,\beta\}} & CB_1 \oplus B_2 \end{array} \end{array} \right\}. \end{aligned} \tag{2.13}$$

As we mentioned earlier, our argument does not involve references to elements of sets, so one can proceed with the dual, in projective relative homotopy theory, automatically. As an illustration, for a given Λ-module homomorphism $\alpha : A_1 \to A_2$ and a given Λ-module B, one alternatively views the projective relative homotopy group $\underline{\pi}_n(\alpha,B)$, $n \geq 1$, as follows.

Theorem 2.6. *Suppose given* $\alpha : A_1 \to A_2$. *For each* B, *consider the diagram*

$$\begin{array}{ccccccc} & & \ker\langle\alpha,\eta\rangle & \xrightarrow{\iota} & A_1 \oplus PA_2 & \xrightarrow{\langle\alpha,\eta\rangle} & A_2 \\ & & \rho|\downarrow & & \rho\downarrow & & \downarrow\sigma \\ P\Omega^n B & \xrightarrow{\eta_n} & \Omega^n B & \xrightarrow{\mu_n} & P\Omega^{n-1}B & \xrightarrow{\eta_{n-1}} & \Omega^{n-1}B \end{array} \tag{2.14}$$

where $\eta_0 : PB \twoheadrightarrow B$ *is the projection of a projective ancestor* PB *onto* B, μ_1 *is the inclusion map with* ΩB, *called the loop space of* B, *as the kernel,* $\eta : PA_2 \twoheadrightarrow A_2$ *is the projection of a projective ancestor* PA_2 *onto* A_2, *and* ι *is the expected inclusion map. Then,*

$$\underline{\pi}_n(\alpha,B) \cong \operatorname{RHom}_\Lambda\left(\ker\langle\alpha,\eta\rangle, \Omega^n B\right)/\iota^*\eta_{n_*} \operatorname{Hom}_\Lambda\left(A_1 \oplus PA_2, P\Omega^n B\right), \tag{2.15}$$

where

$$\begin{aligned} &\operatorname{RHom}_\Lambda\left(\ker\langle\alpha,\eta\rangle, \Omega^n B\right) \\ &= \left\{ \begin{array}{l} \rho| \in \operatorname{Hom}_\Lambda\left(\ker\langle\alpha,\eta\rangle, \Omega^n B\right) \mid \rho| \text{ is the restriction} \\ \text{of a commutative square } \begin{array}{ccc} A_1 \oplus PA_2 & \xrightarrow{\langle\alpha,\eta\rangle} & A_2 \\ \rho\downarrow & & \downarrow\sigma \\ P\Omega^{n-1}B & \xrightarrow{\eta_{n-1}} & \Omega^{n-1}B \end{array} \end{array} \right\}. \end{aligned} \tag{2.16}$$

3. Various cases for $\beta: B_1 \to B_2$

Here, we have Theorem 2.5, which does not only give us an alternative way of computing relative homotopy groups for a map $\beta: B_1 \to B_2$, but also shows a close connection between the (injective) relative homotopy groups $\overline{\pi}_n(A,\beta)$ and the (injective) homotopy groups $\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$. The latter indicates the possibility of developing analogous concepts of fibration and cofibration in module theory to those existing ones in algebraic topology. Before further commenting on this matter, we demonstrate a few calculations through analyzing these phenomena on $\operatorname{RHom}_\Lambda(\Sigma^n A, \operatorname{coker}\{\iota,\beta\})$.

First, we examine the case that the map $\beta: B_1 \to B_2$ is the zero map. The homotopy exact sequence of a map $\beta: B_1 \to B_2$ (see [1, Theorem 13.15]), thus,

$$\cdots \xrightarrow{\partial} \overline{\pi}_n(A,B_1) \xrightarrow{\beta_*} \overline{\pi}_n(A,B_2) \xrightarrow{J} \overline{\pi}_n(A,\beta) \xrightarrow{\partial} \overline{\pi}_{n-1}(A,B_1) \xrightarrow{\beta_*} \cdots$$
$$\xrightarrow{\partial} \overline{\pi}_1(A,B_1) \xrightarrow{\beta_*} \overline{\pi}_1(A,B_2) \xrightarrow{J} \overline{\pi}_1(A,\beta) \xrightarrow{\partial} \overline{\pi}(A,B_1) \xrightarrow{\beta_*} \overline{\pi}(A,B_2), \tag{3.1}$$

