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Research Article

Mutually Compactificable Topological Spaces

Martin Maria Kovár

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Two disjoint topological spaces X, Y are (T_2-) mutually compactificable if there exists a compact (T_2-) topology on $K = X \cup Y$ which coincides on X, Y with their original topologies such that the points $x \in X$, $y \in Y$ have open disjoint neighborhoods in K. This paper, the first one from a series, contains some initial investigations of the notion. Some key properties are the following: a topological space is mutually compactificable with some space if and only if it is θ-regular. A regular space on which every real-valued continuous function is constant is mutually compactificable with no S_2-space. On the other hand, there exists a regular non-$T_{3.5}$ space which is mutually compactificable with the infinite countable discrete space.

1. The notation and terminology

Throughout the paper we mostly use the standard topological notions as in [1], with a few exceptions that will be described in this section. The source of notions related to the construction of the Wallman compactification is Császár's book [2]. By a space we always mean a topological space. All spaces are assumed with no separation axioms in general. Especially, compactness, paracompactness and their modifications are understood without T_2 or any other separation axiom. At this point we should note that compactness without Hausdorff axiom is referred to as *quasicompactness* in some topological literature (see [1, 3]).

A topology of a space X we will usually denote by τ or τ_X (in the case that we will work simultaneously with more topological spaces). In a space X a point $x \in X$ is in the *θ-closure* of a set $A \subseteq X$ ($x \in \mathrm{cl}_\theta A$) if every closed neighborhood of x intersects A. A filter base Φ in X has a *θ-cluster point* $x \in X$ if $x \in \bigcap\{\mathrm{cl}_\theta F \mid F \in \Phi\}$. The filter base Φ

θ-converges to its *θ-limit* x if for every closed neighborhood H of x there is $F \in \Phi$ such that $F \subseteq H$. We say that a space X is *(countably) θ-regular* if every (countable) filter base in X with a θ-cluster point has a cluster point [4, 5]. The points x, y in a space X are *T_0-separable* if there is an open set containing only one of the points x, y. The points x, y are *T_2-separable* if they have open disjoint neighborhoods. In this notation, the space X is said to be S_2 [2] if every two T_0-separable points of X are T_2-separable. We say that the space X is R_1 [4] if for every $x, y \in X$ satisfying $\mathrm{cl}\{x\} \neq \mathrm{cl}\{y\}$ the sets $\mathrm{cl}\{x\}$, $\mathrm{cl}\{y\}$ have disjoint neighborhoods. Obviously, the separation axioms R_1 and S_2 are equivalent and slightly weaker than Hausdorff separation axiom T_2. However, in many cases they mean almost the same as Hausdorff. Let X be a space. Two disjoint sets $A, B \subseteq X$ are said to be *pointwisely separated in* X if every $x \in A$, $y \in B$ are T_2-separable in X. In this paper, we say that a space is *(strongly) locally compact* if its every point has a compact (closed) neighborhood. However, we should note that in some literature the meaning of "locally compact" could be different—for instance, the notion could also mean that every point of the space has a local base consisting of compact neighborhoods. Of course, for Hausdorff spaces all these modifications coincide. There also exist more slightly different definitions of regularity in the literature. In this paper, we say that a space X is *regular* if for every $x \in X$ and every open $U \subseteq X$ with $x \in U$ there exists open set $V \subseteq X$ such that $x \in V$ and $\mathrm{cl}_X V \subseteq U$ (hence we do not assume T_1 for the definition of regularity). A filter in a space X is said to be *ultra closed* (*ultra open*, resp.) if it is maximal among all filters in X having a base consisting of closed (open, resp.) sets [2]. By the Wallman compactification of X we mean that the set $\omega X = X \cup \{y \mid y$ is a nonconvergent ultra-closed filter in $X\}$. The sets $\mathcal{S}(U) = U \cup \{y \mid y \in \omega X \setminus X,\ U \in y\}$, where U is open in X, constitute an open base of ωX (see [2]).

