Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 74639, 10 pages
doi:10.1155/2007/74639

Review Article

Nonrepetitive Colorings of Graphs—A Survey

Jarosław Grytczuk

Received 8 August 2006; Revised 4 November 2006; Accepted 29 November 2006

Recommended by George E. Andrews

A vertex coloring f of a graph G is *nonrepetitive* if there are no integer $r \geq 1$ and a simple path $v_1,\ldots,v_{2r}$ in G such that $f(v_i) = f(v_{r+i})$ for all $i = 1,\ldots,r$. This notion is a graph-theoretic variant of nonrepetitive sequences of Thue. The paper surveys problems and results on this topic.

1. Introduction

Let f be a coloring of the vertices of a graph G. A simple path $v_1,\ldots,v_{2r}$ in G is *repetitive* if $f(v_i) = f(v_{r+i})$ for all $i = 1,\ldots,r$. A coloring f is *nonrepetitive* if no path in G is repetitive. The minimum number of colors needed is denoted by $\pi(G)$ and is called occasionally the *Thue chromatic number* of a graph G. Notice that it is not obvious that this parameter is bounded even for paths P_n. A motivation for studying nonrepetitive colorings came from the following theorem of Thue.

Theorem 1.1 (Thue [1]). *If P_n is a path on $n \geq 4$ vertices, then $\pi(P_n) = 3$.*

This result has wide applications in different branches of mathematics. Rediscovered many times, it is presently regarded as the starting point of *Combinatorics on Words*, or *Symbolic Dynamics*. We refer the interested reader to several monographs or surveys on this topic, restricting ourselves here to graph-theoretic aspects (cf. [2–6]).

The original proof of Theorem 1.1 supplies explicit construction of a nonrepetitive coloring of P_n. Suppose $C = \{a,b,c\}$ is the set of colors and let $s(a) = abcab$, $s(b) = acabcb$, $s(c) = acbcacb$. It can be proved that if $c_1 \cdots c_n$ is a nonrepetitive coloring of P_n, with $c_i \in C$, then $s(c_1)\cdots s(c_n)$ is a nonrepetitive coloring of the longer path. The theorem follows by induction.

A different proof was given by Shelton and Soni [7–9]. Their method is nonconstructive and gives a stronger assertion that the set of 3-colorings of an infinite path is perfect (with a natural product topology).

Theorem 1.1 is clearly the best possible, but it is worth mentioning that a finite (though weaker) bound can be obtained by a probabilistic argument, based on the Lovász local lemma (cf. [10]). This approach works well in more general situations, where no other method is known, whether constructive or not (cf. [10–13]).

2. Bounded degree

Let C_n be a cycle on n vertices. Theorem 1.1 implies easily that $\pi(C_n) \le 4$. By inspection, one may find that $\pi(C_n) = 4$ for $n = 5,7,9,10,14,17$. Curiously, these are the only values where the equality holds, as proved by Currie [14]. So, the picture is complete for graphs of maximum degree at most 2. For graphs of higher degree, the situation is not so clear.

Let $\pi(d)$ be the supremum of $\pi(G)$, where G ranges over all graphs of maximum degree at most d. The above remarks show that $\pi(2) = 4$. This is the only known exact value of $\pi(d)$ for $d \ge 2$. Notice that it is not obvious *a priori* that $\pi(d)$ is finite for any $d \ge 3$.

Theorem 2.1 (Alon et al. [15]). *There exist absolute constants c_1, c_2 such that for every integer $d \ge 1$,*

$$c_1 \frac{d^2}{\log d} \le \pi(d) \le c_2 d^2. \tag{2.1}$$

The upper bound was proved by the local lemma while the lower bound follows from a construction based on random graphs (cf. [10]). We give here the proof of the upper bound providing an explicit constant. Recall that a *dependency graph* of random events $A_1,\dots,A_n$ is any graph $D = (V,E)$ on the set of vertices $V = \{A_1,\dots,A_n\}$, such that each event A_i is mutually independent of the events $\{A_j : A_iA_j \notin E\}$.

Lemma 2.2 (The local lemma, cf. [10]). *Let $A_1,\dots,A_n$ be events in any probability space with dependency graph $D = (V,E)$. Let $V = V_1 \cup \cdots \cup V_k$ be a partition such that all members of each part V_r have the same probability p_r. Suppose that the maximum number of vertices from V_s adjacent to a vertex from V_r is at most Δ_{rs}. If there are real numbers $0 \le x_1,\dots,x_k < 1$ such that $p_r \le x_r \prod_{s=1}^{k}(1-x_s)^{\Delta_{rs}}$, then* $\Pr(\bigcap_{i=1}^{n} \overline{A_i}) > 0$.

