Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 54217, 27 pages
doi:10.1155/2007/54217

Research Article

Derived Categories and the Analytic Approach to General Reciprocity Laws—Part II

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Received 13 November 2006; Accepted 13 April 2007

Recommended by Pentti Haukkanen

Building on the topological foundations constructed in Part I, we now go on to address the homological algebra preparatory to the projected final arithmetical phase of our attack on the analytic proof of general reciprocity for a number field. In the present work, we develop two algebraic frameworks corresponding to two interpretations of Kubota's n-Hilbert reciprocity formalism, presented in a quasi-dualized topological form in Part I, delineating two sheaf-theoretic routes toward resolving the aforementioned (open) problem. The first approach centers on factoring sheaf morphisms eventually to yield a splitting homomorphism for Kubota's n-fold cover of the adelized special linear group over the base field. The second approach employs linked exact triples of derived sheaf categories and the yoga of gluing t-structures to evolve a means of establishing the vacuity of certain vertices in diagrams of underlying topological spaces from Part I. Upon assigning properly designed t-structures to three of seven specially chosen derived categories, the collapse just mentioned is enough to yield n-Hilbert reciprocity.

1. Introduction

As we conveyed in detail in [1], the motivation for the present investigation is Erich Hecke's 80-year-old open problem asking for an analytic proof of the general reciprocity law for a global algebraic number field, k. Hecke issued his challenge at the end of [2, 3] where he gave the definitive classical Fourier-analytic treatment of the quadratic case. This proof was recast in unitary group representation-theoretic terms some forty years later by Weil [4]. Not long after that, Kubota [5] gave an explicitly low-dimensional cohomological treatment of Weil's representation theory and, a few years later, went on to address the open higher degree case [6]. Specifically, Kubota demonstrated that Hilbert reciprocity, that is, the cover of $SL_2(k)_{\mathbf{A}}$ by the nth roots of unity, μ_n (assumed to lie in k, which is to say that k is totally imaginary), split on $SL_2(k)$, the rational points. (See [1, Section 1] for further details.) Perpetuating our jargon from Part I [1], we call this arrangement Kubota's n-Hilbert reciprocity formalism and observe, as regards our greater objective, that Hecke's open problem will be settled if this splitting can be derived without presupposing any higher reciprocity law. Accordingly, we delineated in Part I what we take the liberty to describe as a quasi-dualization of this Kubota formalism, exchanging the given setting of algebraic groups for that of sheaves and sheaf complexes on topological spaces closely associated to these groups. We also demonstrated in Part I that in this quasi-dual setting, n-Hilbert reciprocity can be reached along a couple of different paths, such as by means of factoring a morphism in a natural derived sheaf category through another morphism. We address this homological algebraic theme in the first four sections, Sections 2–5, of the present work.

Moreover, Proposition 5.1, one of the central results of [1], yields inter alia that n-Hilbert reciprocity follows if we can prove that the image of a certain mapping, Ω, is located entirely within a locally closed set, $\coprod_{\ell=1}^{\infty} X_{1;\ell}$, sitting inside our primary topological space, $\widetilde{X}_{\mathbf{A}}^2$. In Section 6, below, we translate this condition to the level of derived sheaf complexes on the indicated neighboring topological spaces. This permits us to prove that the stated condition will be realized if $n-1$ of n derived categories $\mathfrak{D}_{\overline{Y}_{\xi_0}}$, $\xi_0 \in \mu_n$, defined below, are void. We go on to address this matter in Sections 7, and 8 in terms of the behavior of t-structures on three derived categories, including a $\mathfrak{D}_{\overline{Y}_{\xi_0}}$, arranged in a seven-vertex diagram of interlaced, or linked, exact triples accruing to the underlying spaces, including $\overline{Y}_{\xi_0}$, $\xi_0 \neq 1$. Whenever reasonable, we allow for the likelihood that such t-structures should be perverse. We systematically glue and unglue t-structures in this seven-vertex arrangement so as to precipitate relations on them as a consequence of a uniqueness criterion pertaining to the middle vertex, $\mathfrak{D}_{\widetilde{X}_{\mathbf{A}}^2}$. Beyond this, and more importantly, we gain the wherewithal to identify arithmetically motivated conditions on the operative initial t-structures directly geared toward the collapse of the $\mathfrak{D}_{\overline{Y}_{\xi_0}}$ and $\overline{Y}_{\xi_0}$ for $1 \neq \xi_0 \in \mu_n$, as mentioned above. In part III of this sequence, projected for the near future, we address some preliminary ideas covering how to phrase these conditions in computationally accessible ways in anticipation of the final arithmetical phase of our campaign (tacitly assuming this path to be more lucrative than the first strategem which, however, has to be kept viable). We propose two broad approaches in this content: one dealing with a putative index, $\chi(\_)$, or rather $\chi(t(\_))$, acting on t-structures, and the other dealing with the potential of applying a beautiful result due to Bridgeland [7] to the effect that under certain conditions, a set of t-structures on a derived category can be endowed with a metric and in fact be made into a finite-dimensional complex manifold. But before we get to all this, it is useful to draw a quick sketch of the bigger picture as it now takes shape, supplementing the historically framed discussion in Sections 2 and 3 of Part I.

Despite its origins [2, 3] in Hecke's classical Fourier analysis, as we already indicated, our sheaf-theoretic framework for the projected analytic proof of higher reciprocity is built on the representation theory and low-dimensional cohomology of Weil and Kubota, and we initiated this quasi-dualization of Kubota's n-Hilbert reciprocity formalism by means of the following moves in Part I. First, we restructured the splitting of

Kubota's adelic 2-cocycle, $c_{\mathbf{A}}^{(n)} \in H^2(\mathrm{SL}_2(k)_{\mathbf{A}}, \mu_n)$, on $\mathrm{SL}_2(k)$, as an assertion about associated topological spaces designed to convey suppressed group structures, both ordinary and twisted, by a "doubling" manoeuvre. In our earlier and now readopted numbering (cf. [1]), this first level of our quasi-dualization is contained in [1, the diagrams (4.8), (4.9), and (4.20)], in $\mathfrak{Top}$, the category of topological spaces. As already hinted, the splitting of $c_{\mathbf{A}}^{(n)}$ on $\mathrm{SL}_2(k) =: X_0$ can, in this arrangement, be rendered as the existence of a specific continuous mapping $\Omega = \bigotimes_{\xi_0} \Omega_{\xi_0}$ for which, for each $\xi_0 \in \mu_n =: \mu$, we have $\Omega_{\xi_0} : X_0 \to \coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}$ in keeping with (2.1). The object $X_{\xi_0;\ell}$ is the set of all ordered quadruples $(\sigma, \sigma'; \xi, \xi') \in \mathrm{SL}_2(k)_{\mathbf{A}}^2 \times \mu^2 =: \widetilde{X}_{\mathbf{A}}^2$, for which $c_{\mathbf{A}}^{(n)}(\sigma, \sigma') = \xi_0$; this sets up the next level of our quasi-dualization, of bounded sheaf complexes collected into derived categories "above" these first-level topological spaces.

Given this architecture, we will see presently that [1, Proposition 5.1] effectively provides that n-Hilbert reciprocity amounts to the condition that if $\xi_0 \neq 1$, the space $\mathrm{SL}_2(k)^2 \times \mu^2 =: \widetilde{X}_0^2$ fails to meet $\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}$. As far as the earlier-mentioned second route is concerned, in what follows we propose to head for this result through careful manipulation of certain t-structures that may be imparted to the appropriate derived categories. Thus, our primary future objectives include producing suitable, arithmetically conditioned, initial t-structures ensuring the a forteriori collapse of t-structures on the derived categories $\mathfrak{D}_{\overline{Y}_{\xi_0}}$, for $\xi_0 \neq 1$, where $\overline{Y}_{\xi_0}$ is the closure of $\widetilde{X}_0^2 \cap \coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}$ (and $\mathfrak{D}_{\overline{Y}_{\xi_0}} = D^b(\overline{Y}_{\xi_0})$). This turns out to be part and parcel of n-Hilbert reciprocity.

2. A reprise of material from Part I

Diagram (2.1), reproduced below, sits at the heart of our quasi-dualized formulation of the splitting of $\widetilde{\mathrm{SL}}_2(k)_{\mathbf{A}}^{(n)} = \mathrm{SL}_2(k)_{\mathbf{A}} \times_{c_{\mathbf{A}}^{(n)}} \mu_n = \widetilde{X}_{\mathbf{A}}$ on the rational points, $\mathrm{SL}_2(k) = X_0$, translated to the category $\mathfrak{Top}$, of topological spaces; as already stated above, $c_{\mathbf{A}}^{(n)}$ is Kubota's adelic 2-cocycle defining the given cover of $\mathrm{SL}_2(k)_{\mathbf{A}}$ by the nth roots of unity:

$$
\begin{tikzcd}
\mu^2 \arrow[d, "m_\ell"'] & & \coprod_{\ell=1}^{\infty} X_{\xi_o;\ell} \arrow[ll, "j^0 \otimes j^0"'] \arrow[dl, "m_{\xi_0; c_{\mathbf{A}}^{(n)}}"] & \\
\mu & \widetilde{X}_{\mathbf{A}} \arrow[l, "j^0"'] & & X_0^2 \arrow[ul, dotted, "\Omega_{\xi_0}"'] \arrow[dl, "m_0"] \\
& & X_0 \arrow[ull, dotted, "s_{\mathbf{A}}"] \arrow[uull, dotted, "s_{\mathbf{A}} \otimes s_{\mathbf{A}}"'] &
\end{tikzcd}
\tag{2.1}
$$

Proposition 5.1 of [1] provides that the important thing is to construct Ω_{ξ_0}, or derive its existence, noting that it has to map into $\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}$ whose constituent spaces are locally closed (see [1, Corollary 4.6]) in anticipation of the probable appearance of perverse sheaves in the future. All the vertices live in $\mathfrak{Top}$, so, while the group structures on μ,

X_0, $\widetilde{X}_{\mathbf{A}}$ are suppressed ab initio, the respective multiplications are recovered by the mappings m_ℓ, m_0, $m_{\xi_0;c_{\mathbf{A}}^{(n)}}$. Beyond this, j^0 and $j^0 \otimes j^0$ are just the morphisms opposite to the obvious imbeddings (see [1, Section 4]), and as always, the dotted arrows denote maps to be constructed (with Ω_{ξ_0} being the one that counts).

If $\mathcal{F}$ is an a priori unspecified sheaf on $\widetilde{X}_{\mathbf{A}}$, identified with its sheaf space when necessary, that is, $\mathcal{F} \approx {}^{\grave{e}t}\mathcal{F}$, and if $i \otimes 1 : X_0 \to \widetilde{X}_{\mathbf{A}}$ is the imbedding $\sigma \mapsto (\sigma, 1)$, consider, as part of the next level of our quasi-dualization of Kubota's formalism, the following sheaf diagram:

$$\begin{tikzcd}
 & & (m_{\xi_0;c_{\mathbf{A}}^{(n)}})_? m^?_{\xi_0;c_{\mathbf{A}}(n)}\mathcal{F} & & \\
(j^0)^? j^0_? \arrow[r, "\iota"] & \mathcal{F} \arrow[ur, "\nu"] \arrow[dr, "\iota^0"'] & & & ((i\otimes 1)\circ m_0)_?((i\otimes 1)\circ m_0)^?\mathcal{F} \\
 & & (i\otimes 1)_? \circ (i\otimes 1)^?\mathcal{F} \arrow[urr, "\nu^0"'] & &
\end{tikzcd} \tag{2.2}$$

Here, ? stands for $*$ or !, making allowances for future appearances of Verdier's R when we pass to derived categories in those cases where we have only left exactness (at the sheaf level) to begin with. Diagram (2.2) typifies what we will call the contravariant option vis-a-vis (2.1) in the sense that, by comparison, the arrows point backwards. In what follows, we also consider the respective covariant options for these constructs. Additionally, note that the sheaf morphisms ι, ν, ι^0, ν^0 are named with the underlying continuous functions j^0, $m_{\xi_0;c_{\mathbf{A}}^{(n)}}$, $i \otimes 1$, m_0 in mind: a policy we try to follow as often as possible. Lastly, as in [1], we simplify the cumbersome notation of diagram (2.2), thus

$$\begin{tikzcd}
 & & \mathcal{B} & \\
\mathcal{A} \arrow[r, "\iota"] & \mathcal{F} \arrow[ur, "\nu"] \arrow[dr, "\iota^0"'] & & \mathcal{C} \\
 & & \mathcal{D} \arrow[ur, "\nu^0"'] &
\end{tikzcd} \tag{2.3}$$

