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Research Article
About Some Linear Operators

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Using the method of Jakimovski and Leviatan from their work in 1969, we construct a general class of linear positive operators. We study the convergence, the evaluation for the rate of convergence in terms of the first modulus of smoothness and we give a Voronovskaja-type theorem for these operators.

1. Introduction

The aim of this paper is to construct a class of linear operators in more general conditions. The method was inspired by Jakimovski and Leviatan (see [1]). We do not study the convergence of these operators with the well-known theorem of Bohman-Korovkin. The evaluation theorems for the rate of convergence are different from the well-known theorem of Shisha-Mond. We prove the Voronovskaja-type theorem for these operators. In the end, we give particularizations of these operators.

We recall some notions and results which we will use in this paper.

Let $\mathbb{N}$ be the set of positive integer numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a given interval I, we will use the following function sets: $B(I) = \{f \mid f : I \to \mathbb{R}, f \text{ bounded on } I\}$, $C(I) = \{f \mid f : I \to \mathbb{R}, f \text{ continuous on } I\}$, and $C_B(I) = B(I) \cap C(I)$.

For any $x \in I$, consider the functions $\psi_x : I \to \mathbb{R}$ defined by $\psi_x(t) = t - x$ and $e_i : I \to \mathbb{R}$, $e_i(t) = t^i$ for any $t \in I$, $i \in \{0,1,2,3,4\}$.

For $f \in C_B(I)$, by the first-order modulus of smoothness of f is meant the function $\omega(f;\cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = \sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\}. \tag{1.1}$$

In the following, we take into account the properties of the first-order modulus of smoothness and the properties of the linear positive functional.

Lemma 1.1. *Let $f \in C_B(I)$. Then, $\omega(f;\cdot)$ has the following properties:*
(a) $\omega(f;0) = 0$,
(b) *$\omega(f;\cdot)$ is an increasing function,*
(c) *$\omega(f;\cdot)$ is a uniform continuous function on I,*
(d) *for any $\delta > 0$, $x,t \in I$, one has $|f(t) - f(x)| \leq [1+\delta^{-2}\psi_x^2(t)]\omega(f;\delta)$.*

Lemma 1.2. *Let $A : E(I) \to \mathbb{R}$ be a linear positive functional. Then,*
(a) *for $f,g \in E(I)$ with $f(x) \leq g(x)$ for any $x \in I$, one has*

$$A(f) \leq A(g); \tag{1.2}$$

(b) *$|A(f)| \leq A(|f|)$ for any $f \in E(I)$, where $E(I)$ is a subset of the set of real functions defined on I.*

In [2] we have demonstrated the following theorem.

Theorem 1.3. *Let I be an interval $x \in I$, and let the function $f : I \to \mathbb{R}$ be s times differentiable in x. According to the Taylor Expansion Theorem, one has*

$$f(t) = \sum_{i=0}^{s} \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x), \tag{1.3}$$

where μ is a bounded function and $\lim_{t\to x} \mu(t-x) = 0$. If $f^{(s)}$ is a continuous function on I, then for any $\delta > 0$ and $x \in I$ one has

$$|(\mu(t-x)| \leq \frac{1}{s!}[1+\delta^{-2}\psi_x^2(t)]\omega(f^{(s)};\delta). \tag{1.4}$$

2. Preliminaries

In this section, we construct a general class of linear and positive operators and we demonstrate for these operators an approximation theorem and a Voronovskaja-type theorem.

Let I, J be intervals and $I \cap J$ is a nonempty interval. For any $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$, consider the function $\varphi_{m,k} : J \to \mathbb{R}$ with the property $\varphi_{m,k}(x) \geq 0$ for any $x \in J$ and the linear and positive functional $A_{m,k} : E(I) \to \mathbb{R}$.