2. Preliminaries and introduction

The main stream of topology deals with spaces satisfying various separation axioms, but the most commonly accepted baseline is that all considered spaces should be at least Hausdorff. It seems it was Bourbaki [3], who made Hausdorff axiom a part of the definition of compactness. The reasons for this approach consist deeply in the history of mathematics and in the way how the topology has developed from the other mathematical disciplines such as geometry or analysis. On the other hand, the topological approach to the order and domain theory, motivated by certain recent developments in computer science as well as some recent alternative approaches to quantum physics and relativity witness that the Hausdorff topological structure might not be rich enough for modeling the diversity of the nature, despite the fact that Hausdorff spaces are well established and widely accepted. Hence, there is a request, perhaps silent and not so strong yet, but really unignorable and continuously growing, for some more systematic research of non-Hausdorff topological structures.

However, remaining strictly inside the topological discipline, we may find another sufficient arguments for studying the non-Hausdorff spaces. One of the most simple could be the following. There are more than three hundred different topologies (more precisely, 355) on a four-element set, and this number grows very rapidly with the power of the underlying set. For instance, a 14-element set already carries 98484324257128207032183 different T_0-topologies [6], but just one among them—the discrete topology—is Hausdorff. One can hardly claim that this Hausdorff topology is the most interesting one. Also, it is rather difficult to believe that this disproportion can be completely reverted if someone replace the finite set by an infinite one. Simply, it is unsatisfactory that we know so little about these spaces. On the other hand, by omitting Hausdorff axiom we will lose a very handy and strong tool for our theorems and proofs. In order be able to make any progress, some other alternative properties of spaces should be recognized and studied. Indeed, one of such properties is the θ-regularity, introduced by Janković in [4]. Its relations to separation and covering properties were studied in detail, among others, by the author in [5]. A practical utility of this topological property can also be demonstrated in the papers [7, 8].

The property of θ-regularity is known to replace regularity and Hausdorffness in several covering theorems concerning paracompactness. But it is also very naturally connected with compactness and certain partial "Hausdorff-like" separation in general compact spaces. It can be proved (in a not very difficult way) that each θ-regular space can be generated by splitting of some compact space in to two—to the considered θ-regular space itself and to its complement, in a way that any two points—one from the space and the other from the complement—have open disjoint neighborhoods. Note that then the complement is again a θ-regular space. Thus any θ-regular space may be considered as an uncomplete part of a certain compact space and naturally one may ask, which two θ-regular spaces match to each other so they can backwards recreate the original compact space. Comparing various θ-regular spaces and splitting their class into subclasses containing θ-regular spaces with the equivalent behavior we may even obtain a certain scale, measuring a "level of noncompactness" of a fixely considered space.

In this paper, we will study especially the initial properties of the above-mentioned fundamental construction. We will describe some θ-regular spaces which are compatible in the considered sense and will give some necessary counterexamples. We will also give a basic discussion regarding a "Hausdorff" variant of the construction, requiring the generating compact space to be in addition Hausdorff. First, we will show that even for regular or regular T_1 spaces (i.e., those which are θ-regular and Hausdorff) one would get a different theory. Second, "Hausdorff" variant of the theory will be studied in another, separate line of papers. Further, in this paper we will not study the classes containing the θ-regular spaces of equivalent behavior, because this topic is covered by another three proceeding papers, "The classes of mutual compactificability," "The compactificability classes of certain spaces," and "The compactificability classes—the behavior at infinity."

As our starting point, we will summarize some older results from [5, 9] (with exception of the condition (iv) which has never been published before but it has its natural place here).