Theorem 2.3. *If G is a graph of maximum degree at most d, then $\pi(G) \le 16d^2$.*

Proof. Let G be a graph of maximum degree at most d. Consider a random coloring of the vertices of G with $N = 16d^2$ colors. For each path P in G, let A_P be the event that the first half of P is colored the same as the second half. Define a dependency graph so that A_P is adjacent to A_Q if and only if the paths P and Q have a common vertex. Let V_r be the set of all events A_P with P having $2r$ vertices. Clearly we have $p_r = N^{-r}$. Now, for each fixed vertex v, there are at most sd^{2s} paths going through v in G. Hence, a fixed path with $2r$ vertices intersects at most $2rsd^{2s}$ paths with $2s$ vertices in G, and we may take $\Delta_{rs} = 2rsd^{2s}$.

Next set $x_s = (3d)^{-2s}$, and notice that $(1 - x_s) \geq e^{-2x_s}$, as $x_s \leq 1/2$. We get

$$x_r \prod_s (1 - x_s)^{\Delta_{rs}} \geq (3d)^{-2r} \prod_s e^{-4rs5^{-2s}} > (3d)^{-2r} \exp\left(-2r \sum_{s=1}^{\infty} \frac{2s}{3^{2s}} \right). \tag{2.2}$$

Now for every $\theta > 1$, the series $\sum_{s=1}^{\infty}(s/\theta^s)$ converges to $\theta/(\theta - 1)^2$ (substitute $x = 1/\theta$ in the identity $\sum_{s=1}^{\infty} sx^s = x/(1 - x)^2$ which follows by differentiating $1 + x + x^2 + \cdots = 1/(1 - x)$, and multiplying the resulting identity by x). Hence the series $\sum_{s=1}^{\infty}(2s/3^{2s})$ converges to 9/32, and we get

$$x_r \prod_s (1 - x_s)^{\Delta_{rs}} \geq \left(3e^{9/32}d\right)^{-2r} > (4d)^{-2r} = p_r. \tag{2.3}$$

By Lemma 2.2, the proof is complete. □

3. Bounded treewidth

We start with a simple result for trees. A *palindrome* is a sequence that reads the same forward and backward. A sequence $a_1 \cdots a_n$ is *palindrome-free* if none of its blocks is a palindrome. For this property to hold, it is sufficient and necessary that a_i, a_{i+1}, a_{i+2} are pairwise different for each $1 \leq i \leq n - 2$. If $a_1 \cdots a_n$ is a nonrepetitive sequence, with $c_i \in \{a, b, c\}$, then $a_1a_2da_3a_4d \cdots a_n$ is nonrepetitive and palindrome-free. Hence by Theorem 1.1, every path P_n has a 4-coloring which is nonrepetitive and palindrome-free.

Theorem 3.1. *$\pi(T) \leq 4$ for every tree T.*

Proof. Choose a root v_0 of T and arrange the vertices into levels L_i according to the distance from v_0, that is, $v \in L_i$ if and only if $d(v, v_0) = i$, $0 \leq i \leq n$. Let $a = a_0a_1 \cdots a_n$ be a nonrepetitive and palindrome-free sequence, with $a_i \in \{a, b, c, d\}$. Define a vertex coloring f by $f(v) = a_i$ if $v \in L_i$. We claim that f is nonrepetitive. Indeed, suppose that there is a path $P = v_1 \cdots v_{2r}$ in T such that $w' = f(v_1) \cdots f(v_r)$ is the same as $w'' = f(v_{r+1}) \cdots f(v_{2r})$. Since a is nonrepetitive, there must be a vertex in P, say v_h, whose neighbors v_{h-1}, v_{h+1} are on the same level L_i. Without loss of generality, we may assume that $1 < h \leq r$ and that v_h is the root of T. Then the sequence $w = w'w''$ looks as follows:

$$w = a_{h-1} \cdots a_1a_0a_1 \cdots a_{h-1}a_h \cdots a_{2r-h}. \tag{3.1}$$

If $h < r$, then a palindrome $a_1a_0a_1$ lies entirely in the first half w' of w. Since $w' = w''$, this palindrome appears in w'' and hence in a, which is a contradiction. If $h = r$, we get

$$w' = a_{r-1} \cdots a_1a_0, \qquad w'' = a_1 \cdots a_{r-1}a_r. \tag{3.2}$$

Again, the equality $w' = w''$ implies that $a_i = a_{r-i}$ for all $0 \leq i \leq r$. Hence the word $a_0 \cdots a_n$ is a palindrome, which is a contradiction. This completes the proof. □

In [16] Kündgen and Pelsmajer extended this theorem to k-trees. A *k-tree* is any graph that can be obtained, starting from a clique on k vertices, by repeating the following recursive step: add a new vertex and join it to k vertices of any existing clique. Thus, 1-trees are just the usual trees. The *treewidth* of a graph G is the least integer k such that G is a subgraph of a k-tree.