In the following, let $E(I)$ and $F(J)$ be subsets of the set of real functions defined on I, J respectively, such that the series $\sum_{k=0}^{\infty} \varphi_{m,k}(x)A_{m,k}(f)$ is convergent for any $f \in E(I)$ and any $x \in J$. We suppose that $\psi_x^i \in E(I)$ for any $x \in I \cap J$ and any $i \in \{0,1,\dots,s+2\}$.

In what follows $s \in \mathbb{N}_0$, s is even.

Definition 2.1. For $m \in \mathbb{N}$, define the operator $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x)A_{m,k}(f) \tag{2.1}$$

for any $f \in E(I)$ and $x \in J$.

Proposition 2.2. *The operators $(L_m)_{m\geq 1}$ are linear and positive on $E(I \cap J)$.*

Proof. The proof follows immediately. □

Definition 2.3. For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, define T_i by

$$(T_i L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{\infty} \varphi_{m,k}(x)A_{m,k}(\psi_x^i) \tag{2.2}$$

for any $x \in I \cap J$.

Theorem 2.4. *If $f \in E(I)$ is an s-times differentiable function in $x \in I \cap J$, with $f^{(s)}$ continuous in x, and if there exist α_s, $\alpha_{s+2} \in [0,\infty)$ and $m(s) \in \mathbb{N}$ such that*

$$\alpha_{s+2} < \alpha_s + 2 \tag{2.3}$$

and $(T_s L_m)(x)/m^{\alpha_s}$, $(T_{s+2}L_m)(x)/m^{\alpha_{s+2}}$ are bounded for any $m \in \mathbb{N}$, $m \geq (s)$, then

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s} \frac{1}{i! m^i}(T_i L_m)(x) f^{(i)}(x)\right] = 0. \tag{2.4}$$

Assume that f is an s times differentiable function on I with $f^{(s)}$ continuous on I and an interval $K \subset I \cap J$ exists such that there exist $m(s) \in \mathbb{N}$ and the constants $k_j(K) \in \mathbb{R}$ depending on K, so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and $x \in K$, one has

$$\frac{(T_j L_m)(x)}{m^{\alpha_j}} \leq k_j(K), \tag{2.5}$$

where $j \in \{s, s+2\}$. Then, the convergence given in (2.4) is uniform on K and

$$\begin{aligned} m^{s-\alpha_s}&\left|(L_m f)(x) - \sum_{i=0}^{s} \frac{1}{i! m^i}(T_i L_m)(x) f^{(i)}(x)\right| \\ &\leq \frac{1}{s!}\left(k_s(K) + k_{s+2}(K)\right)\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right) \end{aligned} \tag{2.6}$$

for any $x \in K$ and $m \geq m(s)$.

Proof. According to Taylor's Theorem, we have

$$f(t) = \sum_{i=0}^{s} \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x), \tag{2.7}$$

where μ is a bounded function and $\lim_{t\to x}\mu(t-x) = 0$.

Hence, from (2.7), we have

$$A_{m,k}(f) = \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} A_{m,k}(\psi_x^i) + A_{m,k}(\psi_x^s \mu_x), \tag{2.8}$$

where $\mu_x : I \to \mathbb{R}$, $\mu_x(t) = \mu(t-x)$, for any $t \in I \cap J$.

Multiplying by $\varphi_{m,k}(x)$ and summing over $k \in \mathbb{N}_0$, we obtain

$$(L_m f)(x) = \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!} (L_m \psi_x^i)(x) + \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x). \tag{2.9}$$

Thus,

$$m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i! m^i}(T_i L_m)(x)\right] = (R_m f)(x), \tag{2.10}$$

where

$$(R_m f)(x) = m^{s-\alpha_s} \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x). \tag{2.11}$$

Then,

$$|(R_m f)(x)| \le m^{s-\alpha_s} \sum_{k=0}^{\infty} \varphi_{m,k}(x) |A_{m,k}(\psi_x^s \mu_x)| \tag{2.12}$$

and taking Lemma 1.2 into account, we obtain

$$|(R_m f)(x)| \le m^{s-\alpha_s} \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s |\mu x|). \tag{2.13}$$