yields a short exact sequence

$$\overline{\pi}_n(A,B_2) \rightarrowtail^{J} \overline{\pi}_n(A,\beta) \overset{\partial}{\twoheadrightarrow} \overline{\pi}_{n-1}(A,B_1) \tag{3.2}$$

as $\beta_* = 0$. In addition, the special feature of the zero map suggests that (3.2) actually splits, thus, the relative homotopy group $\overline{\pi}_n(A,\beta)$ is the direct sum of the other two.

Theorem 3.1. *Assume that $\beta: B_1 \to B_2$ is the zero map. Then, for each A,*

$$\overline{\pi}_n(A,\beta) \cong \overline{\pi}_{n-1}(A,B_1) \oplus \overline{\pi}_n(A,B_2), \text{ canonically}, \quad n \geq 1. \tag{3.3}$$

Before proceeding with its proof, we note that the theorem can also be derived using the conventional method, namely, compute $\overline{\pi}_n(A,\beta)$ through the commutative square

$$\begin{array}{ccc} \Sigma^{n-1}A & \overset{\iota_{n-1}}{\hookrightarrow} & C\Sigma^{n-1}A \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\beta=0} & B_2 \end{array} \tag{3.4}$$

Proof. In diagram (2.11), thus,

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \overset{\iota_{n-1}}{\hookrightarrow} & C\Sigma^{n-1}A & \overset{\epsilon_n}{\twoheadrightarrow} & \Sigma^n A & \overset{\iota_n}{\hookrightarrow} & C\Sigma^n A \\ \downarrow \rho & & \downarrow \sigma & & \downarrow \sigma' & & \\ B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & CB_1 \oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & \operatorname{coker}\{\iota,\beta\} & & \end{array} \tag{3.5}$$

we first note that $\operatorname{coker}\{\iota,\beta\} = \operatorname{coker}\{\iota,0\} = \Sigma B_1 \oplus B_2$ and that $\kappa = \{\langle \kappa_1, 0\rangle, \langle 0, 1_{B_2}\rangle\}$, where $\iota: B_1 \hookrightarrow CB_1$ is the inclusion of B_1 into an injective container CB_1, κ_1 is the quotient map to ΣB_1, called the suspension of B_1, and 1_{B_2} is the identity map on B_2. So (2.11) is essentially

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xhookrightarrow{\iota_n} & C\Sigma^n A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle \sigma=\{\sigma_1,\sigma_2\}} & & \downarrow{\scriptstyle\sigma'=\{\sigma'_1,\sigma'_2\}} & & \\ B_1 & \xrightarrow{\{\iota,0\}} & CB_1\oplus B_2 & \xrightarrow{\kappa} & \Sigma B_1\oplus B_2 & & \end{array} \tag{3.6}$$

Moreover, it is the natural combination of the two commutative diagrams

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xhookrightarrow{\iota_n} & C\Sigma^n A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle\sigma'_1} & & \\ B_1 & \xhookrightarrow{\iota} & CB_1 & \xrightarrow{\kappa_1} & \Sigma B_1 & & \end{array} \tag{3.7}$$

$$\begin{array}{ccccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xhookrightarrow{\iota_n} & C\Sigma^n A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle\sigma'_2} & & \\ B_1 & \xrightarrow{\beta=0} & B_2 & \overset{1_{B_2}}{=\!=\!=} & B_2 & & \end{array} \tag{3.8}$$

Thus we define

$$\phi : \mathrm{RHom}_\Lambda\left(\Sigma^n A, \mathrm{coker}\{\iota,\beta\}\right)/\kappa_*\iota_n^* \mathrm{Hom}_\Lambda\left(C\Sigma^n A, CB_1\oplus B_2\right) \longrightarrow \overline{\pi}_{n-1}(A,B_1)\oplus\overline{\pi}_n(A,B_2) \tag{3.9}$$