Theorem 2.1. *The following statements are equivalent for a space X.*

(i) *X is θ-regular.*

(ii) *The sets X, $\omega X \setminus X$ are pointwisely separated in ωX.*

(iii) *There exists a compact space K containing X as a subspace such that the sets X, $K \setminus X$ are pointwisely separated in K.*

(iv) *For every $x \in X$, there is a compact set $K(x)$ such that if any open U contains $K(x)$, then x has a closed neighbourhood H with $H \subseteq U$.*
(v) *For every open cover Ω of X and every $x \in X$, there exists an open neighbourhood U of x such that $\mathrm{cl}_X U$ can be covered by finitely many elements of Ω.*

Proof. The equivalence of (i), (ii), (iii) follows from [9, Theorem 1] and the equivalence of (i) and (ii) is an immediate consequence of [5, Theorem 1]. Since the implication (iv)⇒(v) is clear, it remains to show that (iii) implies (iv).

Suppose (iii). Denote $K(x) = \mathrm{cl}_\theta\{x\}$. Then $K(x) \subseteq X$ is a closed subspace of K and hence compact. Let $U \subseteq X$ be open in X and let $K(x) \subseteq U$. Let $W \subseteq K$ be open in K such that $U = W \cap X$. Then $K \setminus W$ is compact and for every $y \in K \setminus W$, the points x and y have disjoint neighborhoods in K. Therefore, there exist open $V \subseteq K$ such that $x \in V$ and $\mathrm{cl}_K V \subseteq W$. Then $H = \mathrm{cl}_K V \cap X$ is a closed neighbourhood of x in X such that $H \subseteq U$. That implies (iv). □

Now, let X be a space. We denote by $\mathfrak{B}_\theta(X)$ the family of all sets $B \subseteq X$ such that $\mathrm{cl}_\theta\{x\} \subseteq B$ for every $x \in B$. The elements of $\mathfrak{B}_\theta(X)$ are called θ-*saturated* sets.

Corollary 2.2. *The following statements are fulfilled for a space X.*
(i) *$\mathfrak{B}_\theta(X)$ is a complete Boolean set algebra.*
(ii) *If X is θ-regular and $Y \in \mathfrak{B}_\theta(X)$ is a subspace of X, then Y is also θ-regular.*

For the proof, see [5, Theorem 4].

Corollary 2.3. *A space is regular if and only if the space is θ-regular and S_2.*

For the proof, see [5, Corollary 2].

Our main results are contained in the following section.

3. The mutual compactificability

Over the years, a rather fascinating question had been asked and studied: *when it is possible to replace a growth of a space in some compactification by another space?* Some related results were already obtained by Hausdorff and Kuratowski in the thirties at least for metrizable spaces. For Hausdorff locally compact spaces the question was reopened again and studied, among others, by Magill [10]; however, a general characterization still seems to be lacking. Note that a brief introduction to the topic (in the realm of Hausdorff locally compact spaces, including Magill's main theorem) can now be found by the reader in the book [1]. The role of a space and its growth is strongly nonsymmetric, which sometimes may be considered as a disadvantage. However, the symmetrical case also could be studied. The following definition is a certain compromise between the requirement of a sufficient level of generality and the convenience of leaving some kind of separation, latently contained in every compact space, still as a part of the game.

Definition 3.1. Let X, Y be spaces with $X \cap Y = \varnothing$. The space X is said to be *compactificable* by the space Y or, in other words, X, Y are called *mutually compactificable* if there exists a compact topology on $K = X \cup Y$ extending the topologies of X and Y such that the sets X, Y are pointwisely separated in K.

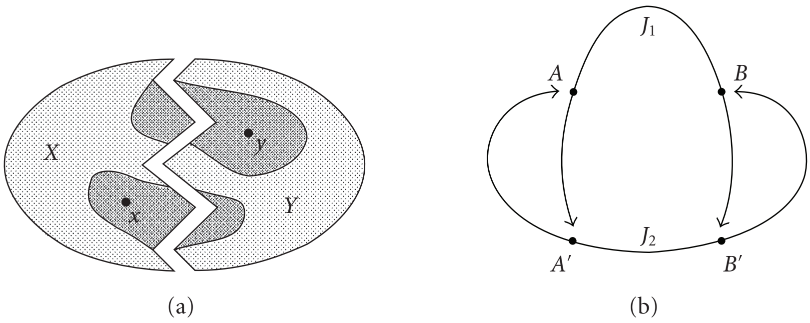

Figure 3.1

The idea of the mutual compactificability is illustrated by Figure 3.1(a). But note that the space K in the previous definition may be, and also need not be, a compactification of any of the spaces X, Y. The next example is very simple.

