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Research Article

Nonderogatory Unicyclic Digraphs

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A digraph is nonderogatory if its characteristic polynomial and minimal polynomial are equal. We find a characterization of nonderogatory unicyclic digraphs in terms of Hamiltonicity conditions. An immediate consequence of this characterization ia that the complete product of difans and diwheels is nonderogatory.

1. Introduction

A digraph is nonderogatory if its characteristic polynomial and minimal polynomial are equal. Relations between nonderogatory digraphs and their automorphism groups have appeared in the literature (see [1] and [2, Theorem 5.10]). More recently, several articles (see, e.g., [1, 3–8]) address the problem of determining when certain operations between nonderogatory digraphs are again nonderogatory.

The aim of this paper is to characterize unicyclic nonderogatory digraphs (i.e., nonderogatory digraphs which have exactly one cycle), in terms of Hamiltonicity conditions. The results obtained by Gan [4] concerning the complete product of difans and diwheels are an immediate consequence of this characterization.

A digraph (directed graph) Γ consists of a finite set of vertices together with a set of arcs, which are ordered pairs of vertices. The vertices and arcs are denoted V_Γ and E_Γ, respectively. If $(u,v) \in E_\Gamma$, then u and v are adjacent and (u,v) is an arc starting at vertex u and terminating at vertex v. A walk in Γ from vertex u to vertex v is a finite vertex sequence

$$u = u_0, u_1, u_2, \dots, u_k = v \tag{1.1}$$

(sometimes denoted by $u_0 \to u_1 \to \cdots \to u_k$), where (u_{t-1}, u_t) is an arc of D for all $1 \le t \le k$. The number k is the length of the walk. If $u = v$, we say that the walk is a closed walk. A path is a walk in which no vertex is repeated, and a closed path is a cycle.

Suppose that $\{u_1,\dots,u_n\}$ is the set of vertices of Γ. The adjacency matrix of Γ, denoted A_Γ, is the square matrix of order n whose entry ij is the number of arcs starting at u_i and terminating at u_j. The characteristic polynomial of the digraph Γ, denoted by $\Phi_\Gamma(x)$ (or simply Φ_Γ), is the characteristic polynomial of the adjacency matrix A_Γ, that is, $\Phi_\Gamma(x) = |xI - A_\Gamma|$, where I is the identity matrix and $|M|$ is the determinant of M.

By the Cayley-Hamilton theorem, Φ_Γ is an annihilating polynomial of A_Γ, which means that $\Phi_\Gamma(A_\Gamma) = 0$. The monic polynomial of least degree which annihilates A_Γ is called the minimal polynomial of Γ and will be denoted by $\mu_\Gamma(x)$ (or simply μ_Γ). Recall that if

$$\Phi_\Gamma(x) = (x-\lambda_1)^{q_1}(x-\lambda_2)^{q_2}\cdots(x-\lambda_r)^{q_r}, \tag{1.2}$$

where $q_1,\dots,q_r$ are positive integers, then

$$\mu_\Gamma(x) = (x-\lambda_1)^{p_1}(x-\lambda_2)^{p_2}\cdots(x-\lambda_r)^{p_r}, \tag{1.3}$$

where $1 \le p_i \le q_i$ for all $i = 1,\dots,r$.

Definition 1.1. Say that Γ is a nonderogatory digraph if $\Phi_\Gamma(x) = \mu_\Gamma(x)$.

Recall that a linear directed graph L is a digraph in which every vertex has indegree and outdegree equal to 1; in other words, its components are cycles. The coefficient theorem for digraphs relates the coefficients of the characteristic polynomial of a digraph Γ with the set of linear directed subgraphs of Γ.

Theorem 1.2 (see [2, Theorem 1.2]). *Let Γ be a digraph with n vertices. Then the characteristic polynomial Φ_Γ of Γ is*

$$\Phi_\Gamma = x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_n, \tag{1.4}$$

where $c_i = \sum_{L\in\mathcal{L}_i}(-1)^{p(L)}$. Here, $\mathcal{L}_i$ is the set of linear directed subgraphs L of Γ with exactly i vertices, and $p(L)$ is the number of components of L.