Let v_0 be any vertex of a connected graph G. A *levelling* with root v_0 is a function $\lambda : V(G) \to \mathbb{Z}$ defined by $\lambda(v) = d(v, v_0)$. The following lemma can be proved similarly as Theorem 3.1, using a nonrepetitive and palindrome-free sequence over 4 symbols.

Lemma 3.2 (palindrome lemma [16]). *For every levelling λ of a graph G, there is a 4-coloring of the vertices of G such that every repetitive path $u_1, \ldots, u_{2r}$ satisfies $\lambda(v_i) = \lambda(v_{i+r})$ for all $i = 1, \ldots, r$.*

The following theorem asserts that $\pi(G)$ is bounded for graphs of bounded treewidth.

Theorem 3.3 (Kündgen and Pelsmajer [16]). *$\pi(G) \leq 4^k$ for every k-tree G.*

Proof. We proceed by induction on k. The case $k = 1$ was proved in the previous theorem. So assume that the assertion holds up to $k - 1$, for some $k \geq 2$. Let $v_0, v_1, \ldots, v_n$ be a simplicial ordering of a k-tree G, that is, for every $1 \leq i \leq n$, the neighbors of v_i with indices smaller than i induce a clique in G. Let λ be a levelling of G with root v_0. Let $L_i = \{v \in V(G) : \lambda(v) = i\}$ and let G_i be a subgraph of G induced by the set L_i. Notice that each graph G_i is a subgraph of a $(k-1)$-tree. So by the inductive assumption, there exists a coloring h of the vertices of G by at most 4^{k-1} colors such that each subgraph G_i is colored nonrepetitively. Let g be a 4-coloring satisfying Lemma 3.2. Define a new coloring f by $f(v) = (g(v), h(v))$ for every vertex $v \in V(G)$. Clearly f uses at most 4^k colors. We claim that f is nonrepetitive. To prove this, assume that $P = u_1, \ldots, u_{2r}$ is a shortest repetitive path in G. Let $m = \max\{\lambda(u_i) : 1 \leq i \leq 2r\}$ and let $u_i, \ldots, u_j$ be a connected component of $P \cap L_m$, for some $1 \leq i \leq j \leq 2r$. By the inductive assumption and Lemma 3.2, we may assume that $1 \leq i \leq j \leq r$ and $1 < i$ or $j < r$. Suppose that $1 < i \leq j < r$. Then $u_{i-1}, u_{j+1} \in L_{m-1}$. By the simplicial ordering property, u_{i-1} and u_{j+1} are adjacent. By Lemma 3.2, the same happens in the second half of P. Hence the path $u_1, \ldots, u_{i-1}, u_{j+1}, \ldots, u_{i-1+r}, u_{j+1+r}, \ldots, u_{2r}$ is a shorter repetitive path in G. Verification of other cases is similar. □

A similar result (with a weaker bound) was obtained independently by Barát and Varjú [17] (cf. [18]). The proof uses fraternal orientations of k-trees, obtained by directing edges according to a simplicial ordering of G.

4. Planar graphs

Perhaps the most intriguing problem about nonrepetitive colorings is to decide whether $\pi(G)$ is bounded for planar graphs.

Conjecture 4.1. *There exists an integer N such that $\pi(G) \leq N$ for every planar graph G.*