Example 3.2. Two open real intervals are mutually compactificable by embedding them properly into the real plane (as it is schematically demonstrated by Figure 3.1(b)).

Now we directly obtain the following.

Theorem 3.3. *For a space X, there exists a space Y such that X, Y are mutually compactificable if and only if X is θ-regular.*

Definition 3.4. Let X and Y be disjoint and mutually compactificable. Then any compact topology on $K = X \cup Y$ which induces the original topologies on X, Y such that X, Y are in K pointwisely separated is called $\mathscr{C}$-*acceptable*.

Regarding the two previous definitions, there arise a number of natural and—in the author's opinion—interesting questions. For instance, if one have two mutually compactificable spaces X and Y, are there more $\mathscr{C}$-acceptable topologies on $K = X \cup Y$? If one have a $T_{3.5}$ space, it is always compactificable by its Čech-Stone remainder, which again is a $T_{3.5}$ space. Analogously, a θ-regular space is always compactificable by its Wallman remainder, which again is θ-regular. We may say that the classes of $T_{3.5}$ spaces or θ-regular spaces are closed, in some sense, with respect to mutual compactificability. But is every regular space compactificable by some regular space? In other words, is the class of regular spaces closed with respect to mutual compactificability? And further: is there any outstanding class of spaces such that any two spaces from this class are mutually compactificable? If so, how large is that class? Can it be enlarged? Which spaces are compactificable by a compact space? Which spaces have equivalent behavior with respect to the mutual compactificability: that is, for which pairs of spaces X and Y it follows that X is compactificable by Z if and only if Y is compactificable by Z? Some of these questions we will try to answer in the present paper or in the three proceeding papers. Now, let us start with the following simple, but important lemma.

Lemma 3.5. *Let X, Y be disjoint mutually compactificable S_2 spaces with some $\mathscr{C}$-acceptable (but not necessarily S_2) topology on $K = X \cup Y$. Denote $H = \mathrm{cl}_K X \cap \mathrm{cl}_K Y$, $R = K \setminus \mathrm{cl}_K X$, $S = K \setminus \mathrm{cl}_K Y$. Then R, S are open regular locally compact subspaces of K and H is a closed S_2 subspace of K. (Hence, the spaces H, R, S are all $T_{3.5}$-separable.)*

Proof. Let us show that H is S_2. Take $x, y \in H$. If $x \in X$ and $y \in Y$, we are done. Thus suppose that $x, y \in H \cap X$ and suppose x, y are T_0-separable in H. Then, obviously, they are T_0-separable also in X. Since X is S_2 there exist $U, V \in \tau_K$ such that $x \in U \cap X$, $y \in V \cap X$ and $U \cap V \cap X = \varnothing$. Then $U \cap V \cap H \subseteq U \cap V \cap \mathrm{cl}_K X = \varnothing$ which implies that $U \cap H$, $V \cap H$ are τ_H-open disjoint neighborhoods of x and y. Hence, H is S_2. Since X, Y are point wise separated in a compact space K, by Theorem 2.1 it follows that they are both θ-regular and hence regular by Corollary 2.3. Therefore, $R \subseteq Y$ and $S \subseteq X$ are also regular. We will show that R, S are strongly locally compact. Let $x \in S$. There exist $V \in \tau_X$ with $x \in V$ and $\mathrm{cl}_X V \subseteq S$. We will show that V is open in K. There is some $Q \in \tau_K$ such that $V = Q \cap X$. But $V = V \cap S = Q \cap X \cap S = Q \cap S \in \tau_K$ since $Q, S \in \tau_K$. Since H is compact, X is S_2 and Y is point wise separated from X, for every $t \in H$ there are $U_t, V_t \in \tau_K$ such that $t \in U_t$, $x \in V_t$ and $U_t \cap V_t \cap X = \varnothing$. Let $\{U_{t_1}, U_{t_2}, \ldots, U_{t_m}\}$ be a finite cover of H. We put $P = V \cap (\bigcap_{i=1}^{m} V_{t_i})$ and $U = R \cup (\bigcup_{i=1}^{m} U_{t_i})$. It follows that $x \in P$ and $P \cap U = \varnothing$. Further, $K = S \cup R \cup H \subseteq S \cup (\bigcup_{i=1}^{m} U_{t_i}) = S \cup U$. Hence, $P \subseteq K \setminus U \subseteq S$ which implies that $\mathrm{cl}_K P \subseteq S$. It follows that $\mathrm{cl}_K P$ is a compact closed neighborhood of x in S, so S is strongly locally compact. The fact that also R is strongly locally compact can be shown analogously. □