With (2.2), (2.3) situated in the abelian category $\mathfrak{A} := \mathfrak{Sh}/_{\widetilde{X}_{\mathbf{A}}}$ of sheaves on the central topological space $\widetilde{X}_{\mathbf{A}}$, we can (in the contravariant case, with little loss of generality) articulate the theme we investigate first in what follows. Proposition 7.1 of [1] asserts that if we associate sheaf complexes $\mathcal{A}^\bullet$, $\mathcal{B}^\bullet$, $\mathcal{C}^\bullet$, $\mathcal{D}^\bullet$, $\mathcal{F}^\bullet$ to the respective sheaves in (2.3), using the usual "concentration in degree-zero" convention, and go over to the derived category $\mathfrak{D} := \mathcal{D}(\mathfrak{A})$, then the condition $\mathrm{Hom}_{\mathfrak{D}}(Z^\bullet[-1], \mathcal{C}^\bullet) = 0$, where $Z^\bullet$ is a cone completing some distinguished triangle based on $\mathcal{F}^\bullet \xrightarrow{\nu} \mathcal{B}^\bullet$, abusing notation a bit, is enough to yield

$$\begin{tikzcd}
 & & \mathcal{B}^\bullet \arrow[dr, dotted, "\Phi"] & \\
\mathcal{A}^\bullet \arrow[r, "\iota"] \arrow[drr, dotted, "\varphi"'] & \mathcal{F}^\bullet \arrow[ur, "\nu"] \arrow[dr, "\iota^0"] & & \mathcal{C}^\bullet \\
 & & \mathcal{D}^\bullet \arrow[ur, "\nu^0"'] &
\end{tikzcd} \tag{2.4}$$

which is to say, the factorization $\nu^0 \circ \iota^0 = \Phi \circ \nu$. This appearance of Φ yields, in turn, that (2.1) and (2.2) can be linked as follows:

$$\begin{tikzcd}
(m_{\xi_0;c_{\mathbf{A}}^{(n)}})_? m^?_{\xi_0;c_{\mathbf{A}}^{(n)}} \mathcal{F} \arrow[rr, "\Phi"] \arrow[dr] & & ((i\otimes 1)\circ m_0)_? ((i\otimes 1)\circ m_0)^? \mathcal{F} \arrow[dl] \\
 & \widetilde{X}_{\mathbf{A}} = \widetilde{\mathrm{SL}}(k)^{(n)}_{\mathbf{A}} & \\
\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell} \arrow[ur, "m_{\xi_0;c_{\mathbf{A}}^{(n)}}"] & & X_0^2 = \mathrm{SL}_2(k)^2 \arrow[ul, "(i\otimes 1)\circ m_0"'] \arrow[ll, dotted, "\Omega_{\xi_0}"']
\end{tikzcd} \tag{2.5}$$

where we briefly use the notation Φ ambiguously (see directly below). The two unnamed maps in (2.5) are just the usual projections from sheaf spaces to their underlying sites, so we again have a diagram in $\mathfrak{Top}$. Our goal is to investigate the implications of [1, Proposition 7.1] and flesh out (2.5) in preparation for the future task of building $\mathcal{F}$, taking the indicated arithmetical requirements into account, for making the proper assignments to the ?'s, and for delineating the morphisms ι, ν, ι^0, ν^0.

As far as the covariant option is concerned, with (2.1) as our starting point, the first move is to reverse arrows in (2.4) (whence in (2.2), (2.3)):

$$\begin{tikzcd}
 & & \mathcal{B}^\bullet \arrow[dl, "\eta"'] & \\
\mathcal{A}^\bullet & \mathcal{F}^\bullet \arrow[l] & & \mathcal{C}^\bullet \arrow[ul, dotted, "\Psi"'] \arrow[dl, "\nu_0"] \\
 & & \mathcal{D}^\bullet \arrow[ul, "\iota_0"] &
\end{tikzcd} \tag{2.6}$$

We obtain, parallel to [1, Proposition 7.1], the following.

Proposition 2.1. *If $Z^\bullet$ is a cone of η, that is, if $\mathcal{B}^\bullet \xrightarrow{\eta} \mathcal{F}^\bullet \to Z^\bullet \xrightarrow{+1}$ is a distinguished triangle in $\mathfrak{D}$, and if* $\mathrm{Hom}_{\mathfrak{D}}(\mathcal{C}^\bullet, Z^\bullet) = 0$, *then Ψ as drawn in (2.6) exists, yielding the factorization $\iota_0 \circ \nu_0 = \eta \circ \Psi$. (Even though the proof of this assertion is just dual to that given* *for [1, Proposition 7.1], we include it here for good form and to make an attempt at self-containment.)*

Proof. Go to the associated long exact Hom-sequence part of which is

$$\begin{aligned} &\cdots \longrightarrow \operatorname{Hom}_{\mathfrak{D}}\left(\mathscr{C}^{\bullet}, Z^{\bullet}[-1]\right) \longrightarrow \operatorname{Hom}_{\mathfrak{D}}\left(\mathscr{C}^{\bullet}, \mathscr{B}^{\bullet}\right) \xrightarrow{\eta_*} \operatorname{Hom}_{\mathfrak{D}}\left(\mathscr{C}^{\bullet}, \mathscr{F}^{\bullet}\right) \longrightarrow \\ &\longrightarrow \operatorname{Hom}_{\mathfrak{D}}\left(\mathscr{C}^{\bullet}, \mathfrak{C}^{\bullet}\right) \longrightarrow \cdots . \end{aligned} \tag{2.7}$$

The vanishing of $\operatorname{Hom}_{\mathfrak{D}}(\mathscr{C}^{\bullet}, \mathfrak{C}^{\bullet})$ directly yields the surjectivity of the map η_* defined, as always, by the rule $\eta_*(\sigma) = \eta \circ \sigma$, for any $\sigma : \mathscr{C}^{\bullet} \to \mathscr{B}^{\bullet}$. So, since $\iota_0 \circ \nu_0 \in \operatorname{Hom}_{\mathfrak{D}}(\mathscr{C}^{\bullet}, \mathscr{F}^{\bullet})$, we get a morphism $\Psi : \mathscr{C}^{\bullet} \to \mathscr{B}^{\bullet}$ with $\eta \circ \Psi = \iota_0 \circ \nu_0$, as required. ☐

Finally, with (2.6) and Proposition 2.1 in place, we obtain the covariant counterpart to (2.5), namely,

$$\begin{tikzcd} \left(m_{\xi_0;c_{\mathbf{A}}^{(n)}}\right)_{?} m^{?}_{\xi_0;c_{\mathbf{A}}^{(n)}} \mathscr{F} \arrow[dr] & & \left((i\otimes 1)\circ m_0\right)_{?}\left((i\otimes 1)\circ m_0\right)^{?} \mathscr{F} \arrow[ll, "\Psi"'] \arrow[dl] \\ & \widetilde{X}_{\mathbf{A}} = \widetilde{\mathrm{SL}}_2(k)^{(n)}_{\mathbf{A}} & \\ \coprod_{\ell=1}^{\infty} X_{\xi_0;\ell} \arrow[ur, "m_{\xi_0;c_{\mathbf{A}}^{(n)}}"] & & X_0^2 = \mathrm{SL}_2(k)^2 \arrow[ul, "(i\otimes 1)\circ m_0"'] \arrow[ll, dotted, "\Omega_{\xi_0}"'] \end{tikzcd} \tag{2.8}$$

with Ψ suffering the same ambiguity as Φ in (2.5). While Propositions 7.1 of [1], and 2.1 address the category $\mathfrak{D}$, the diagrams (2.5), (2.8) are supposed to exist in $\mathfrak{A} = \mathfrak{Sh}/_{\widetilde{X}_{\mathbf{A}}}$. Thus, the next order of business is to remove these ambiguities, and we address this matter in the next section.

3. Preliminaries on the interplay between $\mathfrak{D}$ and $\mathfrak{A}$

Utilizing the less cumbersome notation of (2.3), we can rewrite (2.5) and (2.8) as

$$\begin{tikzcd} \mathscr{B} \arrow[rr, "\Phi"] \arrow[dr] & & \mathscr{C} \arrow[dl] \\ & \widetilde{X}_{\mathbf{A}} & \\ \coprod_{\ell=1}^{\infty} X_{\xi_0;c_{\mathbf{A}}^{(n)}} \arrow[ur] & & X_0^2 \arrow[ul] \arrow[ll, dotted, "\Omega_{\xi_0}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{B} \arrow[dr] & & \mathscr{C} \arrow[ll, "\Psi"'] \arrow[dl] \\ & \widetilde{X}_{\mathbf{A}} & \\ \coprod_{\ell=1}^{\infty} X_{\xi_0;c_{\mathbf{A}}^{(n)}} \arrow[ur] & & X_0^2 \arrow[ul] \arrow[ll, dotted, "\Omega_{\xi_0}"'] \end{tikzcd} \tag{3.1}$$

The trouble is that, as per (2.4), (2.6) and Propositions 7.1 of [1], and 2.1, the morphisms Φ, Ψ map between the sheaf complexes $\mathscr{B}^{\bullet}$ and $\mathscr{C}^{\bullet}$ rather than the sheaves $\mathscr{B}$ and $\mathscr{C}$, so (3.1) involve an obliteration of distinctions between categories. This can be rectified, however, by setting

$$\delta : \mathfrak{A} = \mathfrak{Sh}/_{\widetilde{X}_{\mathbf{A}}} \longrightarrow \mathfrak{D} = \mathscr{D}(\mathfrak{A}), \tag{3.2}$$

$${}^{\acute{e}t}\mathscr{B} \approx \mathscr{B} \longmapsto \mathscr{B}^{\bullet}, \tag{3.3}$$

(provisionally) employing the standard concentration of $\mathscr{B}^{\bullet}$ in degree zero, and setting

$$\epsilon : \mathfrak{D} \longrightarrow \mathfrak{A}, \tag{3.4}$$

$$\mathscr{C}^{\bullet} \longmapsto \mathscr{C}^{\bullet} =: \mathscr{C} \approx {}^{\acute{e}t}\mathscr{C}. \tag{3.5}$$

Of course, ϵ can be made to pick off the sheaf in any degree, or do something more sophisticated than that, should the need arise, and (3.4), as also (3.2), should be regarded as provisional. Later considerations should determine what the specifics must be as regards δ and ϵ. However, we certainly have that $\delta : \mathfrak{A} \to \mathfrak{D}$ and $\epsilon : \mathfrak{D} \to \mathfrak{A}$, and this permits us to amend and complete (3.1) to

$$\begin{tikzcd}
{}^{\acute{e}t}\mathscr{B} \arrow[d, "\delta"'] & & & \\
\mathscr{B}^{\bullet} \arrow[rr, "\Phi"] \arrow[dr] & & \mathscr{C}^{\bullet} \arrow[r, "\epsilon"] \arrow[dl] & {}^{\acute{e}t}\mathscr{C} \arrow[dddlll, bend left, "\beta"] \\
& \widetilde{X}_{\mathbf{A}} & & \\
\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell} \arrow[ur] & & X_0^2 \arrow[ll, dotted, "\Omega_{\xi_0}"'] \arrow[ul] \arrow[uuull, bend right, "\alpha"'] &
\end{tikzcd} \tag{3.6}$$

$$\begin{tikzcd}
{}^{\acute{e}t}\mathscr{B} \arrow[r, "\epsilon"] \arrow[ddr, bend right, "\beta"'] & \mathscr{B}^{\bullet} \arrow[dr] & & \mathscr{C}^{\bullet} \arrow[ll, "\Psi"'] \arrow[dl] & {}^{\acute{e}t}\mathscr{C} \arrow[l, "\delta"'] \\
& & \widetilde{X}_{\mathbf{A}} & & \\
& \coprod_{\ell=1}^{\infty} X_{\xi_0;\ell} \arrow[ur] & & X_0^2 \arrow[ll, dotted, "\Omega_{\xi_0}"'] \arrow[ul] \arrow[uur, bend right, "\alpha"'] &
\end{tikzcd} \tag{3.7}$$

so that we need only arrange for the continuity of

$$\Omega_{\xi_0} := \beta \circ \epsilon \circ \Phi \circ \delta \circ \alpha \tag{3.8}$$

or

$$\Omega_{\xi_0} := \beta \circ \epsilon \circ \Psi \circ \delta \circ \alpha, \tag{3.9}$$

respectively, according as α, β live in (3.6) or (3.7). Once Ω_{ξ_0} is continuous (for each ξ_0), it is fit for insertion into (2.1), ending the game.