There are some heuristic arguments supporting Conjecture 4.1. Let $\chi_k(G)$ be the least number of colors needed for a coloring of G so that no path on at most $2k$ vertices is repetitive. Thus, $\chi_1(G) = \chi(G)$ is the usual chromatic number, while $\chi_2(G)$ is known as the *star-chromatic number*. As observed independently by Kierstead and Kündgen and Nešetřil and Ossona de Mendez (personal communication), $\chi_k(G)$ is bounded for planar graphs, for every fixed $k \geq 1$. This is not trivial even for $k \geq 2$. To see this, suppose that $v_1, \ldots, v_n$ is a linear ordering of the vertices of G. Let $S(v_j)$ be the set of vertices v_i, with $i < j$, such that there is a path $v_j = v_{j_1}, \ldots, v_{j_r} = v_i$ satisfying $r \leq k+1$ and $i < j_m$ for all $1 \leq m \leq r-1$. Define $\mathrm{col}_k^*(G) = \min_L \max_{1 \leq j \leq n}(|S(v_j)| + 1)$ over all linear orderings L of G. Thus for $k = 1$, the number $\mathrm{col}_1^*(G)$ is the usual coloring number $\mathrm{col}(G)$ (e.g., $\mathrm{col}(G) \leq 6$ for every planar graph G).

Theorem 4.2. *$\chi_k(G) \leq \mathrm{col}_k^*(G)$, for every $k \geq 1$ and for every graph G.*

Proof. Let $L = \{v_1 < \cdots < v_n\}$ be a linear ordering of the vertices of G witnessing that $\mathrm{col}_k^*(G) = N$. Color the vertices by N colors greedily in that order so that each vertex v_j is colored differently than any of the vertices in S_j. We claim that this coloring is nonrepetitive on paths with at most $2k$ vertices. Suppose there is a repetitive path $P = u_1, \ldots, u_{2r}$, $r \leq k$. Let u_j be the earliest vertex on P in the order L. We may assume that $1 \leq j \leq r$. Then the vertex u_{j+r} is joined to u_j by a path of length at most k all of whose vertices are not earlier than u_j in the order L. Hence $u_j \in S(u_{j+r})$, and therefore the vertices u_j and u_{j+r} are colored differently. This contradicts repetitivity of P. □

A result of Kierstead and Yang [19] asserts that $\mathrm{col}_k^*(G)$ is bounded for every class of graphs closed under taking topological minors and having bounded coloring number $\mathrm{col}(G)$. In particular $\chi_k(G)$ is bounded for planar graphs, for every $k \geq 1$. The resulting bounds grow with k to infinity, but this may be due to the fact that the greedy coloring from the proof of Theorem 4.2 has much stronger properties. Indeed, it guarantees that every path of length at most k has a uniquely colored vertex.

Moreover, the proof of Theorem 4.2 works also for the *list version* of the problem, where a color for every vertex v is chosen from a preassigned list of colors L_v. A desired coloring exists for every $k \geq 1$, provided that $|L_v| \geq \mathrm{col}_k^*(G)$ for every vertex $v \in V(G)$. For the list version of $\pi(G)$, a greedy coloring argument will not work, even for the simplest case of paths. However, notice that the probabilistic proof of Theorem 2.3 is still valid if the colors are chosen from arbitrary lists of sufficiently large size.

Conjecture 4.3. *Every path P_n has a nonrepetitive coloring from lists of size at least three.*

Let F be a fixed graph and let $\mathcal{M}(F)$ be the class of graphs not containing F as a minor. Nešetřil and Ossona de Mendez [20] proved (by a different method) that for every such class and for every integer $k \geq 1$, there is a constant $N = N(F, k)$ such that every graph from the class satisfies $\chi_k(G) \leq N$. Again, a stronger property holds guaranteeing that every path of bounded length has a uniquely colored vertex.

Conjecture 4.4. *The Thue chromatic number $\pi(G)$ is bounded in every proper minor-closed class of graphs.*

A deep theorem of Robertson and Seymour asserts that if F is a planar graph, then $\mathcal{M}(F)$ has bounded treewidth. Hence Theorem 3.3 implies that planar graphs form the smallest minor-closed class for which $\pi(G)$ may be unbounded.

5. Subdivided graphs

Theorem 1.1 implies that every graph G has a subdivision which has a nonrepetitive 5-coloring. Indeed, subdivide each edge uv of G with a different odd number of vertices. Color the original vertices red, the middle vertices blue, and the remaining paths by colors $\{a,b,c\}$ in a nonrepetitive way. If there is a repetitive path P, then red and blue vertices must occupy the same positions in both halves of P. But this is impossible since any two subdivided edges have different numbers of vertices. Barát and Wood [21] improved this bound using Lemma 3.2.