by $\phi([\{\sigma'_1,\sigma'_2\}]) = ([\rho],[\sigma'_2])$ and show that ϕ is an isomorphism. First, suppose given $[\{\sigma'_1,\sigma'_2\}] \in \mathrm{RHom}_\Lambda(\Sigma^n A,\mathrm{coker}\{\iota,\beta\})/\kappa_*\iota_n^*\mathrm{Hom}_\Lambda(C\Sigma^n A, CB_1\oplus B_2)$ and assume that $\{\sigma'_1,\sigma'_2\}\in\kappa_*\iota_n^*\mathrm{Hom}_\Lambda(C\Sigma^n A,CB_1\oplus B_2)$. Then there exists $\{\chi_1,\chi_2\} : C\Sigma^n A\to CB_1\oplus B_2$ such that $\{\sigma'_1,\sigma'_2\} = \kappa\circ\{\chi_1,\chi_2\}\circ\iota_n$. Equivalently, one has $\sigma'_1 = \kappa_1\circ\chi_1\circ\iota_n$ in (3.7) and $\sigma'_2 = 1_{B_2}\circ\chi_2\circ\iota_n = \chi_2\circ\iota_n$ in (3.8). The former says that the map σ'_1 factors through κ_1; therefore, by Proposition 2.3, $\rho = \theta\iota_{n-1}$ for some $\theta : C\Sigma^{n-1}A\to B_1$. Hence $[\rho] = 0$ in $\overline{\pi}_{n-1}(A,B_1)$. The latter says that $[\sigma'_2] = 0$ in $\overline{\pi}_n(A,B_2)$. So ϕ is well defined.

To show that ϕ is monomorphic, suppose given

$$[\{\sigma'_1,\sigma'_2\}] \in \mathrm{RHom}_\Lambda\left(\Sigma^n A,\mathrm{coker}\{\iota,\beta\}\right)/\kappa_*\iota_n^*\mathrm{Hom}_\Lambda\left(C\Sigma^n A, CB_1\oplus B_2\right) \tag{3.10}$$

and assume that $\phi([\{\sigma'_1,\sigma'_2\}]) = ([\rho],[\sigma'_2]) = (0,0)\in\overline{\pi}_{n-1}(A,B_1)\oplus\overline{\pi}_n(A,B_2)$. That is, $\rho = \gamma\iota_{n-1}$ for some $\gamma : C\Sigma^{n-1}A\to B_1$ and $\sigma'_2 = \eta\iota_n$ for some $\eta : C\Sigma^n A\to B_2$, respectively. The former says that the map ρ factors through ι_{n-1} in (3.7); therefore, by Proposition 2.3, $\sigma'_1 = \kappa_1\tau$ for some $\tau : \Sigma^n A\to CB_1$. Moreover, $\tau = \nu\iota_n$ for some $\nu : C\Sigma^n A\to CB_1$, due to the facts that CB_1 is injective and that ι_n is monomorphic. Therefore, $\sigma'_1 = \kappa_1\nu\iota_n$ and hence $\{\sigma'_1,\sigma'_2\} = \{\kappa_1\circ\nu\circ\iota_n,\eta\circ\iota_n\} = \{\kappa_1\circ\nu\circ\iota_n, 1_{B_2}\circ\eta\circ\iota_n\} = \kappa\circ\{\nu,\eta\}\circ\iota_n \in \kappa_*\iota_n^*\mathrm{Hom}_\Lambda(C\Sigma^n A, CB_1\oplus B_2)$.

Finally, suppose given $([\rho],[\sigma'_2])\in\overline{\pi}_{n-1}(A,B_1)\oplus\overline{\pi}_n(A,B_2)$. We use the map ρ to complete a diagram (3.7)—since CB_1 is injective and ι_{n-1} is monomorphic, there exists a map

$\sigma_1 : C\Sigma^{n-1}A \to CB_1$ such that $\iota\rho = \sigma_1\iota_{n-1}$ and σ_1' is then the induced map:

$$\begin{array}{ccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\ \downarrow{\scriptstyle \rho} & \circlearrowleft & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_1'} \\ B_1 & \xhookrightarrow{\iota} & CB_1 & \xrightarrow{\kappa_1} & \Sigma B_1 \end{array} \tag{3.11}$$