But we say no more about this for the moment and proceed, next, to take a closer look at Φ (and Ψ).

4. The meaning of a vanishing Hom-group

We already observed at the end of [1, Section 7] that the condition $\mathrm{Hom}_{\mathfrak{D}}(Z^\bullet[-1], \mathscr{C}^\bullet) = 0$ translates to a requirement on the relevant sheaf complexes involving chain homotopy, and this will constitute our point of departure for what follows in the present section as well as the next. We begin, however, by positing that in these two sections we require the various sheaves populating the degrees of the upcoming derived sheaf complexes to take their values in the category of vector spaces; imposing this restriction allows us to render the various morphisms situated in these derived categories as simple, ordinary arrows, sparing us the task of having to deal with, for example, fraction constructs of the type $\bullet \not\leftarrow \bullet \rightarrow \bullet$. (In this connection, see [8, pages 72-73] and [9, page 485]: it is easy to prove that a sheaf of vector spaces is injective.) Also, following, for example, [10], we write $\bullet \not\twoheadrightarrow \bullet$ for a quasi-isomorphism. Now we get the following.

Proposition 4.1. $\mathrm{Hom}_{\mathfrak{D}}(Z^\bullet[-1], \mathscr{C}^\bullet) = 0$ *if and only if every $\mathfrak{D}$-morphism $f : Z^\bullet \rightarrow \mathscr{C}^\bullet$ admits a quasi-isomorphism $s : \mathscr{C}^\bullet \rightarrow \mathscr{E}^\bullet$, for some chain complex $\mathscr{E}^\bullet$, such that $s \circ f$ is chain homotopic to 0.*

Proof. This is an immediate consequence of [11, pages 38-39]. □

It follows that, in order to obtain the existence of Φ in (2.4), it is enough to have that for any such $f : Z^\bullet \rightarrow \mathscr{C}^\bullet$, there should exist $s : \mathscr{C}^\bullet \rightarrow \mathscr{E}^\bullet$ and a chain map

$$\Psi : Z^\bullet[-1] \longrightarrow \mathscr{E}^\bullet[-1] \tag{4.1}$$

such that

$$s^n \circ f^n = -d^{n-1}_{\mathscr{E}^\bullet} \circ \psi^n + \psi^{n-1} \circ d^n_{Z^\bullet[-1]} \tag{4.2}$$

for all $n \in \mathbb{Z}$, or, in the usual notation,

$$s \circ f \simeq_{\psi} 0. \tag{4.3}$$

This leads to the ladder diagram

$$\begin{tikzcd}
{} \arrow[d] & \cdots & {} \arrow[d] & \cdots \arrow[d] & & \cdots \arrow[d] \\
Z^{-2} \arrow[d] & = & Z^{-1}[-1] \arrow[d] \arrow[r, "\psi^{-1}"] \arrow[dr, "s^{-1}\circ f^{-1}"] & \mathscr{E}^{-1}[-1] \arrow[d] & = & \mathscr{E}^{-2} \arrow[d] \\
Z^{-1} \arrow[d] & = & Z^{\circ}[-1] \arrow[d] \arrow[r, "\psi^{\circ}"] \arrow[dr, "s^{0}\circ f^{0}"] & \mathscr{E}^{\circ}[-1] \arrow[d] & = & \mathscr{E}^{-1} \arrow[d] \\
Z^{0} \arrow[d] & = & Z^{1}[-1] \arrow[d] \arrow[r, "\psi^{1}"] \arrow[dr, "s^{1}\circ f^{1}"] & \mathscr{E}^{1}[-1] \arrow[d] & = & \mathscr{E}^{0} \arrow[d] \\
Z^{1} \arrow[d] & = & Z^{2}[-1] \arrow[d] \arrow[r, "\psi^{2}"] \arrow[dr, "s^{2}\circ f^{2}"] & \mathscr{E}^{2}[-1] \arrow[d] & = & \mathscr{E}^{1} \arrow[d] \\
Z^{2} \arrow[d] & = & Z^{3}[-1] \arrow[d] \arrow[r, "\psi^{3}"] & \mathscr{E}^{3}[-1] \arrow[d] & = & \mathscr{E}^{2} \arrow[d] \\
{} & \cdots & {} & \cdots & & \cdots
\end{tikzcd} \tag{4.4}$$

where all the vertical maps are the indicated differentials in the appropriate degrees. Here, we have also taken into account the sign convention (cf. [12, page 31])

$$d^{n}_{X^{\bullet}[k]} = (-1)^{k} d^{n+k}_{X^{\bullet}} \tag{4.5}$$

for an arbitrary object $X^{\bullet}$ in $\mathfrak{D}$; additionally, the fact that we have

$$\begin{tikzcd}
Z^{\bullet}[-1] \arrow[r, "f"] \arrow[dr, swap, "s\circ f"] & \mathscr{C}^{\bullet} \arrow[d, "s", "\shortmid" marking] \\
 & \mathscr{E}^{\bullet}
\end{tikzcd} \tag{4.6}$$

means that for all $n \in \mathbb{Z}$,

$$\begin{tikzcd}
Z^{n+1} \arrow[r, "f^{n}"] \arrow[dr, swap, "s^{n}\circ f^{n}"] & \mathscr{C}^{n} \arrow[d, "s^{n}"] \\
 & \mathscr{E}^{n}
\end{tikzcd} \tag{4.7}$$

in view of the usual shift convention (see [12]), namely,

$$X^n[k] = X^{n+k}. \tag{4.8}$$

Accordingly, (4.2) becomes

$$s^n \circ f^n = -d_{\mathscr{E}^\bullet}^{n-1} \circ \psi^n - \psi^{n+1} \circ d_{Z^\bullet}^{n+1}. \tag{4.9}$$

Bearing in mind the possibility, if not the likelihood, that our erstwhile maps δ, ϵ, of Section 3, might have to be chosen in unorthodox ways later, we consider now what happens if we simply go with the standard choice of locating $\mathfrak{A}$ in $D(\mathfrak{A})$ (indeed, this is really preordained by the construction of the very derived category $D(\mathfrak{A})$ itself). In other words, we associate to any sheaf $\mathscr{F}$ in $\mathfrak{A} = \mathfrak{Sh}/_{\tilde{X}}$ the sheaf complex $\mathscr{F}^\bullet$ (with some abuse of language) defined by

$$\mathscr{F}^n = \begin{cases} \mathscr{F}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0 \end{cases} \tag{4.10}$$

(concentration in degree 0); thus, we certainly have acyclicity in nonzero degrees.

With the preceding convention in place, we have, first, the following.

Proposition 4.2. *If $Z_\nu^\bullet$ is the (actual) mapping cone of $\nu : \mathscr{F}^\bullet \to \mathscr{B}^\bullet$, which is to say that*

$$Z_\nu^\bullet = \mathscr{F}^\bullet[1] \oplus \mathscr{B}^\bullet \tag{4.11}$$

equipped with the differential

$$d_{Z_\nu^\bullet} = \begin{pmatrix} d_{\mathscr{F}^\bullet[1]}^n & 0 \\ \nu^{n+1} & d_{\mathscr{B}^\bullet}^n \end{pmatrix} = \begin{pmatrix} -d_{\mathscr{F}^\bullet}^{n+1} & 0 \\ \nu^{n+1} & d_{\mathscr{B}^\bullet}^n \end{pmatrix} \tag{4.12}$$

(using (4.6)), then $\operatorname{Hom}_{\mathfrak{D}}(Z_\nu^\bullet[-1], \mathscr{C}^\bullet) = 0$.

Proof. Sufficiency is obvious. As for necessity, suppose that we have $\operatorname{Hom}_{\mathfrak{D}}(Z_\nu^\bullet[-1], \mathscr{C}^\bullet) = 0$ and let $f : Z^\bullet[-1] \to \mathscr{C}^\bullet$, where $Z^\bullet$ is any cone for ν (so $Z^\bullet \cong_{\mathfrak{D}} Z_\nu^\bullet$). Then, we have

$$\begin{tikzcd} \mathscr{F}^\bullet \arrow[r] \arrow[d, equal] & \mathscr{B}^\bullet \arrow[r] \arrow[d, equal] & Z^\bullet \arrow[r] \arrow[d, dotted, shift left, "\sigma"] & \mathscr{F}^\bullet[1] \arrow[d, equal] \\ \mathscr{F}^\bullet \arrow[r] & \mathscr{B}^\bullet \arrow[r] & Z_\nu^\bullet \arrow[r] \arrow[u, dotted, shift left, "\sigma^{-1}"] & \mathscr{F}^\bullet[1] \end{tikzcd} \tag{4.13}$$

where the morphism pair (σ, σ^{-1}) realizes the $\mathfrak{D}$-isomorphism between $Z^\bullet$ and $Z_\nu^\bullet$. Thus $(\sigma[-1], \sigma[-1]^{-1})$ gives $Z^\bullet[-1] \cong_{\mathfrak{D}} Z_\nu^\bullet[-1]$ and this immediately yields

$$\begin{tikzcd} Z_\nu^\bullet[-1] \arrow[r, "{\sigma[-1]^{-1}}"] \arrow[dr, dotted, "{f \circ \sigma[-1]^{-1}}"'] & Z^\bullet[-1] \arrow[d, "f"] \\ & Z^\bullet \end{tikzcd} \tag{4.14}$$

But now, $f \circ \sigma[-1]^{-1} \in \mathrm{Hom}_{\mathfrak{D}}(Z_\nu^\bullet[-1], \mathscr{C}^\bullet) = 0$, forcing the relation $f \circ \sigma[-1]^{-1} = 0$. Now, just compose with $\sigma[-1]$ to get $f = 0$. □

Of course, if we look beyond the outlandish notation of derived categories, this is really just the elementary fact that in any reasonable category, $\mathfrak{C}$, we have that $\mathrm{Hom}_{\mathfrak{C}}(X_0, Y) = 0$ if and only if, for all $X \cong X_0, \mathrm{Hom}_{\mathfrak{C}}(X, Y) = 0$. In any event, the upshot (and the raison d'être for Proposition 4.2) is that we can now safely turn our attention to the case where $Z^\bullet = Z_\nu^\bullet$ as given by (4.11) and (4.13), and prove the following useful result.

Proposition 4.3. *Let $\mathscr{F}^\bullet$, $\mathscr{B}^\bullet$, $\mathscr{C}^\bullet$ be concentrated in degree 0 and let $Z_\nu^\bullet$ (resp., $Z^\bullet$) be the (resp., any) mapping cone of $\nu : \mathscr{F}^\bullet \to \mathscr{B}^\bullet$. Then* $\mathrm{Hom}_{\mathfrak{D}}(Z_\nu^\bullet[-1], \mathscr{C}^\bullet) = 0 = \mathrm{Hom}_{\mathfrak{D}}(Z^\bullet[-1], \mathscr{C}^\bullet)$ *if and only if, with $\mathfrak{A}$ the underlying abelian sheaf category, so that $\mathfrak{D} = D(\mathfrak{A})$,* $\mathrm{Hom}_{\mathfrak{D}}(\mathscr{F}, \mathscr{C}) = 0$. *(cf. (4.10)).*