Theorem 5.1 (Barát and Wood [21]). *Every graph G has a subdivision S such that $\pi(S) \leq 4$.*

Proof. Define a subdivision S of a graph G in the following way. Draw the vertices of a graph G in any order $v_1,\dots,v_n$ on a straight line l in the plane, and join the adjacent vertices by simple arcs. For each $1 \leq i \leq n$, draw a line l_i through v_i perpendicular to l. Subdivide the edges of G by adding vertices at the intersection points of the lines l_i with the arcs of a drawing. Let L_i be the set of vertices of S on the line l_i. This gives a levelling λ defined by $\lambda(v) = i$ if and only if $v \in L_i$. Let f be a 4-coloring of S satisfying Lemma 3.2. If there is a repetitive path, then it must cross the lowest level twice, which is clearly impossible. □

The following conjecture would be a nice generalization of Theorem 1.1.

Conjecture 5.2. *Every graph has a subdivision which is nonrepetitively 3-colorable.*

In [22], Conjecture 5.2 was confirmed for trees by using specific properties of Thue sequences. Clearly no result of the above type can hold in general if we restrict the number of vertices subdividing an edge of a graph. It would be interesting to find out if the following conjecture holds.

Conjecture 5.3. *There are constants k and N such that every planar graph has a subdivision, with at most k vertices subdividing an edge, which is nonrepetitively N-colorable.*

It is not excluded that the above statement actually implies Conjecture 4.1.

6. The rhythm threshold

Let $k \geq 2$ be a fixed integer. A vertex coloring f of a graph G is *k-repetitive* if there are an integer $r \geq 1$ and a path on kr vertices $v_1, v_2, \dots, v_{kr}$ such that $f(v_i) = f(v_{i+r}) = \cdots = f(v_{i+(k-1)r})$ for all $1 \leq i \leq r$. Otherwise, f is called *k-nonrepetitive*. In such a coloring, at most $k-1$ identical blocks may appear consecutively on a path in G. Let $\pi_k(G)$ denote the least number of colors in a k-nonrepetitive coloring of G. Notice that for $k \geq 3$, a k-nonrepetitive coloring may not be proper in the usual sense. Another classical result of Thue asserts that every path has a 3-nonrepetitive 2-coloring.

Theorem 6.1 (Thue [23]). *$\pi_3(P_n) = 2$ for every $n \geq 3$.*

The proof is constructive and uses the substitutions $S(a) = ab$ and $S(b) = ba$ in a similar way. Based on this construction, Currie and Fitzpatrick [24] proved that $\pi_3(C_n) = 2$

for all $n \geq 3$. Let $\pi_k(d)$ denote the supremum of $\pi_k(G)$, where G ranges over all graphs of maximum degree d. Extending the results of [15], we proved the following.

THEOREM 6.2 (Alon and Grytczuk [25]). *There exist absolute positive constants c_1, c_2 such that for all $k \geq 2$,*

$$\frac{c_1}{k}\frac{d^{k/(k-1)}}{(\log d)^{1/(k-1)}} \leq \pi_k(d) \leq c_2 d^{k/(k-1)}. \tag{6.1}$$

We also considered what happens if we fix d and let k be large. Define $t = t(d)$ as the minimum number such that $\pi_k(d) \leq t$ for some huge k. By the results for paths and cycles, it follows that $t(2) = 2$. No other value of $t(d)$ is known for $d \geq 2$, but one tempts to conjecture the following.

CONJECTURE 6.3. $t(d) = d$ *for every* $d \geq 1$.

The conjecture is supported by the following probabilistic result.

THEOREM 6.4 (Alon and Grytczuk [25]). $t(d) \leq d + 1$ *for every* $d \geq 1$.

From below, the function $t(d)$ is bounded by $(d+1)/2$. This can be seen by considering d regular graphs of sufficiently large girth. Using at most $d/2$ colors, long paths which are either monochromatic or alternating will appear.

Let $\mathcal{F}$ be any class of graphs. Define the *rhythm threshold* of $\mathcal{F}$ as the least number $t = t(\mathcal{F})$ for which there exists a finite number k such that $\pi_k(G) \leq t$ for every graph G in $\mathcal{F}$. In other words, for every k there is a graph G_k in $\mathcal{F}$ such that any vertex coloring of G_k using less than t colors is k-repetitive. The main problem is to decide whether $t(\mathcal{F})$ is finite for a given class $\mathcal{F}$. In [25], we proved that finiteness of $t(\mathcal{F})$ implies that $\mathcal{F}$ has bounded average degree, but $t(\mathcal{F}) = \infty$ already for 2-degenerate graphs.

CONJECTURE 6.5. $t(\mathcal{F})$ *is finite for every proper minor-closed class of graphs* $\mathcal{F}$.