The lemma and the next theorem will look more clear and understandable if one replace S_2 by Hausdorff separation axiom. However, Császár's separation axiom S_2 is more convenient when decomposing, for instance, regularity (see Corollary 2.3). One can also easily verify that S_2 in combination with compactness implies normality [2] and, hence, $T_{3.5}$ separation property. This is an important point of the following proof. Regular spaces with no nonconstant real-valued function are often presented as interesting examples of non-$T_{3.5}$ spaces. One of such examples was given by Herrlich [11]. Now this example can be also found in Engelking's monograph [1]. The next theorem says also the following: if the regular spaces on which all the continuous functions are constant are a legitimate object of study, then so are the non-S_2 spaces. Another consequence of the theorem is that regular spaces do not form a closed class with respect to mutual compactificability.

Theorem 3.6. *If X is a regular space on which every continuous real-valued function is constant, then X is compactificable by no S_2 space.*

Proof. It follows that X is a connected non-$T_{3.5}$ space. Suppose that there exist a Hausdorff space Y such that $X \cap Y = \varnothing$ and an $\mathscr{C}$-acceptable topology on $K = X \cup Y$. Denote $H = \mathrm{cl}_K X \cap \mathrm{cl}_K Y$, $S = K \setminus \mathrm{cl}_K Y$ and $F = X \setminus S$. It follows from Lemma 3.5 that H is closed, compact, S_2 and hence normal subspace of K. Then $F \subseteq H$ is a $T_{3.5}$ subspace of X. Similarly, $S \subseteq X$ is an open subspace of K which is regular and locally compact by Lemma 3.5 and therefore S is an open, also $T_{3.5}$ subspace of X. Clearly, $S \neq \varnothing$ and $F \neq \varnothing$ since $X = F \cup S$ and X is not $T_{3.5}$. Take $x \in S$. Since X is regular, there exists an open set $U \in \tau_X$ with $x \in U$ and $\mathrm{cl}_X U \subseteq S$. Since X is connected, it follows that $\mathrm{cl}_X U \neq S$ which implies that $A = S \setminus U$ is a closed nonempty subset of S. Let $f : S \to I$ be a continuous function with $f(x) = 0$ and $f(A) = \{1\}$. We put $g(t) = f(t)$ for every $t \in S$ and $g(t) = 1$ for every $t \in F$. Let $P = X \setminus \mathrm{cl}_X U$. It follows that $P \subseteq A \cup F$ is an open subset of X, $f(P) = \{1\}$ and $X = P \cup S$. Since g is continuous on both of the open sets P, S which together cover the space X it follows that g is continuous and, obviously, nonconstant on X. This is a contradiction. □