Proof. Because $Z_\nu^\bullet = \mathscr{F}^\bullet[1] \oplus \mathscr{B}^\bullet$, so that $Z_\nu^\bullet[-1] = \mathscr{F}^\bullet \oplus \mathscr{B}^\bullet[-1]$, we can employ the earlier shift and imbedding connections (4.8), (4.10) to infer that if $n \neq 0, 1$, then $\mathscr{F}^n = 0 = \mathscr{B}^{n-1}$, forcing $Z_\nu^n[-1] = 0 \oplus 0 = 0$. On the other hand, when $n = 0$, $\mathscr{F}^0 = \mathscr{F}$ and $\mathscr{B}^0[-1] = \mathscr{B}^{-1} = 0$, so that $Z_\nu^0[-1] = Z_\nu^{-1} = \mathscr{F} \oplus 0 = \mathscr{F}$, whereas when $n = 1$, $\mathscr{F}^1 = 0$ and $\mathscr{B}^1[-1] = \mathscr{B}^0 = \mathscr{B}$, so that $Z_\nu^1[-1] = Z_\nu^0 = 0 \oplus \mathscr{B} = \mathscr{B}$. Additionally, setting $n = -1$ in (4.12) produces $d_{Z_\nu^\bullet}^{-1} = \left(\begin{smallmatrix} 0 & 0 \\ \nu & 0 \end{smallmatrix}\right)$, because $d^0_{\mathscr{F}^\bullet} : \mathscr{F}^\bullet = \mathscr{F} \to \mathscr{F}^1 = 0$ and $f_{\mathscr{B}^\bullet}^{-1} = \mathscr{B}^{-1} = 0 \to \mathscr{B}^0 = \mathscr{B}$, while $\nu^0 = \nu$, of course. So, $d_{Z_\nu^\bullet}^{-1}$ simply reduces to μ. Putting all this together, we see that, with $Z_\nu^\bullet$ in place of $Z^\bullet$ and with the hypotheses of Proposition 4.1 in place, (4.4) becomes

$$
\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] \\
 & 0 = Z_\nu^{-2} \arrow[r, "\psi^{-1}"] \arrow[d, "-d^{-2}_{Z_\nu^\bullet}"'] \arrow[dr, "s^0 \circ f^0"] & \mathscr{E}^{-2} \arrow[d, "-d^{-2}_{\mathscr{E}^\bullet}"] \\
\mathscr{F} \arrow[r, equal] \arrow[d, "-\nu"'] & Z_\nu^{-1} \arrow[r, "\psi^0"] \arrow[d, "-d^{-1}_{Z_\nu}"'] \arrow[dr, "s^1 \circ f^1"] & \mathscr{E}^{-1} \arrow[d, "-d^{-1}_{\mathscr{E}^\bullet}"] \\
\mathscr{B} \arrow[r, equal] & Z_\nu^{0} \arrow[r, "\psi^1"] \arrow[d, "-d^{0}_{Z_\nu}"'] \arrow[dr, "s^2 \circ f^2"] & \mathscr{E}^{0} \arrow[d, "-d^{0}_{\mathscr{E}^\bullet}"] \\
 & 0 = Z_\nu^{1} \arrow[r, "\psi^2"] \arrow[d, "-d^{1}_{Z_\nu}"'] \arrow[dr, "s^3 \circ f^3"] & \mathscr{E}^{1} \arrow[d, "-d^{1}_{\mathscr{E}^\bullet}"] \\
 & 0 = Z_\nu^{2} \arrow[r, "\psi^3"] \arrow[d] & \mathscr{E}^{2} \arrow[d] \\
 & \cdots & \cdots
\end{tikzcd}
\tag{4.15}
$$

We can read off immediately that if $n \neq 0, 1$, then $\psi^n = 0$ and, more to the point, if $n \neq 1, 2$, then $s^n \circ f^n = 0$. However, (4.7) provides that if $n \neq 0$, then $i^n = 0$ (use (4.10), mutatis mutandis), forcing $s^n \circ f^n = 0$ for all $n \in \mathbb{Z}$ with the only possibly nontrivial annihilation (with $s^n \neq 0$, possibly) occurring in degree zero. It follows that f should satisfy the commutativity

$$\begin{tikzcd} \mathscr{F} = Z_\nu^{-1} \arrow[r, "f^0"] \arrow[dr, "0"'] & \mathscr{C}^0 = \mathscr{C} \arrow[d, "s^0"] \\ & \mathscr{E}^0 \end{tikzcd} \tag{4.16}$$

in degree zero, and this is just a diagram in $\mathfrak{A}$. Furthermore, the degenerate nature of $\mathscr{C}^\bullet$, together with the requirement from Proposition 4.1 that s should be a quasi-isomorphism, yields that $H^n(\mathscr{E}^\bullet) = 0$ if $u \neq 0$ and $H^0(\mathscr{E}^\bullet) = \mathscr{E}^0 \cong H^0(\mathscr{C}^\bullet) = \mathscr{C}^\bullet = \mathscr{C}$. In other words, s^0 is a sheaf isomorphism, forcing immediately that $f^0 = 0$ in $\mathrm{Mor}(\mathfrak{A})$. Finally, it follows from the surjectivity of the (natural) map

$$\begin{gathered} \mathrm{Hom}_{\mathfrak{D}}\left(Z_\nu^\bullet[-1], \mathscr{C}^\bullet\right) \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(\mathscr{F}, \mathscr{C}), \\ \left[f : \mathscr{F}^\bullet \oplus B^\bullet[-1] \longrightarrow \mathscr{C}^\bullet\right] \longmapsto \left[f^0 : \mathscr{F} \longrightarrow \mathscr{C}\right] \end{gathered} \tag{4.17}$$

that $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{F}, \mathscr{C}) = 0$ too. The converse is obvious. □

The implications of this result for getting at Φ in (2.4) and, therefore, for the entire formalism represented by (2.4), (2.2), and (2.3), are dramatic. Specifically, in (2.3) we obtain that the composite morphism $\nu^\circ \circ \iota^0 \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{F}, \mathscr{C})$ has to vanish:

$$\begin{tikzcd} & & \mathscr{B} & \\ \mathscr{A} \arrow[r, "\iota"] & \mathscr{F} \arrow[ur, "\nu"] \arrow[rr, "0"] \arrow[dr, "\iota^\circ"'] & & \mathscr{C} \\ & & \mathscr{D} \arrow[ur, "\nu^0"'] & \end{tikzcd} \tag{4.18}$$

Consequently, if we abuse notation over more and just write Φ for $\Phi^0 = \epsilon \circ \Phi \circ \delta$ in (3.8), we get the sheaf diagram

$$\begin{tikzcd} & & \mathscr{B} \arrow[dr, dotted, "\Phi"] & \\ \mathscr{A} \arrow[r] & \mathscr{F} \arrow[ur, "\nu"] \arrow[rr, "0"] \arrow[dr, "\iota^0"'] & & \mathscr{C} \\ & & \mathscr{D} \arrow[ur, "\nu^0"'] & \end{tikzcd} \tag{4.19}$$

The upshot for future work is that our sheaf $\mathcal{F}$, as well as its "neighbors" $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, should be designed so as to satisfy the conditions

$$\nu^0 \circ \iota^0 = 0 = \Phi \circ \nu. \tag{4.20}$$

Appearances notwithstanding, this requirement is not that preclusive. In fact, already in the abelian category of sheaves of abelian groups on a space, X, it is easy to arrange nontrivial, or nondegenerate, sheaves $\mathcal{F}$, $\mathcal{G}$ on X with $\mathrm{Hom}_{\mathfrak{Sh}/X}(\mathcal{F},\mathcal{G}) = 0$: for example, take $\mathcal{F} = (\mathbb{Z}_3)_X$ and $\mathcal{G} = (\mathbb{Z}_2)_X$. However, the sheaves in (4.19), which is to say in (2.2) completed by Φ, will undoubtedly have to be considerably more sophisticated in view of what we will be asking of them as far as n-Hilbert reciprocity is concerned. The nature of the underlying toplogical spaces (cf. (2.1)) driving our quasi-duality also augurs strongly for this, and (2.5) will obviously also have its due. In light of such objectives, (4.20) begins to appear as an aid rather than an obstacle.

Regarding the parallel covariant option, it is evident that similar calculations can be brought to bear on the matter of ψ's existence (see (2.6)) as a consequence of having $\mathrm{Hom}_{\mathfrak{D}}(\mathcal{C}^\bullet, Z^\bullet) = 0$, where now $Z^\bullet$ is a mapping cone of $\eta : \mathcal{B}^\bullet \to \mathcal{F}^\bullet$ (see Proposition 2.1).

5. Adjointness

The thrust of the foregoing considerations is that locating our quasi-dual Kubota formalism in the abelian category $\mathfrak{Sh}/_{\tilde{X}_{\mathbb{A}}}$ leads to the task of designing $\mathcal{F}$ such that $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{F},\mathcal{C}) = 0$, and by means of Proposition 4.3, to the observation that if we use the "concentration in degree-zero" convention for situating $\mathfrak{A}$ in its derived category, we really do not gain anything. So, let us abandon this convention for the moment, which is to say that we suggest that δ, ϵ, as per (3.2), (3.4), should be rather more sophisticated mappings, and observe by way of synopsis that in this more liberal environment the idea is to design a sheaf complex $\mathcal{F}^\bullet$, of some appropriate arithmetical character, subject to $\mathrm{Hom}_{\mathfrak{D}}(Z_\nu^\bullet[-1]\mathcal{C}^\bullet) = 0$ or, for the covariant option, $\mathrm{Hom}_{\mathfrak{D}}(\mathcal{C}^\bullet, Z_\eta^\bullet) = 0$, with $\mathfrak{D} = D(\mathfrak{A})$. We saw that $Z_\nu^\bullet$ (resp., $Z_\eta^\bullet$) can actually be any mapping cone of $\nu : \mathcal{F}^\bullet \to \mathcal{C}^\bullet$ (resp., $\eta : \mathcal{B}^\bullet \to \mathcal{F}^\bullet$).

Furthermore, recalling (cf. (2.2), (2.3), (2.4)) that $\mathcal{C}^\bullet = (m^?_{\xi_0;c_{\mathbb{A}}^{(n)}})_? m_{\xi_0;c_{\mathbb{A}}^{(n)}} \mathcal{F}^\bullet$ and $\mathcal{B}^\bullet = ((i \otimes 1) \circ m_0)_? ((i \otimes 1) \circ m_0)^? \mathcal{F}^\bullet$, we are faced with the additional task of assigning "values" to ?'s chosen from $*$, !, modulo Verdier's R, all still in the cause of bringing about the vanishing of one of the above Hom-groups. It stands to reason that adjointness should be a major player in this part of the game, and so we devote the present section to this topic.

The general situation we are facing is this if $Y \xrightarrow{f} X$ is a continuous function acting between topological spaces and if $\mathcal{F}$ (resp., $\mathcal{G}$) is a sheaf on X (resp., Y), then f_* and f^*, respectively, direct and inverse image (with their usual definitions), comprise an adjoint pair as follows:

$$\mathrm{Hom}_{\mathfrak{Sh}/X}(\mathcal{F}, f_*\mathcal{G}) \cong \mathrm{Hom}_{\mathfrak{Sh}/Y}(f^*\mathcal{F}, \mathcal{G}). \tag{5.1}$$

If $\mathscr{F}^\bullet$ (resp., $\mathscr{G}^\bullet$) lives in $D^+(\mathfrak{Sh}/_X)$ (resp., $D^+(\mathfrak{Sh}/_Y)$), then this adjointment becomes

$$\mathrm{Hom}_{D^+(\mathfrak{Sh}/_X)}(\mathscr{F}^\bullet, Rf_*\mathscr{G}^\bullet) \cong \mathrm{Hom}_{D^+(\mathfrak{Sh}/_Y)}(f^*\mathscr{F}^\bullet, \mathscr{G}^\bullet), \tag{5.2}$$

where, in general terms, $D^+(\mathfrak{A})$ is the full subcategory of $D(\mathfrak{A})$ consisting of derived sheaf complexes vanishing in sufficiently low degrees; Rf_* is required due to f_* being merely left exact instead of exact. Next, the functor $f_!$, "direct image with proper supports," realizes in $f_!\mathscr{G}$ a subsheaf of $f_*\mathscr{G}$, and then, taking things to the next level once more, the according-derived function $Rf_!$ realizes in $Rf_!\mathscr{G}^\bullet$ a subcomplex of $Rf_*\mathscr{G}^\bullet$. In the derived category, setting this engenders that $Rf_!$ admits an adjoint functor $f^!$, so that

$$\mathrm{Hom}_{D^+(\mathfrak{Sh}/_X)}(Rf_!\mathscr{G}^\bullet, \mathscr{F}^\bullet) \cong \mathrm{Hom}_{D^+(\mathfrak{Sh}/_Y)}(\mathscr{G}^\bullet, f^!\mathscr{F}^\bullet). \tag{5.3}$$

The details of all this, replete with carefully presented definitions and constructions, are given in [12, Chapters II and III].