At present, it is not known if the rhythm threshold is finite for planar graphs. By Theorem 3.3, $t(\mathcal{F})$ is finite if $\mathcal{F}$ has bounded treewidth, which implies as before that $t(\mathcal{F})$ is finite if $\mathcal{F}$ consists of graphs not containing a fixed *planar* graph as a minor. Therefore, planar graphs form the smallest minor-closed class of graphs for which the problem is open.

CONJECTURE 6.6. *The rhythm threshold of planar graphs is finite.*

Curiously, the least possible candidate number is four. Indeed, the class of triangular graphs (obtained iteratively from the triangle by inserting a new vertex into a face and joining it to the three vertices of that face) shows that three colors do not suffice. On the other hand, as proved by Berman and Paul [26], four colors suffice to avoid long monochromatic paths for graphs of arbitrary genus g.

7. Orientations and edge colorings

Let $\vec{G}$ be any orientation of a graph G and let $P = v_1 \cdots v_n$, $n \geq 2$, be a path in G. Denote by $s(P) = s_1 \cdots s_{n-1}$ a sequence of signs, defined by $s_i = +$ if $v_i v_{i+1}$ is a directed edge in

$\vec{G}$, and $s_i = -$ otherwise. An orientation $\vec{G}$ of a graph G is *k-repetitive* if there is a path $P = v_1 \cdots v_{kr+1}$ in G such that the sequence $s(P)$ consists of k identical blocks, that is, $s_i = s_{r+i} = s_{2r+i} = \cdots = s_{(k-1)r+i}$ for all $i = 1,\dots,r$. Let $\vec{\pi}(G)$ be the least integer k such that G admits an orientation without k-repetitive paths. Let $\vec{t}(\mathcal{F}) = \sup\{\vec{\pi}(G) : G \in \mathcal{F}\}$ be the *oriented rhythm threshold* of a class of graphs $\mathcal{F}$.

By Theorem 6.1, we have $\vec{\pi}(P_n) = 3$ for every n, and the fact that $\pi_3(C_n) = 2$ shows that $\vec{t}(\mathcal{F}) = 3$ for graphs of maximum degree at most two. However, as noticed by Alon (personal communication), the oriented rhythm threshold is infinite for 8-regular graphs. It is not clear what happens for planar graphs.

Conjecture 7.1. *The oriented rhythm threshold for planar graphs is finite.*

We show now that the above statement implies Conjecture 6.6. A vertex coloring f of G is *consistent* with orientation $\vec{G}$ if it is a proper coloring of G and all edges between any two color classes are oriented in the same direction (there are no two oriented edges ab, xy in $\vec{G}$ such that $f(a) = f(y)$ and $f(b) = f(x)$). The minimum number k, such that for every orientation $\vec{G}$ there is a k-coloring consistent with $\vec{G}$, is called the *oriented chromatic number* of G, denoted by $\chi_o(G)$.

Theorem 7.2. *Let $\mathcal{F}$ be a class of graphs with bounded oriented chromatic number and finite oriented rhythm threshold $\vec{t}(\mathcal{F})$. Then the rhythm threshold $t(\mathcal{F})$ is finite.*

Proof. Let $m = \max\{\chi_o(G) : G \in \mathcal{F}\}$ and let $k = \vec{t}(\mathcal{F})$. Let $\vec{G}$ be an orientation of a graph $G \in \mathcal{F}$ avoiding k-repetitive paths and let f be a vertex m-coloring consistent with $\vec{G}$. We claim that f is a $(k+1)$-nonrepetitive coloring of G. Indeed, suppose $P = v_1,\dots,v_{(k+1)r}$ is a $(k+1)$-repetitive path in G. By consistency of coloring f, the sequence $s(P) = s_1 \cdots s_{(k+1)n-1}$ must satisfy $s_i = s_{r+i} = s_{2r+i} = \cdots = s_{(k-1)r+i}$ for all $i = 1,\dots,r$. But this means that the path $v_1,\dots,v_{kr+1}$ is k-repetitive in the orientation $\vec{G}$, a contradiction. □

It is well known that $\chi_o(G) \le 80$ for every planar graph G (cf. [27]). This bound is a consequence of the famous result of Borodin [28] asserting that every planar graph has an acyclic 5-coloring (where the *acyclic coloring* is a proper vertex coloring with no 2-colored cycles).