A spanning path in a digraph Γ is called a Hamiltonian path. A path-Hamiltonian digraph is a digraph which contains a Hamiltonian path. Then we have a characterization of acyclic nonderogatory digraphs.

Proposition 1.3. *Let Γ be a digraph with n vertices. The following are equivalent:*

(i) Γ *is an acyclic nonderogatory digraph;*

(ii) Γ *is path-Hamiltonian.*

Proof. Recall that the entry uv of the power matrix A^{n-1} is precisely the number of walks in Γ of length $n-1$ from u to v (see [2, Theorem 1.9]). By Theorem 1.2, if Γ is an acyclic digraph with n vertices, then its characteristic polynomial is simply $\Phi_\Gamma = x^n$. Hence, Γ is nonderogatory if and only if $A^{n-1} \neq 0$. If Γ is acyclic, any walk is necessarily a path, and a path of length $n-1$ is necessarily spanning and hence Hamiltonian. From this, it follows that Γ is a nonderogatory acyclic digraph if and only if there exists a walk of length $n-1$ if and only if Γ is path-Hamiltonian. □

2. Nonderogatory unicyclic digraphs

Let $\Gamma_p(u,v)$ denote the set of walks in Γ of length p from vertex u to vertex v. By [2, Theorem 1.9], the entry uv of A^p is the number of walks from u to v of length p.

Proposition 2.1. *Let Γ be a digraph with n vertices and unique cycle C of length $r \geq 2$. The following conditions are equivalent:*

(1) *Γ is a nonderogatory digraph;*

(2) *there exists $u,v \in V_\Gamma$ such that $|\Gamma_{n-1}(u,v)| \neq |\Gamma_{n-r-1}(u,v)|$.*

Proof. From Theorem 1.2, the characteristic polynomial of Γ is

$$\Phi_\Gamma = x^n - x^{n-r} = x^{n-r}(x^r - 1). \tag{2.1}$$

Since $x^r - 1$ is a product of distinct linear factors, the minimal polynomial μ_Γ has the form

$$\mu_\Gamma = x^p(x^r - 1), \tag{2.2}$$

where $1 \leq p \leq n-r$. Hence, Γ is a nonderogatory digraph if and only if

$$A^{n-r-1}(A^r - I) \neq 0 \tag{2.3}$$

or equivalently,

$$A^{n-1} \neq A^{n-r-1}. \tag{2.4}$$

□

In the next results, we give further insight into condition 2 of Proposition 2.1.

Assume that $\{x_1,\dots,x_r\}$ are the vertices of C, and (x_i,x_{i+1}), for $i = 1,\dots,r-1$, together with (x_r,x_1) are the arcs of C. For each $1 \leq j \leq r$, let $C(x_j)$ denote the closed walk $x_j,\dots,x_r,x_1,\dots,x_j$. For $u,v \in V_\Gamma$, define

$$\begin{aligned}\Gamma_p^\circ(u,v) &= \{\pi \in \Gamma_p(u,v) : \pi \text{ contains } C(x_j) \text{ for some } x_j\ (1 \leq j \leq r)\},\\ \Gamma_p^*(u,v) &= \{\pi \in \Gamma_p(u,v) : \pi \text{ contains } x_j \text{ for some } x_j\ (1 \leq j \leq r)\}.\end{aligned} \tag{2.5}$$

Lemma 2.2. *Let Γ be a digraph with n vertices and unique cycle C of length $r \geq 2$. Then $|\Gamma_{n-1}^\circ(u,v)| = |\Gamma_{n-r-1}^*(u,v)|$ for every $u,v \in V_\Gamma$.*

Proof. Note that, $\pi = u\cdots x_j \cdots v \in \Gamma_{n-r-1}^*(u,v)$ if and only if $\overline{\pi} = u\cdots C(x_j)\cdots v \in \Gamma_{n-1}^\circ(u,v)$. Consequently, the function $\Psi : \Gamma_{n-r-1}^*(u,v) \to \Gamma_{n-1}^\circ(u,v)$ defined as $\Psi(\pi) = \overline{\pi}$ is bijective and the result follows. □

Note that, in particular, $\Gamma_{n-r-1}^*(u,v) = \varnothing$ if and only if $\Gamma_{n-1}^\circ(u,v) = \varnothing$ and in this case,

$$\left|\Gamma_{n-1}^\circ(u,v)\right| = \left|\Gamma_{n-r-1}^*(u,v)\right| = 0. \tag{2.6}$$

Let $\Gamma - C$ be the digraph obtained from Γ by deleting the vertices of C and the arcs incident to them.