We now specialize to the case $Y = X_0^2 = \mathrm{SL}_2(k)^2$, $X = \widetilde{X}_{\mathbf{A}} = \widetilde{\mathrm{SL}}_2(k)_{\mathbf{A}}^{(n)}$, and $f = (i \otimes 1) \circ m_0$, which, for the sake of brevity, we continue to denote by f, under these circumstances, we get immediately that $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{F}, \mathscr{C}) = \mathrm{Hom}_{\mathfrak{A}}(\mathscr{F}, f_? f^? \mathscr{F})$, with $\mathfrak{A} := \mathfrak{Sh}/_{\widetilde{X}_{\mathbf{A}}}$, whereas $\mathrm{Hom}_{\mathfrak{D}_{\mathfrak{A}}^+}(Z_\nu^\bullet[-1], \mathscr{C}^\bullet) = \mathrm{Hom}_{\mathfrak{D}_{\mathfrak{A}}^+}(Z_\nu^\bullet[-1], Rf_? f^? \mathscr{F}^\bullet)$ and $\mathrm{Hom}_{\mathfrak{D}_{\mathfrak{A}}^+}(\mathscr{C}^\bullet, Z_\eta^\bullet) = \mathrm{Hom}_{\mathfrak{D}_{\mathfrak{A}}^+}(Rf_? f^? \mathscr{F}^\bullet, Z_\eta^\bullet)$, where $\mathfrak{D}_{\mathfrak{A}}^+ = D^+(\mathfrak{Sh}/_{\widetilde{X}_{\mathbf{A}}})$. We also set $\mathfrak{B} := \mathfrak{Sh}/_{X_0^2}$ and $\mathfrak{D}_{\mathfrak{B}}^+ := D^+(\mathfrak{Sh}/_{X_0^2})$. Then, we have the following.

Proposition 5.1. *In the setting of sheaves, that is, of sheaf categories, the existence of* Φ *follows if* $\mathrm{Hom}_{\mathfrak{B}}(f^*\mathscr{F}, f^*\mathscr{F}) = 0$. *In the setting of derived sheaf categories, the existence of* Φ *follows if* $\mathrm{Hom}_{\mathfrak{D}_{\mathfrak{B}}^+}(f^*Z_\nu^\bullet[-1], f^*\mathscr{F}^\bullet) = 0$, *while (again for the covariant option) the existence of* ψ *follows if* $\mathrm{Hom}_{\mathfrak{D}_{\mathfrak{B}}^+}(f^!\mathscr{F}^\bullet, f^!Z_\eta^\bullet) = 0$.

Proof. In [1, Proposition 7.1], the existence of Φ follows if $\mathrm{Hom}_{\mathfrak{D}}(Z_\nu^\bullet[-1], \mathscr{C}^\bullet) = 0$, which, by means of Proposition 4.3, is equivalent to having $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{F}, \mathscr{C}) = 0$, that is, $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{F}, f_* f^* \mathscr{F}) = 0$, setting each ? equal to $*$ as regards $\mathscr{C}$. Applying (5.1) with $\mathscr{G} = f^*\mathscr{F}$ immediately gives that $\mathrm{Hom}_{\mathfrak{A}}(f^*\mathscr{F}, f^*\mathscr{F}) = 0$. Going on to the derived category setting, we observe that all the relevant sheaf complexes that have figured in the foregoing assertions are (trivially) situated in $\mathfrak{D}_{\mathfrak{A}}^+$ or $\mathfrak{D}_{\mathfrak{B}}^+$, whence we can safely invoke (5.2) instead of (5.1) to get that in this setting, too, the existence of Φ follows if $0 = \mathrm{Hom}_{\mathfrak{D}_{\mathfrak{A}}^+}(Z_\nu^\bullet[-1], Rf_* f^* \mathscr{F}^\bullet) \cong \mathrm{Hom}_{\mathfrak{D}_{\mathfrak{B}}^+}(f^* Z_\nu^\bullet[-1], f^*\mathscr{F}^\bullet)$. Finally, utilizing (5.3), we obtain that ψ's existence follows if $0 = \mathrm{Hom}_{\mathfrak{D}_{\mathfrak{A}}^+}(Rf_! f^! \mathscr{F}^\bullet, Z_\eta^\bullet) = \mathrm{Hom}_{\mathfrak{D}_{\mathfrak{B}}^+}(f^!\mathscr{F}^\bullet, f^! Z_\eta^\bullet)$, *via* Proposition 2.1. $\square$

Added to the tasks set out in Section 3, the content of Proposition 5.1 is to provide us with marching orders down the first of the two paths mentioned in Section 1, the objective being n-Hilbert reciprocity as a consequence of the indicated factorization(s) of a sheaf- or sheaf-complex morphism in our quasi-dualized Kubota formalism.

6. Another first-level diagram

We now take up the second theme discussed in Section 1, namely, the development of a calculus of t-structures on a network of exact triples of derived categories.

The diagram (2.1), of Part I, restated in Section 2, captures the essence of our restructuring of Kubota's approach to n-Hilbert reciprocity in terms of topological spaces instead of algebraic groups; see also [1, diagram (3.6)]. It is now indicated that we look at this diagram in $\mathfrak{Top}$ (the locale for the first level of our quasi-dual construct) more carefully so as to become able to identify the right-derived categories for the purpose of applying Proposition 8-1 in an *avant-garde* fashion.

One more time, then

$$\begin{tikzcd}
\mu^2 \arrow[d, "m_\ell"'] & \coprod_{\ell=1}^{\infty} X_{\xi_0;\ell} \arrow[l, "i^0 \otimes j^0"'] \arrow[d, "m_{\xi_0;c_{\mathbf{A}}^{(n)}}"] & \\
\mu & \widetilde{X}_{\mathbf{A}} \arrow[l, "j^0"'] & X_0^2 \arrow[ul, dotted, "\Omega_{\xi_0}"'] \arrow[d, "m_0"] \\
 & & X_0 \arrow[ull, dotted, bend left, "s_{\mathbf{A}}"] \arrow[uull, dotted, "s_{\mathbf{A}} \otimes s_{\mathbf{A}}"']
\end{tikzcd} \tag{6.1}$$

for all $\xi_0 \in \mu_n = \mu$, with $\widetilde{X}_{\mathbf{A}} = \mathrm{SL}_2(k)_{\mathbf{A}} \times_{c_{\mathbf{A}}^{(n)}} \mu_n$, $X_0 = \mathrm{SL}_2(k)$; also, as stated in [1, equation (4.19)], $\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}$ is the set of all quadruples $(\sigma,\sigma',\xi,\xi')$ of adelic 2×2 matrices $\sigma,\sigma' \in \mathrm{SL}(k)_{\mathbf{A}}$ and roots of unity $\xi,\xi' \in \mu_n$ such that $c_{\mathbf{A}}^{(n)}(\sigma,\sigma') = \xi_0$. Partitioning the latter collection into sets $X_{\xi_0;\ell}$, indexed on $\ell \geq 1$, entails identifying adèlic pairs (σ,σ') in accord with the particular local action induced by $c_{\mathbf{A}}^{(n)} = \otimes_{\mathfrak{p}} c_{\mathfrak{p}}^{(n)}$, where, with $\mathfrak{p}$ ranging over the places of k, we have that $c_{\mathfrak{p}}^{(n)} \in H^2(\mathrm{SL}_2(k_{\mathfrak{p}}),\mu_n)$. This local action on pairs (σ,σ') yields a notion of length, ℓ (see [1, Section 4]); however, for our upcoming purposes, the facts that each $\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}$ is nothing else than $(c_{\mathbf{A}}^{(n)})^{-1}(\xi_0) \times \mu^2$ and each $X_{\xi_0;\ell}$ is locally closed (see [1, Corollaries 4.5 and 4.6]) are much more salient; for further specifics regarding the $X_{\xi_0;\ell}$ (e.g., their rather cumbersome definition, which we will not need at this time), we refer to [1, Section 4], especially [1, equation (4.18)]. Finally, restating [1, equation (4.19)], we have that

$$\widetilde{X}_{\mathbf{A}}^2 = \left(\mathrm{SL}_2(k)_{\mathbf{A}} \times \mu_n\right)^2 = \coprod_{\xi_0 \in \mu} \coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}. \tag{6.2}$$

Proposition 5.1 of [1] provides that (i) the existence of $s_{\mathbf{A}}$ and $s_{\mathbf{A}} \otimes s_{\mathbf{A}}$ is equivalent to that of $\Omega = \bigotimes_{\xi_0} \Omega\xi_0$, with (ii) $s_{\mathbf{A}}$ and $s_{\mathbf{A}} \otimes s_{\mathbf{A}}$ group homomorphisms if and only if $c_{\mathbf{A}}^{(n)}|_{X_0^2} \equiv 1$, which just says that $\mathrm{im}(\Omega) \subset \coprod_{\ell=1}^{\infty} X_{1;\ell}$, so that we recover Kubota's phrasing of n-Hilbert reciprocity to the letter. Additionally, (iii) id $\otimes s_{\mathbf{A}}$ splits $\widetilde{X}_{\mathbf{A}}$ on X_0 (Kubota redux) if and only if

$$\begin{tikzcd}
\mu & \widetilde{X}_{\mathbf{A}} \arrow[l, "j^0"'] & \coprod_{\ell=1}^{\infty} X_{\xi_0;\ell} \arrow[l, "m_{\xi_0;c_{\mathbf{A}}^{(n)}}"'] \\
X_0 \arrow[u, dotted, "s_{\mathbf{A}}"] & & X_0^2 \arrow[ll, "m_0"'] \arrow[u, dotted, "\Omega_{\xi_0}"']
\end{tikzcd} \tag{6.3}$$

(isolating the critical part of (6.1)) for all $\xi_0 \in \mu$; finally, (iv) of Proposition 5.1 is essentially a generalization of (ii). We can, in light of (ii), state the following proposition.

Proposition 6.1. *If one sets, first,*

$$\widetilde{X}_0 := \mathrm{SL}_2(k) \times \mu \tag{6.4}$$

and then define (with, generally $\overline{X}$ for the closure of X)

$$Y_{\xi_0} := \widetilde{X}_0^2 \cap \overline{\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}}, \tag{6.5}$$

then n-Hilbert reciprocity follows if, for all $\xi_0 \neq 1$ in μ, $Y_{\xi_0} = \varnothing$.

Proof. The stated condition implies immediately that unless $\xi_0 = 1$, the action of Ω_{ξ_0} is null, that is, $\mathrm{im}(\Omega) \subset \coprod_{\ell=1}^{\infty} X_{1;\ell}$. □

Corollary 6.2. *If $\overline{Y}_{\xi_0} = \varnothing$ for all $\xi_0 \neq 1$, then n-Hilbert reciprocity follows.*

Proof. The proof is obvious (and trivial). □

Now let $\check{X}_{\xi_0}$ denote the closure of $\coprod_{\ell=1}^{\infty} X_{\xi_0;\ell}$ in $\widetilde{X}_{\mathbf{A}}^2$, rather than the more cumbersome expression in (6.5), so that $U_{\xi_0} = \widetilde{X}_0^2 \cap \check{X}_{\xi_0}$ and $\overline{Y}_{\xi_0} \subseteq \check{X}_{\xi_0}$. We identify the following attendant (or, in a sense to become clear immediately, neighboring) topological spaces: $\check{U}_{\xi_0}^{\text{open}} = \widetilde{X}_{\mathbf{A}}^2 \setminus \check{X}_{\xi_0}$, $W_{\xi_0}^{\text{open}} = \check{X}_{\xi_0} \setminus \overline{Y}_{\xi_0}$, $U_{\xi_0}^{\text{open}} = \widetilde{X}_{\mathbf{A}}^2 \setminus Y_{\xi_0}$, and, finally, $Z_{\xi_0}^{\text{closed}} = U_{\xi_0} \setminus \check{U}_{\xi_0} = \check{X}_{\xi_0} \setminus \overline{Y}_{\xi_0} = (\widetilde{X}_{\mathbf{A}}^2 \setminus \check{X}_{\xi_0}) \setminus (\widetilde{X}_{\mathbf{A}}^2 \setminus \overline{Y}_{\xi_0})$. Observe that, as sets, W_{ξ_0} and Z_{ξ_0} are the same, that is, they agree with $\check{X}_{\xi_0} \setminus \overline{Y}_{\xi_0}$, but we opt to distinguish them here because we wish to regard W_{ξ_0} as relatively open in $\check{X}_{\xi_0}$ and Z_{ξ_0} as relatively closed in U_{ξ_0}; the latter condition hinges on $\check{U}_{\xi_0}$ being open (in the open set U_{ξ_0}: everything starts with $\overline{Y}_{\xi_0}$ being closed), which clearly results from $\check{X}_{\xi_0}$ being closed.