Similarly, the map σ_2' completes diagram (3.8), precisely,

$$\begin{array}{ccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma_2'\epsilon_n} & & \downarrow{\scriptstyle \sigma_2'} \\ B_1 & \xrightarrow{0} & B_2 & \overset{1_{B_2}}{=\!=\!=} & B_2 \end{array} \tag{3.12}$$

Now ϕ is epimorphic because of the existence of the commutative diagram

$$\begin{array}{ccccc} \Sigma^{n-1}A & \xhookrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \{\sigma_1,\sigma_2'\epsilon_n\}} & & \downarrow{\scriptstyle \{\sigma_1',\sigma_2'\}} \\ B_1 & \xrightarrow{\{\iota,0\}} & CB_1 \oplus B_2 & \xrightarrow{\kappa} & \Sigma B_1 \oplus B_2 \end{array} \tag{3.13}$$

□

Theorem 3.1 also implies a couple of immediate consequences.

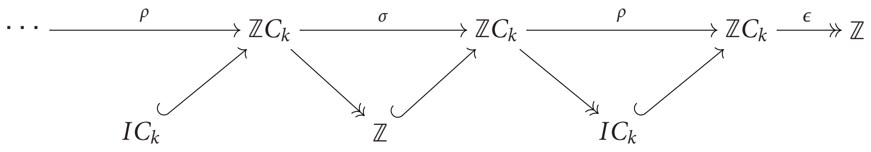
Corollary 3.2. *If $B_1 = 0$, then, for each A, $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_n(A,B_2)$.*

Corollary 3.3. *If $B_2 = 0$, then, for each A, $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_{n-1}(A,B_1)$.*

The dual of Theorem 3.1 and its corollaries say that if we assume that $\alpha : A_1 \to A_2$ is the zero map, then for each B, $\underline{\pi}_n(\alpha,B) \cong \underline{\pi}_{n-1}(A_2,B) \oplus \underline{\pi}_n(A_1,B)$ for $n \geq 1$. Specifically, if $A_2 = 0$, then $\underline{\pi}_n(\alpha,B) \cong \underline{\pi}_n(A_1,B)$, and if $A_1 = 0$, then $\underline{\pi}_n(\alpha,B) \cong \underline{\pi}_{n-1}(A_2,B)$. Notice that as $A_2 = 0$, one sees from diagram (2.14) in Theorem 2.6 that $\underline{\pi}_n(\alpha,B) \cong \underline{\pi}_n(\ker\langle\alpha,\eta\rangle,B)$; however, the isomorphism fails when $A_1 = 0$. As an example, consider the Λ-map $\alpha : 0 \to \mathbb{Z}$, where Λ is the integral group ring of the finite cyclic group C_k with generator τ and $\mathbb{Z}$ is regarded as a trivial C_k-module. Then,

$$\underline{\pi}_n(\alpha,\mathbb{Z}) \cong \overline{\pi}_{n-1}(\mathbb{Z},\mathbb{Z}) = \begin{cases} \mathbb{Z}/k, & \text{for } n \text{ odd}, \\ 0, & \text{for } n \text{ even}. \end{cases} \tag{3.14}$$

(See [3, Theorem 3.1].) On the other hand, the well-known projective resolution of $\mathbb{Z}$, thus,



(3.15)

where the maps ϵ, ρ, σ are the augmentation of $\mathbb{Z}C_k$, multiplication by $\tau - 1$, and multiplication by $\tau^{k-1} + \cdots + \tau + 1$, respectively, gives us that

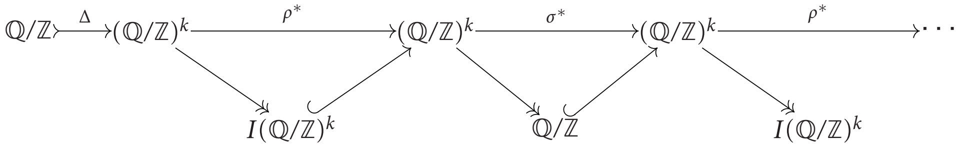
$$\underline{\pi}_n(\ker\langle\alpha,\eta\rangle,\mathbb{Z}) \cong \underline{\pi}_n(\Omega\mathbb{Z},\mathbb{Z}) \cong \underline{\pi}_n(IC_k,\mathbb{Z}) \cong \begin{cases} \underline{\pi}(IC_k,\mathbb{Z}), & \text{for } n \text{ even} \\ \underline{\pi}(IC_k,IC_k), & \text{for } n \text{ odd} \end{cases} = 0, \tag{3.16}$$

because all the maps in $\text{Hom}_\Lambda(IC_k,\mathbb{Z})$ and $\text{Hom}_\Lambda(IC_k,IC_k)$ are p-nullhomotopic.