According to the relation (1.4), for any $\delta > 0$ and $t \in I \cap J$, we have

$$|\mu_x(t)| = |\mu(t-x)| \le \frac{1}{s!}\left[1 + \delta^{-2}\psi_x^2(t)\right]\omega(f^{(s)};\delta), \tag{2.14}$$

and so

$$(\psi_x^s |\mu_x|)(t) \le \frac{1}{s!}\left[\psi_x^s(t) + \delta^{-2}\psi_x^{s+2}(t)\right]\omega(f^{(s)};\delta). \tag{2.15}$$

From (2.13) and (2.15), it results that

$$|(R_m f)(x)| \leq \frac{1}{s!} m^{s-\alpha_s} \left[\sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s) + \delta^{-2} \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^{s+2}) \right] \omega(f^{(s)};\delta). \tag{2.16}$$

Thus,

$$|(R_m f)(x)| \leq \frac{1}{s!} \left[\frac{(T_s L_m)(x)}{m^{\alpha_s}} + \delta^{-2} \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} m^{-2-\alpha_s+\alpha_{s+2}} \right] \omega(f^{(s)};\delta). \tag{2.17}$$

Considering $\delta = 1/\sqrt{m^{2+\alpha_2-\alpha_{s+2}}}$, the inequality above becomes

$$|(R_m f)(x)| \leq \frac{1}{s!} \left[\frac{(T_s L_m)(x)}{m^{\alpha_s}} + \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} \right] \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}} \right). \tag{2.18}$$

Taking into account that $(T_s L_m)(x)/m^{\alpha_s}$ and $(T_{s+2} L_m)(x)/m^{\alpha_{s+2}}$ are bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, and considering the fact that

$$\lim_{m\to\infty} \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}} \right) = \omega(f^{(s)};0) = 0, \tag{2.19}$$

we have that

$$\lim_{m\to\infty} (R_m f)(x) = 0. \tag{2.20}$$

From (2.10) and (2.20), (2.4) follows.

If in addition (2.5) takes place then, (2.18) becomes

$$|(R_m f)(x)| \leq \frac{1}{s!} (k_s(K) + k_{s+2}(K)) \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}} \right), \tag{2.21}$$

for $m \geq m(s)$ and $x \in K$. Therefore, the convergence from (2.4) is uniform on K. Now, (2.10) and (2.21) yield (2.6). □

In the following, we suppose that for any $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we have

$$A_{m,k}(e_0) = 1, \tag{2.22}$$

and for any $x \in I \cap J$ and $m \in \mathbb{N}$

$$\sum_{k=0}^{\infty} \varphi_{m,k}(x) = 1. \tag{2.23}$$

Remark 2.5. Taking (2.22) and (2.23) into account, it results that

$$(T_0 L_m)(x) = 1 \tag{2.24}$$

for any $x \in I \cap J$ and $m \in \mathbb{N}$.

Remark 2.6. In Theorem 2.4, we choose the smallest α_s and α_{s+2}, if they exist.

Remark 2.7. Taking (2.24) into account, we choose $\alpha_0 = 0$.

Remark 2.8. For $s = 0$, $s = 2$, respectively, we state two corollaries which we will use in the section Main results.

Corollary 2.9. *If $f \in E(I)$ is a continuous function in $x \in I \cap J$, and if there exist α_2 and $m(0) \in \mathbb{N}$ such that*

$$0 \leq \alpha_2 < 2 \tag{2.25}$$

and $(T_2L_m)(x)/m^{\alpha_2}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$, then

$$\lim_{m\to\infty} (L_m f)(x) = f(x). \tag{2.26}$$

Assume that f is continuous on I and an interval $K \subset I \cap J$ exists, such that there exist $m(0) \in \mathbb{N}$ and $k_2(K)$ so that for any $m \in \mathbb{N}$, $m \geq m(0)$, and $x \in K$, one has

$$\frac{(T_2L_m)(x)}{m^{\alpha_2}} \leq k_2(K). \tag{2.27}$$

Then, the convergence given in (2.26) is uniform on K and

$$\left| (L_m f)(x) - f(x) \right| \leq (1 + k_2(K))\,\omega\left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{2.28}$$

for any $x \in K$ and $m \geq m(0)$.