We can arrange these spaces in the diagram

$$\begin{tikzcd}
\overline{Y}_{\xi_0} \arrow[d, "\hat{i}_{\xi_0}"'] \arrow[dr, "i_{\xi_0}"] & & \\
\check{X}_{\xi_0} \arrow[r, "\check{i}_{\xi_0}"] & \widetilde{X}^2_{\mathbf{A}} & \check{U}_{\xi_0} \arrow[l, "\check{j}_{\xi_0}"'] \arrow[d, "\hat{j}_{\xi_0}"] \\
W_{\xi_0} \arrow[u, "j"] & & U_{\xi_0} \arrow[ul, "j_{\xi_0}"'] \\
& & Z_{\xi_0} \arrow[u, "i"']
\end{tikzcd} \tag{6.6}$$

In anticipation of the appearance of the methodology of gluing t-structures on the scene, we observe that each of the four morphism pairs in (6.6), that is, $(\hat{i}_{\xi_0}, j)$, $(\check{i}_{\xi_0}, \check{j}_{\xi_0})$, (i_{ξ_0}, j_{ξ_0}), and $(i, \hat{j}_{\xi_0})$, is of the form

$$Y^{\text{closed}} \xrightarrow{i} X \xleftarrow{j} U^{\text{open}} = X \setminus Y \tag{6.7}$$

(for general i, j) and, as we will see in Section 8, it is standard for that such a stratification of X as $Y \amalg U$ gives rise to an exact triple

$$\mathfrak{D}_Y = D^+(\mathfrak{S}\mathfrak{h}/_Y) \xrightarrow{i_*} \mathfrak{D}_X = D^+(\mathfrak{S}\mathfrak{h}/_X) \xrightarrow{j^*} \mathfrak{D}_U = D^+(\mathfrak{S}\mathfrak{h}/_U) \tag{6.8}$$

of derived categories which inflates into a diagram (cf. (7.17) below) with six exact functors arranged into four adjoint pairs; this arrangement carries the germ of gluing data for any pair of t-structures on $\mathfrak{D}_Y$ and $\mathfrak{D}_U$.

Before we address this avant-garde material, however, we recall and collect some general facts about derived categories as triangulated categories and the k-structures they support.

7. Some category-theoretical generalities

References for the material in this section include [13, 14] by Gelfand and Manin, [12] by Kashiwara and Schapira, [8] by Kiehl and Weissauer, and of course [15] by Beĭlinson et al.

An additive category is triangulated if it is equipped with a shift operator $A \mapsto A[1]$, for every object A, giving rise to distinguished triangles characterized by the following conditions. If (A,B,C) is distinguished, with A, B, C objects of the given category, then so is $(B,C,A[1])$; any morphism $A \to B$ can be fitted into a distinguished triangle (A,B,C); ordinary minimal commutative diagrams can be inflated into morphisms of distinguished triangles:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd} \tag{7.1}$$

completes to

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r, "+1"] \arrow[d, dotted] & {} \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r, "+1"] & {} \end{tikzcd} \tag{7.2}$$

where we use the more evocative notation $A \to B \to C \xrightarrow{+1}$ for a distinguished triangle (A,B,C); and, lastly, the so-called *Oktaederaxiom* holds, for an account of which we refer the reader to any of the sources mentioned above. The most striking (and singularly useful) result in this connection is that a distinguished triangle always gives rise to a pair of long exact sequences, a feature already used to some advantage in Section 4.

By definition, a t-structure on a triangulated category, $\mathfrak{D}$, is a pair $(\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0}) =: t(\mathfrak{D})$ of full subcategories such that, writing $\mathfrak{D}^{\leq n} := \mathfrak{D}^{\leq 0}[-n]$ and $\mathfrak{D}^{\geq n} =: \mathfrak{D}^{\geq 0}[-n]$, we have, first, that $\mathfrak{D}^{\leq 0} \subset \mathfrak{D}^{\geq 1}$ and $\mathfrak{D}^{\geq 0} \supset \mathfrak{D}^{\geq 1}$; second, that if $A \in \mathfrak{D}^{\leq 0}$, $B \in \mathfrak{D}^{\geq 1}$, then $\mathrm{Hom}_{\mathfrak{D}}(A, B) = 0$; and third, that, functorially, for every $A \in \mathfrak{D}$, there exist objects $\tau^{\leq 0}A \in \mathfrak{D}^{\leq 0}$ and $\tau^{\geq 1}A \in \mathfrak{D}^{\geq 1}$ such that

$$\tau^{\leq 0}A \longrightarrow A \longrightarrow \tau^{\geq 1}A \xrightarrow{+1} \tag{7.3}$$

is a distinguished triangle. Indeed, it is the case that for all n, there exist functors, the so-called truncation functors,

$$\tau^{\geq n} : \mathfrak{D} \longrightarrow \mathfrak{D}^{\geq n}, \tag{7.4a}$$

$$\tau^{\geq n} : \mathfrak{D} \longrightarrow \mathfrak{D}^{\leq n}, \tag{7.4b}$$

for which the following convenient facts hold true: $\tau^{\leq n}A = 0$ if and only if $A \xrightarrow{\sim} \tau^{\geq n+1}A$; for all $p \geq 0$, we can identify $\tau^{\leq m+p}$ with $\tau^{\leq m}$ and $\tau^{\geq m-p}$ with $\tau^{\geq m}$; also, for all $p \geq 0$, we can identify $\tau^{\geq m}\tau^{\leq m+p}$ with $\tau^{\leq m+p}\tau^{\geq m}$, and then we write $\tau_{m,m+p} := \tau^{\geq m}\tau^{\leq m+p} = \tau^{\leq m+p}\tau^{\geq m}$. We obtain in particular that

$$H^0 := \tau_{0,0} = \tau^{\geq 0}\tau^{\leq 0} = \tau^{\leq 0}\tau^{\geq 0} : \mathfrak{D} \longrightarrow \mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0} \tag{7.5}$$

is a cohomological functor. Writing core $t(\mathfrak{D}))$ for $\mathfrak{D}^{\leq 0} \cap \mathfrak{D}^{\geq 0}$, by definition, the core of the given t-structure, we get

$$H^n := H^0(\_[n]) : \mathfrak{D} \longrightarrow \text{core}\,(t(\mathfrak{D})). \tag{7.6}$$

The latter category, core $(t(\mathfrak{D}))$, is abelian; furthermore, if $\mathfrak{D}$ is the derived category of a given abelian category, $\mathfrak{A}$, which is to say that $\mathfrak{D} = D(\mathfrak{A})$ and $t(\mathfrak{D})$ is the canonical t-structure on $\mathfrak{D}$ (cf. [12, page 33ff.]), so that "truncation" has its original meaning, we obtain that

$$\operatorname{core} t(\mathfrak{D}) = \operatorname{core} t\big(D(\mathfrak{A})\big) = \mathfrak{A}, \tag{7.7}$$

and the formulations (7.5), (7.6) take on their more familiar meaning. For example, if $\mathfrak{A}$ is a sheaf category, we recover cohomology sheaves in this way.

Returning to the general context of a triangulated, but not necessarily derived, category, the definitive result concerning the H^n is the following.

Proposition 7.1. *If $t(\mathfrak{D})$ is nondegenerate, meaning that $\cap_n \mathfrak{D}^{\leq n} = (0) = \cap_n \mathfrak{D}^{\geq n}$, then*
(i) *$A \xrightarrow{\sim} B$ in $\mathfrak{D}$ if and only if $H^n(A) \xrightarrow{\sim} H^n(B)$ for all n*
(ii) *$\mathfrak{D}^{\leq n} = \{A \in \mathfrak{D} \mid H^\nu(A) = 0$ if $\nu > n\}$, $\mathfrak{D}^{\geq n} = \{A \in \mathfrak{D} \mid H^\nu(A) = 0$ if $\nu < n\}$.*

Proof. See, for example, [13, page 135]. □

Regarding the matter of a t-structure's degeneracy, as just mentioned, in light of future considerations, it behooves us to remark that this can occur in two ways: either there exists $A \neq 0$ in $\mathfrak{D}$ such that $A \in \cap_n \mathfrak{D}^{\leq n}$ or $A \in \cap_n \mathfrak{D}^{\geq n}$, or we have that $\cap_n \mathfrak{D}^{\leq n} = \varnothing = \cap_n \mathfrak{D}^{\geq n}$. For our upcoming purposes, the sort of degeneracy that counts is the latter, seeing that we will presently be concerned with t-structures for which

$$\operatorname{core} t(\mathfrak{D}) = \mathfrak{D}^{\leq} \cap \mathfrak{D}^{\geq 0} = \varnothing. \tag{7.8}$$

We take the liberty of referring to this kind of t-structure as strongly degenerate.

Next, suppose that $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ are triangulated categories and that $\mathfrak{C}$ and $\mathfrak{E}$ are equipped with t-structures $t(\mathfrak{C}) = (\mathfrak{C}^{\leq 0}, \mathfrak{C}^{\geq 0})$, $t(\mathfrak{E}) = (\mathfrak{E}^{\leq 0}, \mathfrak{E}^{\geq 0})$. Suppose, too, that we have an inclusion functor $\mathfrak{C} \xrightarrow{P} \mathfrak{D}$ such that $P(\mathfrak{C})$, identified with $\mathfrak{C}$, is a thick subcategory of $\mathfrak{D}$, and a localization functor $\mathfrak{D} \xrightarrow{Q} \mathfrak{E}$ rendering $\mathfrak{E}$ the localization of $\mathfrak{D}$ at the class of quasi-isomorphisms imported from $\mathfrak{C}$. Under these circumstances, we say that

$$\mathfrak{C} \xrightarrow{P} \mathfrak{D} \xrightarrow{Q} \mathfrak{E} \tag{7.9}$$

engenders an exact triple and we obtain the following critical fact.

Proposition 7.2. *If P, Q in the exact triple (7.9) possess both left and right adjoints, then $t(Z)$ and $t(\mathfrak{E})$ determine a t-structure*

$$t(\mathfrak{D}) := t(\mathfrak{C}) \wedge t(\mathfrak{E}) \tag{7.10}$$

on $\mathfrak{D}$ by means of the prescriptions

$$\mathfrak{D}^{\leq 0} = \{A \in^{\perp} (P\mathfrak{C}^{>0}) \mid Q(A) \in \mathfrak{E}^{\leq 0}\}, \tag{7.11a}$$

$$\mathfrak{D}^{\geq 0} = \{A \in (P\mathfrak{C}^{<0})^{\perp} \mid Q(A) \in \mathfrak{E}^{\geq 0}\}, \tag{7.11b}$$

where

$$^{\perp}(P\mathfrak{C}^{>0}) = \{A \in \mathfrak{D} \mid \operatorname{Hom}(A,B) = 0\ \forall B \in P\mathfrak{C}^{>0}\}, \tag{7.12a}$$

$$(P\mathfrak{C}^{<0})^{\perp} = \{A \in \mathfrak{D} \mid \operatorname{Hom}(B,A) = 0\ \forall B \in P\mathfrak{C}^{<0}\}. \tag{7.12b}$$

Proof. See, for example, [13, page 137]. □

Under these circumstances, $t(\mathfrak{D})$ is said to be the result of gluing $t(Z)$ and $t(\mathfrak{E})$ (in the indicated order). Well definition is taken care of by the fact that $P(\mathfrak{C}^{\leq 0}) \subset \mathfrak{D}^{\leq 0}$, $P(\mathfrak{C}^{\geq 0}) \subset \mathfrak{D}^{\geq 0}$, $Q(\mathfrak{D}^{\leq 0}) \subset \mathfrak{E}^{\leq 0}$, $Q(\mathfrak{D}^{\geq 0}) \subset \mathfrak{E}^{\geq 0}$. We take the liberty to complement our notation (7.10) by the diagram

$$\begin{tikzcd} t(\mathfrak{C}) \arrow[d] & t(\mathfrak{C}) \wedge t(\mathfrak{E}) \arrow[d, dotted] & t(\mathfrak{E}) \arrow[d] \\ \mathfrak{C} \arrow[r, "P"] & \mathfrak{D} \arrow[r, "Q"] & \mathfrak{E} \end{tikzcd} \tag{7.13}$$

Going in the other direction, suppose now that (7.9) supports a t-structure at $\mathfrak{D}$, being $t(\mathfrak{D}) = (\mathfrak{D}^{\leq 0}, \mathfrak{D}^{\geq 0})$. Then, the simple manoeuvres $\mathfrak{C}^{\geq 0} := \mathfrak{D}^{\leq 0} \cap \mathfrak{C}$, $\mathfrak{C}^{\geq 0} := \mathfrak{D}^{\geq 0} \cap \mathfrak{C}$, $\mathfrak{E}^{\leq 0} := Q(\mathfrak{D}^{\leq 0})$, and $\mathfrak{E}^{\geq 0} := Q(\mathfrak{D}^{\geq 0})$ provide t-structures on $\mathfrak{C}$, $\mathfrak{E}$, respectively; that is, we get $t(\mathfrak{C}) = t(\mathfrak{D}) \cap \mathfrak{C}$, $t(\mathfrak{E}) = Q(t(\mathfrak{D}))$, in more succinct jargon. We will call this process "ungluing" and encode it by the diagram

$$\begin{tikzcd} t(\mathfrak{D}) \cap \mathfrak{C} \arrow[d, dotted] & t(\mathfrak{D}) \arrow[d] & Q(t(\mathfrak{D})) \arrow[d, dotted] \\ \mathfrak{C} \arrow[r, "P"] & \mathfrak{D} \arrow[r, "Q"] & \mathfrak{E} \end{tikzcd} \tag{7.14}$$

The status of gluing and ungluing as relative inverse operations is captured by the following (formal) result.