A stronger connection holds in case of edge version of the rhythm threshold. Let $t'(\mathcal{F})$ be the *edge rhythm threshold* of the class of graphs $\mathcal{F}$ (defined analogously to $t(\mathcal{F})$). Using a result of Alon and Marshall [29], one can prove (similarly as Theorem 7.2, cf. [25]) that the finiteness of $t'(\mathcal{F})$ implies the finiteness of $t(\mathcal{F})$, provided that the acyclic chromatic number is bounded in $\mathcal{F}$. It is not hard to see that the reverse implication always holds, so the following statement is equivalent to Conjecture 6.6.

Conjecture 7.3. *The edge rhythm threshold for planar graphs is finite.*

8. Conclusion

There exist many interesting variants of nonrepetitive colorings of graphs. One can consider walks, induced paths, or other subgraphs instead of simple paths (cf. [21, 30–32]). We may also investigate Thue type colorings of other combinatorial structures, like hypergraphs, integer lattices, or Euclidean spaces (cf. [33–36]). In general, we look for colorings of a large structure distinguishing specified substructures that are in some sense "adjacent." From this perspective, the topic seems close to traditional graph coloring (at least in spirit). We conclude the paper with a problem illustrating this general philosophy.

Let G be a simple graph and let f be a coloring of its vertices. Two vertex disjoint subgraphs of G are *adjacent* if there is at least one edge between their vertex sets. For two subgraphs A, B of G, we write $f(A) = f(B)$ if there is a color preserving isomorphism between A and B. Consider a coloring f such that $f(A) \neq f(B)$ for each two adjacent connected induced subgraphs A, B of G, and let $\mu(G)$ be the minimum number of colors needed. Thue's theorem gives $\mu(P_n) = 3$ for every $n \geq 4$. Is it possible that $\mu(G)$ stays bounded for planar graphs?

Acknowledgments

The author would like to thank Noga Alon for helpful suggestions and stimulating discussions. The author is supported by Grant KBN 1P03A 017 27.

References

[1] A. Thue, "Über unendliche Zeichenreichen," *Norske Videnskabers Selskabs Skrifter, I Mathematisch-Naturwissenschaftliche Klasse, Christiania*, vol. 7, pp. 1–22, 1906.

[2] J.-P. Allouche and J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press, Cambridge, UK, 2003.

[3] J. Berstel, *Axel Thue's Papers on Repetitions in Words: A Translation*, vol. 20 of *Publications du LaCIM*, Université du Québec a Montréal, Montreal, Quebec, Canada, 1995.

[4] J. Berstel, "Axel Thue's work on repetitions in words," in *Séries Formelles et Combinatoire Algébrique*, P. Leroux and C. Reutenauer, Eds., Publications du LaCIM, pp. 65–80, Université du Québec a Montréal, Montreal, Quebec, Canada, 1992.

[5] M. Lothaire, *Combinatorics on Words*, vol. 17 of *Encyclopedia of Mathematics and Its Applications*, Addison-Wesley, Reading, Mass, USA, 1983.

[6] M. Lothaire, *Algebraic Combinatorics on Words*, vol. 90 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 2002.

[7] R. O. Shelton, "Aperiodic words on three symbols," *Journal für die Reine und Angewandte Mathematik*, vol. 321, pp. 195–209, 1981.

[8] R. O. Shelton, "Aperiodic words on three symbols. II," *Journal für die Reine und Angewandte Mathematik*, vol. 327, pp. 1–11, 1981.

[9] R. O. Shelton and R. P. Soni, "Aperiodic words on three symbols. III," *Journal für die Reine und Angewandte Mathematik*, vol. 330, pp. 44–52, 1982.

[10] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, NY, USA, 2nd edition, 2000.

[11] J. Beck, "An application of Lovász local lemma: there exists an infinite 01-sequence containing no near identical intervals," in *Finite and Infinite Sets, Vol. I, II (Eger, 1981)*, vol. 37 of *Colloquia Mathematica Societatis János Bolyai*, pp. 103–107, North-Holland, Amsterdam, The Netherlands, 1984.

[12] J. D. Currie, "Pattern avoidance: themes and variations," *Theoretical Computer Science*, vol. 339, no. 1, pp. 7–18, 2005.

[13] J. Grytczuk, "Thue-like sequences and rainbow arithmetic progressions," *Electronic Journal of Combinatorics*, vol. 9, no. 1, R44, pp. 1–10, 2002.

[14] J. D. Currie, "There are ternary circular square-free words of length n for $n \geq 18$," *Electronic Journal of Combinatorics*, vol. 9, no. 1, N10, pp. 1–7, 2002.