Hence, in the connection with Theorem 3.6, it is a natural question whether every regular space which is compactificable by some Hausdorff spaces is $T_{3.5}$. Remark that if we replace mutual compactificability by T_2-mutual compactificability, the corresponding answer is (trivially) positive. The source of inspiration for the following counterexample was the article of Thomas [12]. The space constructed in his paper is a relatively simple example of a regular non-$T_{3.5}$ space. We repeat the construction briefly because of completeness and show that this space is compactificable even by the infinite countable discrete space. Since the constructed space is not $T_{3.5}$, it is obviously not T_2-compactificable by any space. Thus the following example completes the result of Theorem 3.6, and we also may see from it that mutual compactificability and T_2-mutual compactificability are different notions even if restricted to the class of Hausdorff or regular T_1 spaces.

Example 3.7. There exists a regular T_1, but non-$T_{3.5}$ space which is compactificable by the infinite countable discrete space.

Construction 3.8. At first, we will construct some subset $X \subseteq \mathbb{R}^2$. Let $\mathbb{E}$ be set of all even integers, $\mathbb{O} = \mathbb{Z} \setminus \mathbb{E}$. For every $m \in \mathbb{E}$, we put $L_m = \{m\} \times [-1,0]$ and for every $n \in \mathbb{O}$, $k \in \mathbb{N}$ we put $C_{n,k} = (\{n+1-1/2^k\} \times [-1,0]) \cup (\{n-1+1/2^k\} \times [-1,0]) \cup \{(x,y) \mid (x-n)^2 + y^2 = (1-1/2^k)^2,\ y \geq 0\}$. Denote $a = (-1,0)$, $b = (1,0)$. We put $X = \{a,b\} \cup (\bigcup_{m\in\mathbb{E}} L_m) \cup (\bigcup_{n\in\mathbb{O},k\in\mathbb{N}} C_{n,k})$. Let us define a topology base σ on X. Any element of σ can have one of the following forms:

(1) $U_1(\varepsilon,v,h) = X \cap ((v-\varepsilon, v+\varepsilon) \times \{h\})$ where $\varepsilon, v, h \in \mathbb{R}$, $\varepsilon > 0$ and $((v-\varepsilon, v+\varepsilon) \times \{h\}) \cap (\{a,b\} \cup \{(n, 1-1/2^k) \mid n \in \mathbb{O},\ k \in \mathbb{N}\}) = \varnothing$,

(2) $U_2(n,k,F) = C_{n,k} \setminus F$, where $n \in \mathbb{O}$, $k \in \mathbb{N}$, and F is finite,

(3) $U_3(m) = \{a\} \cup \{(x,y) \mid (x,y) \in X,\ x < m\}$, where $m \in \mathbb{E}$,

(4) $U_4(m) = \{b\} \cup \{(x,y) \mid (x,y) \in X,\ x > m\}$, where $m \in \mathbb{E}$.

It is not very difficult to show (cf. [12]) that σ forms a base of some topology τ on X, which is T_1 and regular but it is not $T_{3.5}$. Now, we put $Y = \mathbb{E} \times \{1\}$. The construction of the space X and of the set Y is illustrated by Figure 3.2.

Let us define a topology on $K = X \cup Y$. Let H be the set of all line segments of the form $(m-\varepsilon, m+\varepsilon) \times \{h\}$ where $m \in \mathbb{E}$, $\varepsilon, h \in \mathbb{R}$, $0 < \varepsilon < 1$, $-1 \leq h \leq 0$, and let C be the set of all arcs $C_{n,k}$ defined above where $n \in \mathbb{O}$ and $k \in \mathbb{N}$. Firstly, we take the neighborhoods defined in (1) and (2) as elements of the new topology base η of K. Secondly, we define the following additional classes of elements of η:

(3′) $U_3'(m) = \{a\} \cup \{(x,y) \mid (x,y) \in K,\ x < m\}$, where $m \in \mathbb{E}$,

(4′) $U_4'(m) = \{b\} \cup \{(x,y) \mid (x,y) \in K,\ x > m\}$, where $m \in \mathbb{E}$,

(5′) $V_5'(m,F) = L_m \cup (\bigcup_{k\in\mathbb{N}} C_{m-1,k}) \cup (\bigcup_{k\in\mathbb{N}} C_{m+1,k}) \setminus (\bigcup F)$, where $m \in \mathbb{E}$ and $F \subseteq H \cup C$ is finite.