Corollary 2.10. *If $f \in E(I)$ is a two-times differentiable function in $x \in I \cap J$, with $f^{(2)}$ continuous in x, and if there exist α_2, α_4 and $m(2) \in \mathbb{N}$ such that*

$$\begin{gathered} 0 \leq \alpha_2 < 2, \\ 0 \leq \alpha_4 < \alpha_2 + 2, \end{gathered} \tag{2.29}$$

$(T_2L_m)(x)/m^{\alpha_2}$ and $((T_4L_m)(x))/m^{\alpha_4}$ are bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, then

$$\lim_{m\to\infty} m^{2-\alpha_2}\left[(L_m f)(x) - f(x) - \frac{1}{m}(T_1L_m)(x) f^{(1)}(x) - \frac{1}{2m^2}(T_2L_m)(x) f^{(2)}(x) \right] = 0. \tag{2.30}$$

Assume that f is a two-times differentiable function on I with $f^{(2)}$ continuous on I and an interval $K \subset I \cap J$ exists, such that there exist $m(2) \in \mathbb{N}$ and $k_j(K)$, so that for any $m \geq m(2)$ and $x \in K$, one has

$$\frac{(T_jL_m)(x)}{m^{\alpha_j}} \leq k_j(K), \tag{2.31}$$

where $j \in \{2,4\}$. Then, the convergence given in (2.30)is uniform on K.

Remark 2.11. Theorem 2.4, Corollary 2.9, and 2.10 are Voronovskaja-type theorems.

3. Main results

In this section, we construct a general class of linear positive operators. Let $\mathbb{R}_0 = [0,\infty)$ and J be an interval with $J \subset \mathbb{R}_0$. Let the sequence $(a_m)_{m\geq 1}$ so that $a_m x \in J$ for any $m \in \mathbb{N}$ and $x \in J$. The indefinitely differentiable functions $a, b : J \to \mathbb{R}$ have the property:

$$b(x) > 0 \tag{3.1}$$

for any $x \in \mathbb{R}_0$,

$$a(1) \neq 0 \tag{3.2}$$

and for any compact $K \subset J$ the constants $M_1(K)$, $M_2(K)$ depending on K exist, such that

$$\begin{aligned} |a^{(k)}(x)| &\leq M_1(K), \\ |b^{(k)}(x)| &\leq M_2(K) \end{aligned} \tag{3.3}$$

for any $x \in K$ and $k \in \mathbb{N}_0$.

Then, it is known that

$$\begin{aligned} a(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} a^{(n)}(0) x^n, \\ b(x) &= \sum_{p=0}^{\infty} \frac{1}{p!} b^{(p)}(0) x^p \end{aligned} \tag{3.4}$$

for any $x \in J$.

If $u, x, ux \in J$, we calculate

$$a(u)b(ux) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} a^{(n)}(0) u^n \right) \left(\sum_{p=0}^{\infty} \frac{1}{p!} b^{(p)}(0)(ux)^p \right) \tag{3.5}$$

and we take it to the form

$$a(u)b(ux) = \sum_{k=0}^{\infty} p_k(x) u^k, \tag{3.6}$$

where

$$p_k(x) = \sum_{i=0}^{k} \frac{1}{i!(k-i)!} a^{(i)}(0) b^{(k-i)}(0) x^{k-i}. \tag{3.7}$$

Remark 3.1. If $u = 1$, then from (3.6), we obtain

$$a(1)b(a_m x) = \sum_{k=0}^{\infty} p_k(a_m x) \tag{3.8}$$

for any $m \in \mathbb{N}$ and $x \in J$.