[15] N. Alon, J. Grytczuk, M. Hałuszczak, and O. Riordan, “Nonrepetitive colorings of graphs,” *Random Structures and Algorithms*, vol. 21, no. 3-4, pp. 336–346, 2002.
[16] A. Kündgen and M. J. Pelsmajer, “Nonrepetitive colorings of graphs of bounded treewidth,” preprint, 2003.
[17] J. Barát and P. P. Varjú, “On square-free vertex colorings of graphs,” to appear in *Studia Scientiarum Mathematicarum Hungarica*.
[18] J. Grytczuk, “Nonrepetitive graph coloring,” in *Graph Theory, Trends in Mathematics*, pp. 209–218, Birkhäuser, Boston, Mass, USA, 2006.
[19] H. A. Kierstead and D. Yang, “Orderings on graphs and game coloring number,” *Order*, vol. 20, no. 3, pp. 255–264, 2003.
[20] J. Nešetřil and P. Ossona de Mendez, “Tree-depth, subgraph coloring and homomorphism bounds,” *European Journal of Combinatorics*, vol. 27, no. 6, pp. 1022–1041, 2006.
[21] J. Barát and D. Wood, “Notes on nonrepetitive graph colouring,” preprint, 2005, http://arxiv.org/abs/math.CO/0509608.
[22] B. Brešar, J. Grytczuk, S. Klavžar, S. Niwczyk, and I. Peterin, “Nonrepetitive colorings of trees,” *Discrete Mathematics*, vol. 307, no. 2, pp. 163–172, 2007.
[23] A. Thue, “Über die gegenseitigen Lage gleicher Teile gewisser Zeichenreihen,” *Norske Videnskabers Selskabs Skrifter, I Mathematisch-Naturwissenschaftliche Klasse, Christiania*, vol. 1, pp. 1–67, 1912.
[24] J. D. Currie and D. S. Fitzpatrick, “Circular words avoiding patterns,” in *Developments in Language Theory*, vol. 2450 of *Lecture Notes in Comput. Sci.*, pp. 319–325, Springer, Berlin, Germany, 2003.
[25] N. Alon and J. Grytczuk, “Breaking the rhythm on graphs,” to appear in *Discrete Mathematics*.
[26] K. A. Berman and J. L. Paul, “A 4-color theorem for surfaces of genus *g*,” *Proceedings of the American Mathematical Society*, vol. 105, no. 2, pp. 513–522, 1989.
[27] E. Sopena, “Oriented graph coloring,” *Discrete Mathematics*, vol. 229, no. 1–3, pp. 359–369, 2001.
[28] O. V. Borodin, “On acyclic colorings of planar graphs,” *Discrete Mathematics*, vol. 25, no. 3, pp. 211–236, 1979.
[29] N. Alon and T. H. Marshall, “Homomorphisms of edge-colored graphs and Coxeter groups,” *Journal of Algebraic Combinatorics*, vol. 8, no. 1, pp. 5–13, 1998.
[30] B. Brešar and S. Klavžar, “Square-free colorings of graphs,” *Ars Combinatoria*, vol. 70, pp. 3–13, 2004.
[31] J. Barát and P. P. Varjú, “On square-free edge colorings of graphs,” to appear in *Ars Combinatoria*.
[32] J. D. Currie, “Which graphs allow infinite nonrepetitive walks?” *Discrete Mathematics*, vol. 87, no. 3, pp. 249–260, 1991.
[33] J. Grytczuk and W. Śliwa, “Nonrepetitive colorings of infinite sets,” *Discrete Mathematics*, vol. 265, no. 1–3, pp. 365–373, 2003.
[34] D. R. Bean, A. Ehrenfeucht, and G. F. McNulty, “Avoidable patterns in strings of symbols,” *Pacific Journal of Mathematics*, vol. 85, no. 2, pp. 261–294, 1979.
[35] J. D. Currie and J. Simpson, “Non-repetitive tilings,” *Electronic Journal of Combinatorics*, vol. 9, no. 1, R28, pp. 1–13, 2002.
[36] J. D. Currie and C. W. Pierce, “The fixing block method in combinatorics on words,” *Combinatorica*, vol. 23, no. 4, pp. 571–584, 2003.

Jarosław Grytczuk: Algorithmics Research Group, Faculty of Mathematics and Computer Science, Jagiellonian University, 30-387 Kraków, Poland; Department of Didactics of Mathematics and Number Theory, Faculty of Mathematics, Computer Science, and Econometrics, University of Zielona Góra, 65-516 Zielona Góra, Poland
Email addresses: grytczuk@tcs.uj.edu.pl; j.grytczuk@wmie.uz.zgora.pl