Similarly, as $B_1 = 0$, $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$; however, this position alters when $B_2 = 0$; consider the Λ-map $\beta : \mathbb{Q}/\mathbb{Z} \to 0$, where, again, Λ is the integral group ring of the finite cyclic group C_k and $\mathbb{Q}/\mathbb{Z}$ is regarded as a trivial C_k-module. Then,

$$\overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\beta) \cong \overline{\pi}_{n-1}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k, & \text{for } n \text{ odd}, \\ 0, & \text{for } n \text{ even}. \end{cases} \tag{3.17}$$

(See [3, Theorem 2.6].) For $\overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$, we adopt the injective resolution of $\mathbb{Q}/\mathbb{Z}$:



(3.18)

where $\Delta = \epsilon^*$ is the diagonal map, and obtain that

$$\begin{aligned} \overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\text{coker}\{\iota,\beta\}) &\cong \overline{\pi}_n(\mathbb{Q}/\mathbb{Z},\Sigma\mathbb{Q}/\mathbb{Z}) \cong \overline{\pi}_n(\mathbb{Q}/\mathbb{Z},I(\mathbb{Q}/\mathbb{Z})^k) \\ &\cong \begin{cases} \overline{\pi}(\mathbb{Q}/\mathbb{Z},I(\mathbb{Q}/\mathbb{Z})^k), & \text{for } n \text{ even} \\ \overline{\pi}(I(\mathbb{Q}/\mathbb{Z})^k,I(\mathbb{Q}/\mathbb{Z})^k), & \text{for } n \text{ odd} \end{cases} = 0, \end{aligned} \tag{3.19}$$

again because all the maps in $\text{Hom}_\Lambda(\mathbb{Q}/\mathbb{Z},I(\mathbb{Q}/\mathbb{Z})^k)$ and $\text{Hom}_\Lambda(I(\mathbb{Q}/\mathbb{Z})^k,I(\mathbb{Q}/\mathbb{Z})^k)$ are i-nullhomotopic.

Therefore, as one may expect, the classic fibration/cofibration does not hold for arbitrary maps in module theory. The same phenomena arise again even when we generalize B_1 and B_2, respectively, to injective modules.

Theorem 3.4. *Let $\beta : B_1 \to B_2$ be arbitrary. Then, for each A,*

(i) *if B_1 is injective, then $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_n(A,B_2) \cong \overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$;*

(ii) *if B_2 is injective, then $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_{n-1}(A,B_1)$.*

Proof. The first halves of both parts of the theorem, namely, $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_n(A,B_2)$ when B_1 is injective and $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_{n-1}(A,B_1)$ when B_2 is injective, come directly from the (injective) homotopy exact sequence of the map $\beta : B_1 \to B_2$, thus,

$$\cdots \xrightarrow{\partial} \overline{\pi}_n(A,B_1) \xrightarrow{\beta_*} \overline{\pi}_n(A,B_2) \xrightarrow{J} \overline{\pi}_n(A,\beta) \xrightarrow{\partial} \overline{\pi}_{n-1}(A,B_1) \xrightarrow{\beta_*} \overline{\pi}_{n-1}(A,B_2) \xrightarrow{J} \cdots \tag{3.20}$$

To prove that $\overline{\pi}_n(A,\beta) \cong \overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$ when B_1 is injective, one considers diagram (2.11), but now $CB_1 = B_1$. Thus,

$$\begin{array}{ccccccc}
\Sigma^{n-1}A & \overset{\iota_{n-1}}{\hookrightarrow} & C\Sigma^{n-1}A & \overset{\epsilon_n}{\twoheadrightarrow} & \Sigma^n A & \overset{\iota_n}{\hookrightarrow} & C\Sigma^n A \\
\downarrow \rho & & \downarrow \sigma & & \downarrow \sigma' & & \\
B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & B_1 \oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & \text{coker}\{\iota,\beta\} & &
\end{array} \tag{3.21}$$