Proposition 7.3. *If (7.13) is in effect, then* $t(\mathfrak{C}) = [t(\mathfrak{C}) \wedge t(\mathfrak{E})] \cap \mathfrak{C}$ *and* $t(\mathfrak{E}) = Q(t(\mathfrak{C}) \wedge t(\mathfrak{E}))$. *Dually, if (7.14) is in effect, then* $t(\mathfrak{D}) = [t(\mathfrak{D}) \cap \mathfrak{C}] \wedge Q(t(\mathfrak{D}))$.

Proof. Left to the reader. □

The stipulation that the exact triple $\mathfrak{C} \xrightarrow{P} \mathfrak{D} \xrightarrow{Q} \mathfrak{E}$ admits gluing data in the sense that P and Q admit both left and right adjunct functors (which, by the way, is true for P if and only if it is true for Q; see [13, page 137]) is in particular satisfied when $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ are the respective derived sheaf categories attached to a topological space stratification of the type

$$Y \xrightarrow{i} X \xleftarrow{j} U, \tag{7.15}$$

with Y closed, U open, $X = Y \coprod U$. With $\mathfrak{D}_{-} = D^{+}(\mathfrak{Sh}/_{-})$, we then explicitly get that the triple

$$\mathfrak{D}_Y \xrightarrow{i_*} \mathfrak{D}_X \xrightarrow{j^*} \mathfrak{D}_U \tag{7.16}$$

is exact (as we already mentioned in the previous section), and we get gluing data given by

$$\begin{tikzcd} \mathfrak{D}_Y \arrow[r, "i_*"] & \mathfrak{D}_X \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathfrak{D}_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \end{tikzcd} \tag{7.17}$$

where each of the indicated exact functors is in fact left adjoint to the one directly below it. To wit,

$$\operatorname{Hom}_{\mathfrak{D}_Y}(i^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong \operatorname{Hom}_{\mathfrak{D}_X}(\mathcal{F}^\bullet, i_*\mathcal{G}^\bullet), \tag{7.18a}$$

$$\operatorname{Hom}_{\mathfrak{D}_X}(i_*\mathcal{G}^\bullet, \mathcal{F}^\bullet) \cong \operatorname{Hom}_{\mathfrak{D}_Y}(\mathcal{G}^\bullet, i^!\mathcal{F}^\bullet), \tag{7.18b}$$

$$\operatorname{Hom}_{\mathfrak{D}_X}(j_!\mathfrak{H}^\bullet, \mathcal{F}^\bullet) \cong \operatorname{Hom}_{\mathfrak{D}_U}(\mathfrak{H}^\bullet, j^*\mathcal{F}^\bullet), \tag{7.18c}$$

$$\operatorname{Hom}_{\mathfrak{D}_U}(j^*\mathcal{F}^\bullet, \mathfrak{H}^\bullet) \cong \operatorname{Hom}_{\mathfrak{D}_X}(\mathcal{F}^\bullet, Rj_*\mathfrak{H}^\bullet). \tag{7.18d}$$

Beyond this we have the relations

$$i^* j_! = 0, \tag{7.19a}$$

$$j^* i_* = 0, \tag{7.19b}$$

$$i^! R j_* = 0, \tag{7.19c}$$

and the natural transformations

$$\begin{tikzcd} i^* i_* \arrow[r] \arrow[dr, "\cong"'] & \mathrm{id} \arrow[d] \\ & j^! i_* \end{tikzcd} \tag{7.20a}$$

$$\begin{tikzcd} j^* R j_* \arrow[r] \arrow[dr, "\cong"'] & \mathrm{id} \arrow[d] \\ & j^* j_! \end{tikzcd} \tag{7.20b}$$

Finally, there exist morphisms

$$w : i_* i^* \mathcal{F}^\bullet \longrightarrow j_! j^* \mathcal{F}^\bullet[1], \tag{7.21a}$$

$$w' : Rj_* j^* \mathcal{F}^\bullet \longrightarrow i_* \mathcal{F}^\bullet[1], \tag{7.21b}$$

functorial in $\mathcal{F}^\bullet$, such that

$$j_! j^* \mathcal{F}^\bullet \xrightarrow{u} \mathcal{F}^\bullet \xrightarrow{v} i_* i^* \mathcal{F}^\bullet \xrightarrow{w} j_! j^* \mathcal{F}^\bullet[1], \tag{7.22a}$$

$$i_* i^! \mathcal{F}^\bullet \xrightarrow{u'} \mathcal{F}^\bullet \xrightarrow{v} Rj_* j^! \mathcal{F}^\bullet \xrightarrow{w'} i_* i^! \mathcal{F}^\bullet[1], \tag{7.22b}$$

with u, v, w', v' the indicated adjunction morphisms, are distinguished triangles.

It is a particularly marvellous dividend of (7.17), which is to say, of having derived sheaf categories to work with, that (7.11a), (7.11b) simplify as follows.

Proposition 7.4. *If (6.7), (6.8), (7.7) are in effect, then $\mathfrak{D}_X$ acquires a glued t-structure $t(\mathfrak{D}_X) = (\mathfrak{D}_X^{\leq 0}, \mathfrak{D}_X^{\geq 0}) = t(\mathfrak{D}_Y) \wedge t(\mathfrak{D}_u)$ by means of*

$$\mathfrak{D}_X^{\leq 0} = \{\mathcal{F}^\bullet \mid j^*\mathcal{F}^\bullet \in \mathfrak{D}_U^{\leq 0},\ i^*\mathcal{F}^\bullet \in \mathfrak{D}_Y^{\leq 0}\}, \tag{7.23a}$$

$$\mathfrak{D}_X^{\geq 0} = \{\mathcal{F}^\bullet \mid j^*\mathcal{F}^\bullet \in \mathfrak{D}_U^{\geq 0},\ i^!\mathcal{F}^\bullet \in \mathfrak{D}_Y^{\geq 0}\}. \tag{7.23b}$$

Proof. This is [15, Theorem 1.4]. □

And now, we come to perverse sheaves, which we only treat in a cursory manner at this point. Should the need arise, we will return to this matter later.

Seeing that the stratifications of interest are all of the type (6.7), it suffices to look at this situation, that is,

$$X = Y^{\text{closed}} \coprod U^{\text{open}}, \tag{7.24}$$

the most elementary nontrivial case, and present the attendant formalism. By definition, if $\mathcal{S} = \{Y, U\}$ is the stratification given by (7.24) (and (6.7)), a perversity on $\mathcal{S}$ is just a function

$$p : \mathcal{S} \longrightarrow \mathbb{Z}, \tag{7.25}$$

that is, a pair of integers $(p(Y), p(U))$, allowing the reformulation of (7.23a), (7.23b) as follows:

$${}^p\mathfrak{D}_X^{\leq 0} = \{\mathcal{F}^\bullet \mid H^n j^*\mathcal{F}^\bullet = 0 \text{ if } n \geq p(U),\ H^n i^*\mathcal{F}^\bullet = 0 \text{ if } n \geq p(Y)\}, \tag{7.26a}$$

$${}^p\mathfrak{D}_X^{\geq 0} = \{\mathcal{F}^\bullet \mid H^n j^!\mathcal{F}^\bullet = 0 \text{ if } n \leq p(U),\ H^n i^!\mathcal{F}^\bullet = 0 \text{ if } n \leq p(Y)\}, \tag{7.26b}$$

(cf. [13, page 163]). We naturally write ${}^p t(\mathfrak{D}_X) = ({}^p\mathfrak{D}_X^{\leq 0}, {}^p\mathfrak{D}_X^{\geq 0})$ in this circumstance.

8. Linked exact triples of derived categories

Now, we turn our attention to (6.6) and note that, by design, all four morphism pairs $\bullet \longrightarrow \bullet \longleftarrow \bullet$ are of the type (6.7), rendering (6.8) viable. By systematically invoking (6.8), using the standard push-forward and pull-back notation as regards the aforementioned morphism pairs, and bearing in mind that $(f \circ g)_* = f_* \circ g_*$, $(f \circ g)^* = g^* \circ f^*$, we (again) obtain the diagram

$$
\begin{tikzcd}
\mathfrak{D}_{\overline{Y}_{\xi_0}} \arrow[d, "\hat{i}_{\xi_0,*}"'] \arrow[dr, "i_{\xi,0*}"] & & \\
\mathfrak{D}_{\check{X}_{\xi_0}} \arrow[r, "\check{i}_{\xi_0,*}"] \arrow[d, "j^*"'] & \mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}} \arrow[r, "\check{j}^*_{\xi_o}"] \arrow[dr, "j^*_{\xi_0}"'] & \mathfrak{D}_{\check{U}_{\xi_0}} \\
\mathfrak{D}_{W_{\xi_0}} & & \mathfrak{D}_{U_{\xi_0}} \arrow[u, "\hat{j}^*_{\xi_0}"'] \\
 & & \mathfrak{D}_{Z_{\xi_0}} \arrow[u, "i_*"']
\end{tikzcd}
\tag{8.1}
$$

Suppose that we have initial t-structure data situated on the derived sheaf categories $\mathfrak{D}_{\check{X}_{\xi_0}}$, $\mathfrak{D}_{\check{U}_{\xi_0}}$, $\mathfrak{D}_{Z_{\xi_0}}$, denoted by $t(\mathfrak{D}_{\check{X}_{\xi_0}})$, $t(\mathfrak{D}_{\check{U}_{\xi_0}})$, $t(\mathfrak{D}_{Z_{\xi_0}})$, respectively. In anticipation of the arithmetical phase of our investigations, we must allow for the contingency that these t-structures should be perverse, but we will cross this bridge when we get to it. For now, we will work with (7.23a), (7.23b) rather than (7.26a), (7.26b), and the situation at hand is now rendered thus:

$$
\begin{tikzcd}
t(\mathfrak{D}_{\check{X}_{\xi_0}}) \cap \mathfrak{D}_{\overline{Y}_{\xi_0}} \arrow[r, dotted] & \mathfrak{D}_{\overline{Y}_{\xi_0}} \arrow[dd] \arrow[ddr] & & & \\
 & & t(\mathfrak{D}_{\check{X}_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}}) \arrow[d, dotted] & & \\
t(\mathfrak{D}_{\check{X}_{\xi_0}}) \arrow[r] & \mathfrak{D}_{\check{X}_{\xi_0}} \arrow[r] \arrow[d] & \mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}} \arrow[r] \arrow[dr] & \mathfrak{D}_{\check{U}_{\xi_0}} & t(\mathfrak{D}_{\check{U}_{\xi_0}}) \arrow[l] \\
j^*(t(\mathfrak{D}_{\check{X}_{\xi_0}})) \arrow[r, dotted] & \mathfrak{D}_{W_{\xi_0}} & & \mathfrak{D}_{U_{\xi_0}} \arrow[u] & t(\mathfrak{D}_{Z_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}}) \arrow[l, dotted] \\
 & & \begin{array}{c} [t(\mathfrak{D}_{\check{X}_{\xi_0}}) \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}] \wedge \\ \wedge [t(\mathfrak{D}_{Z_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}})] \end{array} \arrow[uu, dotted] & & \\
 & & & \mathfrak{D}_{Z_{\xi_0}} \arrow[uu] & t(\mathfrak{D}_{Z_{\xi_0}}) \arrow[l]
\end{tikzcd}
\tag{8.2}
$$

Here, we have used (7.13), (7.14), and Proposition 7.3, with the obvious assignments to P's and Q's as per (7.9). Since gluing and ungluing should undo each other (Proposition 7.3) and since we require (8.2) to be unambiguous, we should check that the two indicated glued t-structures on $\mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}}$ agree with the following.