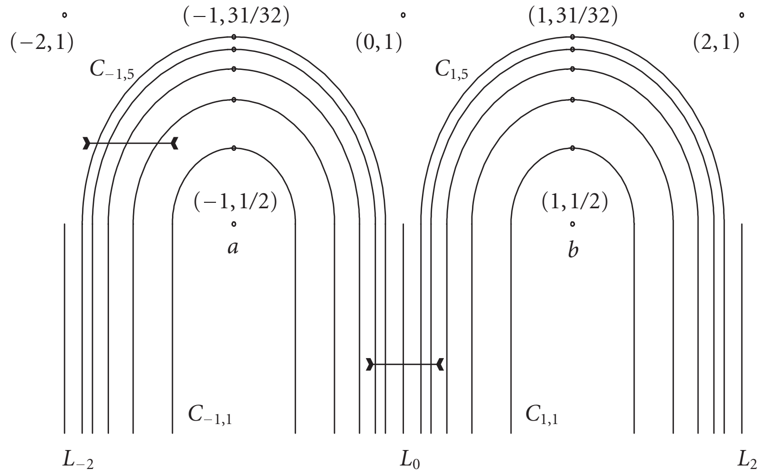

Figure 3.2

We leave to the reader to verify that the sets of (1), (2), (3′), (4′), and (5′) form a topology base on K. Obviously, the topology induced on Y from K is discrete, the topology induced on X coincides with its original topology, and the sets X, Y are disjoint and point wise separated in K. It remains to show that K is compact. Let Ω be an open cover of K. We may assume, without loss of generality, that $\Omega \subseteq \eta$. It follows that there exist the greatest $m_a \in \mathbb{E}$ and the least $m_b \in \mathbb{E}$ such that $U_3'(m_a), U_4'(m_b) \in \Omega$. We may suppose that $m_a \le m_b$ since in another case $\{U_3'(m_a), U_4'(m_b)\}$ is a finite subcover of Ω. Let $M = \{m_a, m_a + 2, \ldots, m_b\}$. For every $m \in M$ the point $(m,1) \in Y$ is covered by some element of Ω, so $V_5'(m, F_m) \in \Omega$ for some finite $F_m \subseteq H \cup C$. Every line segment $(m - \varepsilon, m + \varepsilon) \times \{h\} \in F_m$ meets L_m at $(m,h) \in X$. This point is contained in some element of Ω, so there exist $\delta, w \in \mathbb{R}$, $\delta > 0$ such that $(m,h) \in U_1(\delta, w, h) \in \Omega$. Since the set $(X \cap ((m - \varepsilon, m + \varepsilon) \times \{h\})) \setminus U_1(\delta, w, h)$ is finite, the set $\bigcup(F_m \cap H)$ can be covered by a finite subfamily of Ω. Suppose that $C_{n,k} \in F_m \cap C$. Then $n \in \{m - 1, m + 1\}$. The point $(n, 1 - 1/2^k)$ is at the top of the arc $C_{n,k}$ and it is contained in some element of Ω. Since no neighborhood of type (1) meets $(n, 1 - 1/2^k)$ it follows that there is some finite set F such that $(n, 1 - 1/2^k) \in U_2(n, k, F) \in \Omega$. Hence, $U_2(n, k, F) = C_{n,k} \setminus F$ covers all the arc $C_{n,k}$ with an exception of finitely many points of K. It follows that also the set $\bigcup(F_m \cap C)$ can be covered by a finite subfamily of Ω. Now, we have a finite cover $\Gamma = \{U_3'(m_a), U_4'(m_b)\} \cup \{V_5'(m, F_m) \mid m \in M\} \cup \{\bigcup(F_m \cap (H \cup C)) \mid m \in M\}$ of X whose elements can be covered by finite subfamilies of Ω. Hence, Ω has a finite subcover. Therefore, K is compact.