Remark 3.2. We consider that the conditions $a^{(i)}(0)b^{(k-i)}(0)/a(1) \geq 0$, $i \in \{0,1,\ldots,k\}$ and $k \in \mathbb{N}_0$, hold and then it results that $a(1)p_k(x) \geq 0$ for any $x \in J$ and any $k \in \mathbb{N}_0$.

In the following, let a fixed function $w : \mathbb{R}_0 \to (0,\infty)$, called the weight function, and the set functions

$$E(w) = \{f \mid f : \mathbb{R}_0 \to \mathbb{R} \text{ such that } wf \text{ is bounded on} [0,\infty)\}. \tag{3.9}$$

For $f \in E(w)$, there exists a positive constant M such that $w(x)|f(x)| \leq M$ for any $x \in \mathbb{R}_0$. For $m \in \mathbb{N}$ and $x \in J$, and taking in the end (3.8) into account, we have

$$\begin{aligned} \left| \frac{1}{a(1)b(a_m x)} \sum_{k=0}^{\infty} p_k(a_m x) f\left(\frac{k}{m}\right) \right| &\leq \frac{1}{a(1)b(a_m x)} \sum_{k=0}^{\infty} p_k(a_m x) \left| f\left(\frac{k}{m}\right) \right| \\ &\leq \frac{M}{w(x)} \frac{1}{a(1)b(a_m x)} \sum_{k=0}^{\infty} p_k(a_m x) = \frac{M}{w(x)}, \end{aligned} \tag{3.10}$$

from where it results that the series $(1/a(1)b(a_m x)) \sum_{k=0}^{\infty} p_k(a_m x) f(k/m)$ is convergent.

Definition 3.3. For $m \in \mathbb{N}$, define the operator $L_m : E(w) \to F(J)$ by

$$(L_m f)(x) = \frac{1}{a(1)b(a_m x)} \sum_{k=0}^{\infty} p_k(a_m x) f\left(\frac{k}{m}\right) \tag{3.11}$$

for any $f \in E(w)$ and $x \in J$, where $F(J)$ is a subset of the set of real functions defined on J.

Remark 3.4. The operators $(L_m)_{m \geq 1}$ are linear and positive on $E(w)$.

In the following, we consider that for any $x \in J$, we have $\psi_x^i \in E(w)$, $i \in \{1,2,3,4\}$.

Definition 3.5. For $m \in \mathbb{N}$ and $i \in \{1,2,3,4\}$, define T_i by

$$(T_iL_m)(x) = m^i(L_m\psi_x^i)(x) = m^i\frac{1}{a(1)b(a_mx)}\sum_{k=0}^{\infty}p_k(a_mx)\left(\frac{k}{m}-x\right)^i \tag{3.12}$$

for any $x \in J$.