Proposition 8.1. $t(\mathfrak{D}_{\check{X}_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}}) = [t(\mathfrak{D}_{\check{X}_{\xi_0}}) \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}] \wedge [t(\mathfrak{D}_{Z_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}})]$.

Proof. By means of (7.17), we get that

$$\begin{tikzcd} \mathfrak{D}_{Z_{\xi_0}} \arrow[r, "i_*"] & \mathfrak{D}_{U_{\xi_0}} \arrow[l, shift right=3ex, "i^*"'] \arrow[l, shift left=3ex] \arrow[r, "\hat{j}^*_{\xi_0}"] & \mathfrak{D}_{\check{U}_{\xi_0}} \arrow[l, shift right=3ex, "\hat{j}_{\xi_0,!}"'] \arrow[l, shift left=3ex, "R\hat{j}_{\xi_0,*}"] \end{tikzcd} \tag{8.3a}$$

$$\begin{tikzcd} \mathfrak{D}_{\check{X}_{\xi_0}} \arrow[r, "\check{i}_{\xi_0,*}"] & \mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}} \arrow[l, shift right=3ex, "\check{i}^*_{\xi_0}"'] \arrow[l, shift left=3ex, "\check{i}^!_{\xi_0}"] \arrow[r, "\check{j}^*_{\xi_0}"] & \mathfrak{D}_{\check{U}_{\xi_0}} \arrow[l, shift right=3ex, "\check{j}_{\xi_0,!}"'] \arrow[l, shift left=3ex, "R\check{j}_{\xi_0,*}"] \end{tikzcd} \tag{8.3b}$$

$$\begin{tikzcd} \mathfrak{D}_{\overline{Y}_{\xi_0}} \arrow[r, "i_{\xi_0,*}"] & \mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}} \arrow[l, shift right=3ex, "i^*_{\xi_0}"'] \arrow[l, shift left=3ex, "i_{\xi_0,!}"] \arrow[r, "j^*_{\xi_0}"] & \mathfrak{D}_{U_{\xi_0}} \arrow[l, shift right=3ex, "j_{\xi_0,!}"'] \arrow[l, shift left=3ex, "Rj_{\xi_0,*}"] \end{tikzcd} \tag{8.3c}$$

This implies, by means of (7.23a), (7.23b) in Proposition 7.4, that if we set

$$t(\mathfrak{D}_{Z_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}}) =: t(\mathfrak{D}_{U_{\xi_0}}) = (\mathfrak{D}^{\leq 0}_{U_{\xi_0}}, \mathfrak{D}^{\geq 0}_{U_{\xi_0}}), \tag{8.4}$$

$$t(\mathfrak{D}_{\check{X}_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}}) =: t(\mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}}) = ({}_1\mathfrak{D}^{\leq 0}_{\tilde{X}^2_{\mathbf{A}}}, {}_1\mathfrak{D}^{\geq 0}_{\tilde{X}^2_{\mathbf{A}}}), \tag{8.5a}$$

$$[t(\mathfrak{D}_{\check{X}_{\xi_0}}) \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}] \wedge [t(\mathfrak{D}_{Z_{\xi_0}}) \wedge t(\mathfrak{D}_{\check{U}_{\xi_0}})] =: t({}_2\mathfrak{D}_{\tilde{X}^2_{\mathbf{A}}}) = ({}_2\mathfrak{D}^{\leq 0}_{\tilde{X}^2_{\mathbf{A}}}, {}_2\mathfrak{D}^{\geq 0}_{\tilde{X}^2_{\mathbf{A}}}), \tag{8.5b}$$

$$t(\mathfrak{D}_{\check{X}_{\xi_0}}) \cap \mathfrak{D}_{\overline{Y}_{\xi_0}} =: t(\mathfrak{D}_{\overline{Y}_{\xi_0}}) = (\mathfrak{D}^{\leq 0}_{\overline{Y}_{\xi_0}}, \mathfrak{D}^{\geq 0}_{\overline{Y}_{\xi_0}}), \tag{8.6}$$

$$j^*(t(\mathfrak{D}_{\check{X}_{\xi_0}})) = (j^*\mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}}, j^*\mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}}) =: t(\mathfrak{D}_{W_{\xi_0}}) = (\mathfrak{D}^{\leq 0}_{W_{\xi_0}}, \mathfrak{D}^{\geq 0}_{W_{\xi_0}}), \tag{8.7}$$

where, ab initio, our initial data is given by

$$t(\mathfrak{D}_{\check{X}_{\xi_0}}) = (\mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}}, \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}}), \tag{8.8a}$$

$$t(\mathfrak{D}_{\check{U}_{\xi_0}}) = (\mathfrak{D}^{\leq 0}_{\check{U}_{\xi_0}}, \mathfrak{D}^{\geq 0}_{\check{U}_{\xi_0}}), \tag{8.8b}$$

$$t(\mathfrak{D}_{Z_{\xi_0}}) = (\mathfrak{D}^{\leq 0}_{Z_{\xi_0}}, \mathfrak{D}^{\geq 0}_{z_{\xi_0}}), \tag{8.8c}$$

then we obtain

$$\mathfrak{D}^{\leq 0}_{U_{\xi_0}} = \left\{ \mathcal{G}^\bullet \in \mathfrak{D}_{U_{\xi_0}} \mid \hat{j}^*_{\xi_0}\mathcal{G}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{U}_{\xi_0}},\ i^*\mathcal{G}^\bullet \in \mathfrak{D}^{\leq 0}_{Z_{\xi_0}} \right\}, \tag{8.9a}$$

$$\mathfrak{D}^{\geq 0}_{U_{\xi_0}} = \left\{ \mathcal{G}^\bullet \in \mathfrak{D}_{U_{\xi_0}} \mid \hat{j}^*_{\xi_0}\mathcal{G}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{U}_{\xi_0}},\ i^!\mathcal{G}^\bullet \in \mathfrak{D}^{\geq 0}_{Z_{\xi_0}} \right\}, \tag{8.9b}$$

$${}_1\mathfrak{D}^{\leq 0}_{\tilde{X}^2_{\mathbb{A}}} = \left\{ \mathcal{F}^\bullet \in \mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}} \mid \check{j}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{U}_{\xi_0}},\ \check{i}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}} \right\}, \tag{8.10a}$$

$${}_1\mathfrak{D}^{\geq 0}_{\tilde{X}^2_{\mathbb{A}}} = \left\{ \mathcal{F}^\bullet \in \mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}} \mid \check{j}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{U}_{\xi_0}},\ \check{i}^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}} \right\}, \tag{8.10b}$$

$${}_2\mathfrak{D}^{\leq 0}_{\tilde{X}^2_{\mathbb{A}}} = \left\{ \mathcal{F}^\bullet \in \mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}} \mid j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{U_{\xi_0}},\ i^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\overline{Y}_{\xi_0}} \right\}, \tag{8.11a}$$

$${}_2\mathfrak{D}^{\geq 0}_{\tilde{X}^2_{\mathbb{A}}} = \left\{ \mathcal{F}^\bullet \in \mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}} \mid j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{U_{\xi_0}},\ i^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\overline{Y}_{\xi_0}} \right\}. \tag{8.11b}$$

But then, (8.4)–(8.8) directly imply that

$${}_2\mathfrak{D}^{\geq 0}_{\tilde{X}^2_{\mathbb{A}}} = \left\{ \mathcal{F}^\bullet \mid \hat{j}^*_{\xi_0} j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{U}_{\xi_0}},\ i^* j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{Z_{\xi_0}},\ i^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}} \right\}, \tag{8.12a}$$

$${}_2\mathfrak{D}^{\geq 0}_{\tilde{X}^2_{\mathbb{A}}} = \left\{ \mathcal{F}^\bullet \mid \hat{j}^*_{\xi_0} j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{U}_{\xi_0}},\ i^! j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{Z_{\xi_0}},\ i^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}} \right\}, \tag{8.12b}$$

which means that ${}_1t(\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}}) ={}_2 t(\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}})$ if and only if

$$\begin{aligned} \check{j}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{U}_{\xi_0}},\ \check{i}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}} &\Longleftrightarrow \hat{j}^*_{\xi_0} j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{U}_{\xi_0}},\ i^* j^*_{\xi_0}\mathcal{F}^\bullet \in \\ &\in \mathfrak{D}^{\leq 0}_{Z_{\xi_0}},\ i^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}, \end{aligned} \tag{8.13a}$$

$$\begin{aligned} \check{j}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\xi_0},\ \check{i}^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{U}_{\xi_0}} &\Longleftrightarrow \hat{j}^*_{\xi_0} j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{U}_{\xi_0}},\ i^! j^*_{\xi_0}\mathcal{F}^\bullet \in \\ &\in \mathfrak{D}^{\geq 0}_{X_{\xi_0}},\ i^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}. \end{aligned} \tag{8.13b}$$

And now, $i^* j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{Z_{\xi_0}}$, $i^! j^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{Z_{\xi_0}}$ obtain tautologically from (8.1), and the fact that we have $\check{j}^*_{\xi_0} = \hat{j}^*_{\xi_0} j^*_{\xi_0}$ provides that we need only verify that the conditions $\check{i}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{U}_{\xi_0}}$ (resp., $\check{i}^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}}$) are equivalent to $i^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}$ (resp., $i^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}$). Again, the diagram (8.1) gives that $i_{\xi_0,*} = \check{i}_{\xi_0,*}\hat{i}_{\xi_0,*}$, and this implies, in light of the fact that with $i_{\xi_0} = \check{i}_{\xi_0}\hat{i}_{\xi_0}$ in (6.6) we get $i^*_{\xi_0} = \hat{i}^*_{\xi_0}\check{i}^*_{\xi_0}$ and $i^!_{\xi_0} = \hat{i}^!_{\xi_0}\check{i}^!_{\xi_0}$, and with i_{ξ_0} just inclusion, that

$$\check{i}^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}} \Longleftrightarrow \hat{i}^*_{\xi_0}\check{i}^*_{\xi_0}\mathcal{F}^\bullet = i^*_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\leq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}, \tag{8.14a}$$

$$\check{i}^!\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}} \Longleftrightarrow \hat{i}^!_{\xi_0}\check{i}^!_{\xi_0}\mathcal{F}^\bullet = i^!_{\xi_0}\mathcal{F}^\bullet \in \mathfrak{D}^{\geq 0}_{\check{X}_{\xi_0}} \cap \mathfrak{D}_{\overline{Y}_{\xi_0}}. \tag{8.14b}$$

The proof is complete. □

The thrust of Proposition 8.1 is that assigning initial t-structure data in the form $t(\mathfrak{D}_{\check{X}_{\xi_0}})$, $t(\mathfrak{D}_{\check{U}_{\xi_0}})$, $t(\mathfrak{D}_{Z_{\xi_0}})$ provides a single resultant t-structure, $t(\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}}) = {}_1t(\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}}) = {}_2t(\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}})$, on $\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}}$. We now seek to exploit this identification of ${}_1t(\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}})$ and ${}_2t(\mathfrak{D}_{\tilde{X}^2_{\mathbb{A}}})$ to evolve a workable phrasing of the desired critical degeneration, namely, $Y_{\xi_0} = \varnothing$ (or $\overline{Y}_{\xi_0} = \varnothing$) if $\xi_0 \neq 1$. If we stipulate that this near-future work, as well as the attendant task of designing the aforementioned initial t-structure, is to take place in the context and setting of ordinary truncations of sheaf complexes (determined by perversities), then we may assume that (7.7) applies. But this permits us to prove the following.

Proposition 8.2. *If, for $\xi_0 \neq 1$ in μ, there exists a strongly degenerate t-structure on $\mathfrak{D}_{\overline{Y}_{\xi_0}}$, then, for $\xi_0 \neq 1$, $\overline{Y}_{\xi_0} = \varnothing = Y_{\xi_0}$ (and, by (6.6) and Proposition 6.1, n-Hilbert reciprocity follows).*

Proof. By (7.8), strong degeneracy yields that core $(t(\mathfrak{D}_{\overline{Y}_{\xi_0}})) = \varnothing$ for such a t-structure. But then, the abelian category $(\mathfrak{Sh}/_{\overline{Y}_{\xi_0}})$ is void. However, the only topological space that fails to support even constant sheaves is $\varnothing$, and the result follows. □

We have made a start on characterizing our initial t-structures and developing the according yoga of t-structures in (8.2) to conspire to bring about the collapse of all the $\mathfrak{D}_{\overline{Y}_{\xi_0}}$ except for $\mathfrak{D}_{\overline{Y}_1}$.

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