On the other hand, the following theorem holds.

Theorem 3.9. *Any two disjoint strongly locally compact spaces are mutually compactificable.*

Proof. Let X, Y be two disjoint strongly locally compact spaces, and let $a \in X$, $b \in Y$. We define a topology base σ for $K = X \cup Y$. Every element of σ is a subset of K having one of the following forms:

(1) $U \in \tau_X$, such that $a \notin U$,

(2) $V \in \tau_Y$, such that $b \notin V$,

(3) $U \cup (Y \setminus L)$, where $U \in \tau_X$, $a \in U$, $L \subseteq Y$ is compact and closed in Y, $b \in L$,

(4) $V \cup (X \setminus M)$, where $V \in \tau_Y$, $b \in V$, $M \subseteq X$ is compact and closed in X, $a \in K$.

It is an easy exercise to show that σ forms a base of certain compact topology on K such that the topologies on X, Y induced from K coincide with their original topologies τ_X, τ_Y, respectively. From the construction of σ and from the fact that every point in X, Y has a closed compact neighborhood in X, Y, respectively, it directly follows that X and Y are in K point wise separated. □

The converse, however, is not true, as it is illustrated by the following counterexample. Strong local compactness of a space cannot ensure the same of its counterpart in the mutual compactificability.

Example 3.10. There exists a nonlocally compact space T_2-compactificable by a strongly locally compact space.

Construction 3.11. Let $\mathbb{I} = [0,1] \subseteq \mathbb{R}$ be the closed unit interval. We put $K = \mathbb{I} \times \mathbb{I}$, $Y = \{0\} \times (0,1)$, $X = K \setminus Y$. Then K is compact Hausdorff, Y is strongly locally compact but the point $(0,0) \in X$ has no compact neighborhood in X.

On the other hand, compactness of a space ensures strong local compactness of its counterpart in the mutual compactificability.

Theorem 3.12. *Let X be compactificable by some compact space Y. Then X is strongly locally compact.*

Proof. Let $x \in X$. Since X, Y are point wise separated in K it follows that for each $y \in Y$ there exist $U_y, V_y \in \tau_K$ such that $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \varnothing$. Since Y is compact there exist $y_1, y_2, \ldots, y_n \in Y$ such that $Y \subseteq V = \bigcup_{i=1}^{n} V_{y_i}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$. It follows that $U \cap V = \varnothing$ and then $\mathrm{cl}_K U \subseteq K \setminus V \subseteq K \setminus Y = X$. Hence, $\mathrm{cl}_K U$ is a compact closed neighborhood of x in the space X which implies that X is strongly locally compact. □

Our investigations will proceed in the paper "The classes of mutual compactificability." In that paper, we will introduce the classes of spaces having equivalent behavior with respect to the mutual compactificability. We will see that these classes are arranged in some quasiorder structure, which may be interpreted as a certain "scale of noncompactness." Also we will study the representatives of these classes. Using results obtained in the present paper, we will show that every class, with one exception, contains a T_1 representative, but there still exist classes with no Hausdorff representatives. In another paper "The compactificability classes of certain spaces" we will investigate the behavior of the classes represented by some more familiarly known spaces, constructed from the real line and the Cantor discontinua. Finally, in the currently last paper of the series "The compactificability classes—behavior at infinity" we will see that the behavior of a space at infinity is more determining property for mutual compactificability than its separation properties or cardinality.

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Martin Maria Kovár: Department of Mathematics, Faculty of Electrical Engineering and Communication, University of Technology, Technická 8, 616 69 Brno, Czech Republic
Email address: kovar@feec.vutbr.cz