Since B_1 is injective, the short exact sequence $B_1 \overset{\{\iota,\beta\}}{\rightarrowtail} B_1 \oplus B_2 \overset{\kappa}{\twoheadrightarrow} \text{coker}\{\iota,\beta\}$ splits. Thus there exists a map $\nu : \text{coker}\{\iota,\beta\} \to B_1 \oplus B_2$ such that $\kappa \circ \nu = 1_{\text{coker}\{\iota,\beta\}}$. Applying Theorem 2.5, we define $\chi : \overline{\pi}_n(A,\beta) \to \overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$ by $\chi(\overline{\sigma}') = [\sigma']$ and show that χ is an isomorphism.

First, if $\overline{\sigma}' = 0$ in $\overline{\pi}_n(A,\beta)$, then there is $\theta : C\Sigma^n A \to B_1 \oplus B_2$ such that $\sigma' = \kappa \circ \theta \circ \iota_n$, which also means that $\chi(\overline{\sigma}') = [\sigma'] = 0$ in $\overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$. So χ is well defined. To show that χ is monomorphic, suppose given $\overline{\sigma}' \in \overline{\pi}_n(A,\beta)$ such that $\chi(\overline{\sigma}') = [\sigma'] = 0$, then $\sigma' = \omega \circ \iota_n$ for some $\omega : C\Sigma^n A \to \text{coker}\{\iota,\beta\}$. Thus, $\sigma' = \omega \circ \iota_n = 1_{\text{coker}\{\iota,\beta\}} \circ \omega \circ \iota_n = \kappa \circ \nu \circ \omega \circ \iota_n$, which forces $\overline{\sigma}' = 0$. Thus, χ is monomorphic. Finally, the fact that every $\sigma' : \Sigma^n A \to \text{coker}\{\iota,\beta\}$ yields a commutative diagram

$$\begin{array}{ccccc}
\Sigma^{n-1}A & \overset{\iota_{n-1}}{\hookrightarrow} & C\Sigma^{n-1}A & \overset{\epsilon_n}{\twoheadrightarrow} & \Sigma^n A \\
\downarrow 0 & & \downarrow \nu\sigma'\epsilon_n & & \downarrow \sigma' \\
B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & B_1 \oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & \text{coker}\{\iota,\beta\}
\end{array} \tag{3.22}$$

allows us to conclude that χ is epimorphic. □

Examining the connection between $\overline{\pi}_n(A,\beta)$ and $\overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$, even for the rather simple case that B_2 is injective, we find that, for a map $\sigma' : \Sigma^n A \to \text{coker}\{\iota,\beta\}$ to be related to an element in $\overline{\pi}_n(A,\beta)$, σ' ought to be "*reversible*" in a diagram such as (2.11), that is, σ' must guarantee the existence of a pair (ρ,σ), or equivalently, σ' is the induced map of (ρ,σ). Conversely, for a pair $(\rho,\sigma) : \iota_{n-1} \to \{\iota,\beta\}$ to be related to an element in $\overline{\pi}_n(A,\text{coker}\{\iota,\beta\})$, the *reversible* σ' ought to, simultaneously, factor through not only ι_n but also κ as (ρ,σ) is i-nullhomotopic. These lead precisely to our group

$\mathrm{RHom}_\Lambda(\Sigma^n A, \mathrm{coker}\{\iota,\beta\})/\kappa_* \iota_n^* \mathrm{Hom}_\Lambda(C\Sigma^n A, CB_1 \oplus B_2)$ in Theorem 2.5. One can see this exact targeting through the following exemplification.