Lemma 2.3. *Let Γ be a digraph with n vertices and unique cycle C of length $r \geq 2$. Then*

(1) $\Gamma_{n-1}(u,v)\backslash\Gamma^{\circ}_{n-1}(u,v)$ *is the (possibly empty) set of Hamiltonian paths of Γ from u to v;*

(2) $\Gamma_{n-r-1}(u,v)\backslash\Gamma^{*}_{n-r-1}(u,v)$ *is the (possibly empty) set of Hamiltonian paths of $\Gamma - C$ from u to v.*

Proof. (1) Clearly, a Hamiltonian path of Γ from u to v is a path of length $n-1$ which does not contain C. Conversely, assume that $\pi \in \Gamma_{n-1}(u,v)\backslash\Gamma^{\circ}_{n-1}(u,v)$. Then π is a walk of length $n-1$ from u to v that does not contain the cycle C. Since C is the unique cycle of Γ, then clearly π is a spanning path of Γ from u to v.

(2) First, note that $\Gamma - C$ is a digraph with $n-r$ vertices. It is clear that every Hamiltonian path in $\Gamma - C$ belongs to $\Gamma_{n-r-1}(u,v)\backslash\Gamma^{*}_{n-r-1}(u,v)$. Conversely, if $\sigma \in \Gamma_{n-r-1}(u,v)\backslash \Gamma^{*}_{n-r-1}(u,v)$, then σ is a walk in $\Gamma - C$ of length $n-r-1$. Since $\Gamma - C$ is acyclic, it follows that σ is a Hamiltonian path in $\Gamma - C$. $\square$

For every $u,v \in V_\Gamma$, we define

$$\begin{aligned} h^{\circ}(u,v) &= \left|\Gamma_{n-1}(u,v)\backslash\Gamma^{\circ}_{n-1}(u,v)\right|, \\ h^{*}(u,v) &= \left|\Gamma_{n-r-1}(u,v)\backslash\Gamma^{*}_{n-r-1}(u,v)\right|. \end{aligned} \tag{2.7}$$

By Lemma 2.3, $h^{\circ}(u,v)$ (resp., $h^{*}(u,v)$) is the number of Hamiltonian paths in Γ (resp., $\Gamma - C$) from vertex u to vertex v.

If $\Gamma - C$ is path-Hamiltonian, then its structure is very simple, as we can see in the next result.

Proposition 2.4. *Let Ω be an acyclic path-Hamiltonian digraph with n vertices. Then Ω is obtained from $P_n = u_1 \to u_2 \to \cdots \to u_n$ by adding some edges of the form $u_j \to u_k$, where $k > j$.*

Proof. Let $\pi : u_1 \to u_2 \to \cdots \to u_n$ be a spanning path of Ω and assume that $u_j \to u_k$ belongs to Ω, where $k < j$. Then

$$u_k \longrightarrow \cdots \longrightarrow u_j \longrightarrow u_k \tag{2.8}$$

is a cycle in Ω, which is a contradiction. $\square$

In particular, an acyclic path-Hamiltonian digraph Ω has a unique Hamiltonian path: it starts in the unique vertex of indegree 0 and it ends in the unique vertex of outdegree 0 of Ω.

Now, we can characterize nonderogatory unicyclic digraphs in terms of Hamiltonicity conditions. We distinguish the following two cases:

(a) $\Gamma - C$ is path-Hamiltonian, and so $\Gamma - C$ has the form given in Proposition 2.4;

(b) $\Gamma - C$ is not path-Hamiltonian.