Lemma 3.6. *One has*

$$(L_me_0)(x) = 1, \tag{3.13}$$

$$\begin{aligned}
(L_me_1)(x) &= \frac{a_m}{m}\frac{b^{(1)}(a_mx)}{b(a_mx)}x + \frac{1}{m}\frac{a^{(1)}(1)}{a(1)},\\
(L_me_2)(x) &= \left(\frac{a_m}{m}\right)^2\frac{b^{(2)}(a_mx)}{b(a_mx)}x^2 + \frac{1}{m}\frac{a_m}{m}\frac{a(1)+2a^{(1)}(1)}{a(1)}\frac{b^{(1)}(a_mx)}{b(a_mx)}x + \frac{1}{m^2}\frac{a^{(1)}(1)+a^{(2)}(1)}{a(1)},\\
(L_me_3)(x) &= \left(\frac{a_m}{m}\right)^3\frac{b^{(3)}(a_mx)}{b(a_mx)}x^3 + \frac{1}{m}\left(\frac{a_m}{m}\right)^2\frac{3a(1)+3a^{(1)}(1)}{a(1)}\frac{b^{(2)}(a_mx)}{b(a_mx)}x^2\\
&\quad + \frac{1}{m^2}\frac{a_m}{m}\frac{a(1)+6a^{(1)}(1)+3a^{(2)}(1)}{a(1)}\frac{b^{(1)}(a_mx)}{b(a_mx)}x + \frac{1}{m^3}\frac{a^{(1)}+3a^{(2)}(1)+a^{(3)}(1)}{a(1)},\\
(L_me_4)(x) &= \left(\frac{a_m}{m}\right)^4\frac{b^{(4)}(a_mx)}{b(a_mx)}x^4 + \frac{1}{m}\left(\frac{a_m}{m}\right)^3\frac{6a(1)+4a^{(1)}(1)}{a(1)}\frac{b^{(3)}(a_mx)}{b(a_mx)}x^3\\
&\quad + \frac{1}{m^2}\left(\frac{a_m}{m}\right)^2\frac{7a(1)+18a^{(1)}(1)+6a^{(2)}(1)}{a(1)}\frac{b^{(2)}(a_mx)}{b(a_mx)}x^2\\
&\quad + \frac{1}{m^3}\frac{a_m}{m}\frac{a(1)+14a^{(1)}(1)+18a^{(2)}(1)+4a^{(3)}(1)}{a(1)}\frac{b^{(1)}(a_mx)}{b(a_mx)}x\\
&\quad + \frac{1}{m^4}\frac{a^{(1)}(1)+7a^{(2)}(1)+6a^{(3)}(1)+a^{(4)}(1)}{a(1)}
\end{aligned} \tag{3.14}$$

for any $x \in J$ and $m \in \mathbb{N}$.

Proof. The relation (3.13) results from (3.8). The proof of relations (3.14) follows immediately by differentiating (3.6) with respect to u, and after that take 1 for u and a_mx for x. □

Lemma 3.7. *For $x \in J$ and $m \in \mathbb{N}$, the following hold*

$$
\begin{aligned}
(T_0L_m)(x) &= 1,\\
(T_1L_m)(x) &= -m\left(1-\frac{a_m}{m}\frac{b^{(1)}(a_mx)}{b(a_mx)}\right)x+\frac{a^{(1)}(1)}{a(1)},
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
(T_2L_m)(x) = {}&-m^2\left[1-\left(\frac{a_m}{m}\right)^2\frac{b^{(2)}(a_mx)}{b(a_mx)}\right]x^2\\
&+m^2\left(1-\frac{a_m}{m}\frac{b^{(1)}(a_mx)}{b(a_mx)}\right)\left(2x^2-\frac{1}{m}\frac{a(1)+2a^{(2)}(1)}{a(1)}x\right)\\
&+mx+\frac{a^{(1)}(1)+a^{(2)}(1)}{a(1)},
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
(T_4L_m)(x) = {}&-m^4\left[1-\left(\frac{a_m}{m}\right)^4\frac{b^{(4)}(a_mx)}{b(a_mx)}\right]x^4\\
&+m^4\left[1-\left(\frac{a_m}{m}\right)^3\frac{b^{(3)}(a_mx)}{b(a_mx)}\right]\left(4x^4-\frac{1}{m}\frac{6a(1)+4a^{(1)}(1)}{a(1)}x^3\right)\\
&+m^4\left[1-\left(\frac{a_m}{m}\right)^2\frac{b^{(2)}(a_mx)}{b(a_mx)}\right]\left(-6x^4+4\frac{1}{m}\frac{3a(1)+3a^{(1)}(1)}{a(1)}x^3\right.\\
&\qquad\left.-\frac{1}{m^2}\frac{7a(1)+18a^{(1)}(1)+6a^{(2)}(1)}{a(1)}x^2\right)\\
&+m^4\left(1-\frac{a_m}{m}\frac{b^{(1)}(a_mx)}{b(a_mx)}\right)\left(4x^4-6\frac{1}{m}\frac{a(1)+2a^{(1)}(1)}{a(1)}x^3\right.\\
&\qquad+4\frac{1}{m^2}\frac{a(1)+6a^{(1)}+3a^{(2)}(1)}{a(1)}x^2\\
&\qquad\left.-\frac{1}{m^3}\frac{a(1)+14a^{(1)}+18a^{(2)}(1)+4a^{(3)}(1)}{a(1)}x\right)\\
&+3m^2x^2+\frac{a(1)+10a^{(1)}(1)+6a^{(2)}(1)}{a(1)}mx\\
&+\frac{a^{(1)}(1)+7a^{(2)}(1)+6a^{(3)}(1)+a^{(4)}(1)}{a(1)}.
\end{aligned}
\tag{3.17}
$$