Theorem 3.5. *Assume that $\beta : B_1 \twoheadrightarrow B_2$ is epimorphic. If the inclusion map $B_1/\ker\beta \hookrightarrow CB_1/\ker\beta$, where CB_1 is an injective container of B_1, induces a splitting short exact sequence, namely, $B_1/\ker\beta \hookrightarrow CB_1/\ker\beta \twoheadrightarrow CB_1/B_1$, then, for each A,*

$$\overline{\pi}_n(A,\beta) \cong \overline{\pi}_{n-1}(A,B_1) \oplus \overline{\pi}_n(A,B_2), \quad n \geq 1. \tag{3.23}$$

Proof. First, one can show that, when $\beta : B_1 \twoheadrightarrow B_2$ is epimorphic, $\mathrm{coker}\{\iota,\beta\}$ is isomorphic to $CB_1/\ker\beta$. In addition, since the short exact sequence $B_1/\ker\beta \hookrightarrow CB_1/\ker\beta \twoheadrightarrow CB_1/B_1$ splits, $CB_1/\ker\beta \cong CB_1/B_1 \oplus B_1/\ker\beta \cong CB_1/B_1 \oplus B_2 = \Sigma B_1 \oplus B_2$. Hence, diagram (2.11) becomes

$$\begin{array}{ccccccc}
\Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xrightarrow{\iota_n} & C\Sigma^n A \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma=\{\sigma_1,\sigma_2\}} & & \downarrow{\scriptstyle \sigma'=\{\sigma_1',\sigma_2'\}} & & \\
B_1 & \xrightarrow{\{\iota,\beta\}} & CB_1 \oplus B_2 & \xrightarrow{\kappa} & \Sigma B_1 \oplus B_2 & &
\end{array} \tag{3.24}$$

where $\kappa = \{\langle \kappa_1, 0\rangle, \langle 0, -1_{B_2}\rangle\}$, $\iota : B_1 \hookrightarrow CB_1$ is the inclusion of B_1 into an injective container CB_1, κ_1 is the quotient map to ΣB_1, called the suspension of B_1, and 1_{B_2} is the identity map on B_2. In addition, as diagram (3.6) in Theorem 3.1, (3.24) is the natural combination of the two commutative diagrams

$$\begin{array}{ccccccc}
\Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xrightarrow{\iota_n} & C\Sigma^n A \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_1'} & & \\
B_1 & \xrightarrow{\iota} & CB_1 & \xrightarrow{\kappa_1} & \Sigma B_1 & &
\end{array} \tag{3.25}$$

$$\begin{array}{ccccccc}
\Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A & \xrightarrow{\iota_n} & C\Sigma^n A \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_2'} & & \\
B_1 & \xrightarrow{\beta} & B_2 & \xrightarrow{-1_{B_2}} & B_2 & &
\end{array} \tag{3.26}$$

Hereafter, the proof that $\phi : \overline{\pi}_n(A,\beta) \to \overline{\pi}_{n-1}(A,B_1) \oplus \overline{\pi}_n(A,B_2)$ defined by $\phi([\{\sigma_1',\sigma_2'\}]) = ([\rho],[\sigma_2'])$ is an isomorphism is mostly like that given for Theorem 3.1, only one notices that the argument for ϕ being epimorphic is quite subtle; suppose given $([\rho],[\sigma_2']) \in \overline{\pi}_{n-1}(A,B_1) \oplus \overline{\pi}_n(A,B_2)$. First, we use the map ρ to assure the existence of a diagram (3.25)—since CB_1 is injective and ι_{n-1} is monomorphic, there is a map $\sigma_1 : C\Sigma^{n-1}A \to CB_1$ such that $\iota\rho = \sigma_1\iota_{n-1}$. Thus, we have the induced map σ_1' in the diagram

$$\begin{array}{ccccc}
\Sigma^{n-1}A & \xrightarrow{\iota_{n-1}} & C\Sigma^{n-1}A & \xrightarrow{\epsilon_n} & \Sigma^n A \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_1'} \\
B_1 & \xrightarrow{\iota} & CB_1 & \xrightarrow{\kappa_1} & \Sigma B_1
\end{array} \tag{3.27}$$

Furthermore, the fact that the map $\beta : B_1 \to B_2$ factors through CB_1 as

$$B_1 \xrightarrow{\iota} CB_1 \longrightarrow CB_1/\ker\beta \longrightarrow B_1/\ker\beta \xrightarrow{\cong} B_2, \qquad \beta = \omega\circ\iota \tag{3.28}$$

leads to the existence of a commutative square

$$\begin{array}{ccc} \Sigma^{n-1}A & \overset{\iota_{n-1}}{\hookrightarrow} & \Sigma^{n-1}A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\sigma_2=\omega\sigma_1} \\ B_1 & \overset{\beta}{\twoheadrightarrow} & B_2 \end{array} \tag{3.29}$$