Theorem 2.5. *Let Γ be a digraph with n vertices and unique cycle C of length $r \geq 2$.*

(1) *If $\Gamma - C$ is not path-Hamiltonian, then*

$$\Gamma \text{ is nonderogatory} \Longleftrightarrow \Gamma \text{ is path-Hamiltonian.} \tag{2.9}$$

(2) *If $\Gamma - C$ is path-Hamiltonian with unique Hamiltonian path from u to v, then*

$$\Gamma \text{ is nonderogatory} \Longleftrightarrow h^{\circ}(u,v) \neq 1. \tag{2.10}$$

Proof. By Proposition 2.1, Γ is nonderogatory if and only if there exists $u, v \in V_\Gamma$ such that $|\Gamma_{n-1}(u,v)| \neq |\Gamma_{n-r-1}(u,v)|$. Note that $\Gamma_{n-1}(u,v)$ and $\Gamma_{n-r-1}(u,v)$ can be expressed as disjoint unions

$$\begin{aligned} \Gamma_{n-1}(u,v) &= \Gamma^{\circ}_{n-1}(u,v) \cup \left[\Gamma_{n-1}(u,v) \backslash \Gamma^{\circ}_{n-1}(u,v)\right], \\ \Gamma_{n-r-1}(u,v) &= \Gamma^{*}_{n-r-1}(u,v) \cup \left[\Gamma_{n-r-1}(u,v) \backslash \Gamma^{*}_{n-1}(u,v)\right]. \end{aligned} \tag{2.11}$$

It follows from Lemma 2.2 that

$$\Gamma \text{ is nonderogatory} \Longleftrightarrow h^{\circ}(u,v) \neq h^{*}(u,v) \quad \text{for some } u, v \in V_\Gamma. \tag{2.12}$$

(1) Assume that $\Gamma - C$ is not path-Hamiltonian. Then $h^{*}(u,v) = 0$ for every $u, v \in V_\Gamma$. It follows from (2.12) and Lemma 2.3 that

$$\begin{aligned} \Gamma \text{ is nonderogatory} &\Longleftrightarrow h^{\circ}(u,v) \neq 0 \quad \text{for some } u, v \in V_\Gamma \\ &\Longleftrightarrow \Gamma \text{ is path-Hamiltonian.} \end{aligned} \tag{2.13}$$

(2) Assume that $\Gamma - C$ is path-Hamiltonian with unique Hamiltonian path from vertex u to vertex v. Then $h^{*}(u,v) = 1$ and $h^{*}(w,z) = 0$ for every other pair of vertices w and z in V_Γ. It follows from (2.12) that if $h^{\circ}(u,v) \neq 1$, then Γ is nonderogatory. Now, assume that $h^{\circ}(u,v) = 1$. We will show that $h^{\circ}(w,z) = 0$ for every other pair of vertices w and z in V_Γ, which implies that Γ is derogatory. Let π be a Hamiltonian path in Γ from u to v and suppose that there is a Hamiltonian path σ in Γ from w to z. If $w \neq u$, then there exists an arc $w' \to u$ in Γ. Since π induces a path from u to w', then a cycle in Γ is formed different from C. This is a contradiction, so w must equal u. Similarly, $z = v$. □

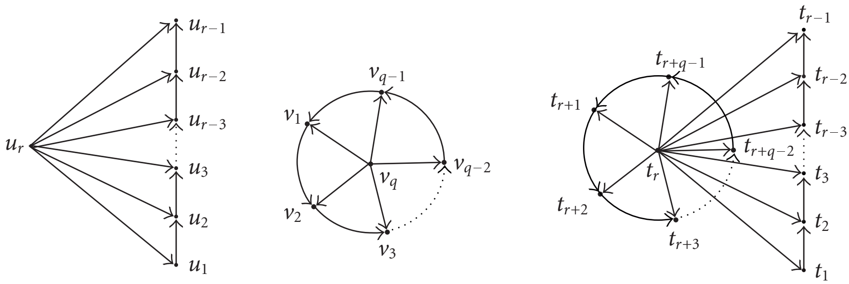
Example 2.6. In this example we turn our attention to a well-known operation between digraphs, the so-called coalescence of digraphs, considered in [8].

Let $F_r \cdot W_q$ be the coalescence of F_r and W_q with respect to the hub of both digraphs (see Figure 2.1).