Proof. The proof follows immediately from (3.12) and Lemma 3.6. □

Theorem 3.8. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be a function, $f\in E(w)$. If $x\in\mathbb{R}_0$, f is continuous in x, α_2 and $m(0)\in\mathbb{N}$ exist such that*

$$1\le\alpha_2<2 \tag{3.18}$$

and $m^{2-\alpha_2}|1-(a_m/m)^i(b^{(i)}(a_mx)/b(a_mx))|$ is bounded for any $m\in\mathbb{N}$, $m\ge m(0)$, where $i\in\{1,2\}$, then

$$\lim_{m\to\infty}(L_mf)(x)=f(x). \tag{3.19}$$

Assume that f is continuous on $\mathbb{R}_0$ and a compact interval $K \subset \mathbb{R}_0$ exists, such that there exist $m(0) \in \mathbb{N}$ and $l_i(K)$ so that for any $m \in \mathbb{N}$, $m \geq m(0)$, and $x \in K$, one has

$$m^{2-\alpha_2}\left|1-\left(\frac{a_m}{m}\right)^i\frac{b^{(i)}(a_mx)}{b(a_mx)}\right| \leq l_i(K), \tag{3.20}$$

where $i \in \{1,2\}$.

Then, the convergence given in (3.19) is uniform in K and

$$|(L_mf)(x)-f(x)| \leq M(K)\omega\left(f;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{3.21}$$

for any $x \in K$ and any $m \geq m(0)$, where $M(K)$ is a constant depending on K.

Proof. Because $m^{2-\alpha_2}|1-(a_m/m)^i(b^{(i)}(a_mx)/b(a_mx))|$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$, it results that $(T_2L_m)(x)/m^{\alpha_2}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$. Taking relation (3.16) into account, we apply now the Corollary 2.9. The proof is similar on a compact interval K. □

Theorem 3.9. *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be a function, $f \in E(w)$. If $x \in \mathbb{R}_0$, f is a two times differentiable function in x with $f^{(2)}$ continuous in x, α_2, α_4 and $m(2) \in \mathbb{N}$ exist such that*

$$1 \leq \alpha_2 < 2, \tag{3.22}$$

$$2 \leq \alpha_4 < \alpha_2+2, \tag{3.23}$$

$m^{4-\alpha_4}|1-(a_m/m)^i(b^{(i)}(a_mx)/b(a_mx))|$ is bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, where $i \in \{1,2,3,4\}$, then

$$\lim_{m\to\infty} m^{2-\alpha_2}\left[(L_mf)(x)-f(x)-\frac{1}{m}(T_1L_m)(x)f^{(1)}(x)-\frac{1}{2m^2}(T_2L_m)(x)f^{(2)}(x)\right]=0. \tag{3.24}$$