The combination of the two yields a commutative diagram, thus,

$$\begin{array}{ccccc} \Sigma^{n-1}A & \overset{\iota_{n-1}}{\hookrightarrow} & C\Sigma^{n-1}A & \overset{\epsilon_n}{\twoheadrightarrow} & \Sigma^{n}A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\{\sigma_1,\sigma_2\}} & & \downarrow{\scriptstyle\{\sigma_1',\theta\}} \\ B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & CB_1\oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & \Sigma B_1\oplus B_2 \end{array} \tag{3.30}$$

with $\{\sigma_1',\theta\}$ being the induced map, where $\theta : \Sigma^n A \to B_2$, and $\sigma_2 = -\theta\circ\epsilon_n$. Finally ϕ is epimorphic, due to the existence of the diagram

$$\begin{array}{ccccc} \Sigma^{n-1}A & \overset{\iota_{n-1}}{\hookrightarrow} & C\Sigma^{n-1}A & \overset{\epsilon_n}{\twoheadrightarrow} & \Sigma^{n}A \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\{\sigma_1,\sigma_2+(\theta-\sigma_2')\epsilon_n\}} & & \downarrow{\scriptstyle\{\sigma_1',\sigma_2'\}} \\ B_1 & \overset{\{\iota,\beta\}}{\rightarrowtail} & CB_1\oplus B_2 & \overset{\kappa}{\twoheadrightarrow} & \Sigma B_1\oplus B_2 \end{array} \tag{3.31}$$

which is commutative because $\{\sigma_1,\sigma_2+(\theta-\sigma_2')\circ\epsilon_n\}\circ\iota_{n-1} = \{\sigma_1\circ\iota_{n-1},\sigma_2\circ\iota_{n-1}\} = \{\sigma_1,\sigma_2\}\circ\iota_{n-1} = \{\iota,\beta\}\circ\rho$ and $\kappa\circ\{\sigma_1,\sigma_2+(\theta-\sigma_2')\circ\epsilon_n\} = \{\kappa_1\circ\sigma_1,-1_{B_2}\circ(\sigma_2+(\theta-\sigma_2')\circ\epsilon_n)\} = \{\kappa_1\circ\sigma_1,-1_{B_2}\circ(\sigma_2+\theta\circ\epsilon_n-\sigma_2'\circ\epsilon_n)\} = \{\sigma_1'\circ\epsilon_n,\sigma_2'\circ\epsilon_n)\} = \{\sigma_1',\sigma_2'\}\circ\epsilon_n$. □

It appears that the (injective) relative homotopy groups $\overline{\pi}_n(A,\beta)$ always have a close connection with the (injective) homotopy groups $\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$. Precisely speaking, though $\overline{\pi}_n(A,\beta)$ may not always be isomorphic to $\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$, it is indeed isomorphic to $\operatorname{RHom}_\Lambda(\Sigma^n A,\operatorname{coker}\{\iota,\beta\})/\kappa_*\iota_n^*\operatorname{Hom}_\Lambda(C\Sigma^n A, CB_1\oplus B_2)$, a group that proceeds from $\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$ and has a structure similar to $\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$ and which we tentatively call a *strong (injective) homotopy group*, $S\overline{\pi}_n(A,\operatorname{coker}\{\iota,\beta\})$. Should a suitable, general, definition become available, the concepts of *cofibration* in the injective homotopy theory of modules and, by duality, *fibration* in the projective homotopy theory of modules will both be within reach.

Acknowledgment

The author would like to express his deep appreciation to Professor Peter Hilton for proofreading the paper and for making valuable suggestions.

References

[1] P. Hilton, *Homotopy Theory and Duality*, chapter 13, Gordon and Breach Science, New York, NY, USA, 1965.
[2] C. J. Su, "The category of long exact sequences and the homotopy exact sequence of modules," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 22, pp. 1383–1395, 2003.
[3] C. J. Su, "Some examples of nontrivial homotopy groups of modules," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 3, pp. 189–195, 2001.

C. Joanna Su: Department of Mathematics and Computer Science, Providence College, Providence, Rhode Island 02918, USA
Email address: jsu@providence.edu