$F_r \cdot W_q$ is unicyclic with unique cycle $C = t_{r+1}, t_{r+2}, \ldots, t_{r+q-2}, t_{r+q-1}, t_{r+1}$. Note that $F_r \cdot W_q - C = F_r$ which is path-Hamiltonian with unique Hamiltonian path $t_r \to t_1 \to \cdots \to t_{r-1}$. Moreover, $h^{\circ}(t_r, t_{r-1}) = 0$. Hence, by Theorem 2.5, $F_r \cdot W_q$ is nonderogatory.

Example 2.7. Consider the digraph D shown in Figure 2.2.

D is unicyclic with unique cycle $C = u_5, u_2, u_3, u_4, u_5$ of length 4. Observe that neither $D - C$ nor D is path-Hamiltonian. Consecuently, by Theorem 2.5, D is derogatory.



Difan F_r (a) Diwheel W_q (b) Coalescence $F_r \cdot W_q$ (c)

Figure 2.1

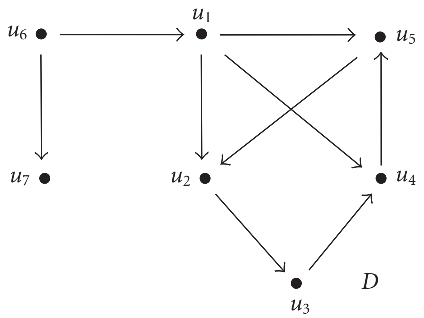


Figure 2.2

The results of Gan [4] follow immediately from Theorem 2.5, as we can see in the next example. But first recall that the complete product $D_1 \otimes D_2$ of digraphs D_1 and D_2 is the digraph obtained by joining every vertex of D_1 to every vertex of D_2 with an arc directed from the vertex of D_1 to the vertex of D_2.

Example 2.8. (1) Consider the complete product $F_q \otimes W_r$ (see Figure 2.3).

Note that $F_q \otimes W_r$ is unicyclic with unique cycle $C = v_1, v_2, \dots, v_{r-1}, v_1$; $(F_q \otimes W_r) - C$ is path-Hamiltonian with unique Hamiltonian path $v_{r+q} \to v_{r+1} \to v_{r+2} \to \cdots \to v_{r+q-1} \to v_r$ and $h^\circ(v_{r+q}, v_r) = 0$. Thus, by Theorem 2.5, $F_q \otimes W_r$ is nonderogatory.

(2) Consider the complete product $W_q \otimes F_r$ (see Figure 2.4).

Note that $W_q \otimes F_r$ is unicyclic with unique cycle $C = v_{r+1}, v_{r+2}, \dots, v_{r+q-1}, v_{r+1}$; $(W_q \otimes F_r) - C$ is path-Hamiltonian with unique Hamiltonian path $v_{r+q} \to v_r \to v_1 \to \cdots \to v_{r-2} \to v_{r-1}$ and $h^\circ(v_{r+q}, v_{r-1}) > 1$. Thus, by Theorem 2.5, $W_q \otimes F_r$ is nonderogatory.

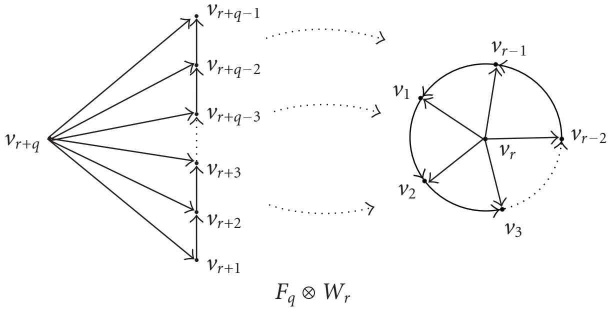


Figure 2.3

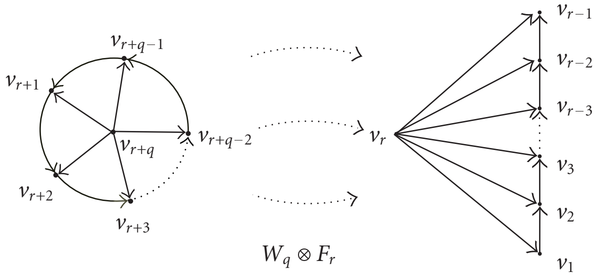


Figure 2.4

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