In addition, if the limit $\lim_{m\to\infty}((T_2L_m)(x)/m^{\alpha_2})$ exists and

$$\lim_{m\to\infty}\frac{(T_2L_m)(x)}{m^{\alpha_2}} = B_2(x) \in \mathbb{R}, \tag{3.25}$$

then

$$\lim_{m\to\infty} m^{2-\alpha_2}\left[(L_mf)(x)-f(x)-\frac{1}{m}(T_1L_m)(x)f^{(1)}(x)\right]=\frac{1}{2}B_2(x)f^{(2)}(x). \tag{3.26}$$

Assume that f is a two-times differentiable function on $\mathbb{R}_0$ with $f^{(2)}$ continuous on $\mathbb{R}_0$ and a compact interval $K \subset \mathbb{R}_0$ exists, such that there exist $m(2) \in \mathbb{N}$ and $l_i(K)$ so that for any $m \geq m(2)$ and $x \in K$, one has

$$m^{4-\alpha_4}\left|1-\left(\frac{a_m}{m}\right)^i\frac{b^{(i)}(a_mx)}{b(a_mx)}\right| \leq l_i(K), \tag{3.27}$$

where $i \in \{1,2,3,4\}$. Then, the convergence given in (3.24) is uniform on K.

Proof. From (3.23), it results that $4-\alpha_4 > 2-\alpha_2$, and then we have that $m^{2-\alpha_2}|1-(a_m/m)^i(b^{(i)}(a_m x)/b(a_m x))|$, $i\in\{1,2\}$ are bounded for any $m \geq m(2)$. So $(T_2 L_m)(x)/m^{\alpha_2}$ is bounded for any $m \geq m(2)$. Using the same idea from the proof of Theorem 3.8, we have that $(T_2 L_m)(x)/m^{\alpha_2}$ and $(T_4 L_m)(x)/m^{\alpha_4}$ are bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, and then we apply Corollary 2.10. □

Now, we give some applications where $a_m = m$ for any $m \in \mathbb{N}$. In the following, by particularization and applying Theorems 3.8 and 3.9, we can obtain approximation theorems and Voronovskaja-type theorems for some known operators. Because every application is a simple substitute in the theorems of this section, we will not replace anything.

Application 3.10. If $a(x) = 1$ and $b(x) = e^x$, $x \in \mathbb{R}_0$, we obtain the Mirakjan-Favard-Szász operators (see [3–5]).

Application 3.11. If $a(x) = g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = e^x$, $x \in \mathbb{R}_0$, we obtain the operators considered by Jakimovski and Leviatan in the paper [1].

Application 3.12. If $a(x) = g(x) = 1$ and $b(x) = \cosh x = \sum_{k=0}^{\infty}(1/(2k)!)x^{2k}$, $x \in \mathbb{R}_0$, then we get the operators considered by Leśniewicz and Rempulska in the paper [6].

Application 3.13. If $a(x) = g(x) = 1$ and $b(x) = \sinh x = \sum_{k=0}^{\infty}(1/(2k+1)!)x^{2k+1}$, $x \in \mathbb{R}_0$, we get the operators

$$(A_m f)(x) = \begin{cases} \dfrac{1}{\sinh mx}\displaystyle\sum_{k=0}^{\infty}\frac{(mx)^{2k+1}}{(2k+1)!}f\left(\frac{2k+1}{m}\right) & \text{if } x > 0, \\ f(0) & \text{if } x = 0, \end{cases} \tag{3.28}$$

where $m \in \mathbb{N}$ and $x \in \mathbb{R}_0$. The operators of this type are introduced and studied by Rempulska and Skorupka in the paper [7].

Application 3.14. If $a(x) = b(x) = g(x) = \cosh x$, $x \in \mathbb{R}_0$, we obtain the operators studied by Ciupa in [8].

Application 3.15. If $a(x) = g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \cosh x$, $x \in \mathbb{R}_0$, we get the operators constructed by Ciupa in the paper [9], and studied in [9, 10].

Application 3.16. If $a(x) = 1$ and $b(x) = b_m((1/m)x)$, $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$, we obtain the operators studied in the paper [11].

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