Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 97250, 11 pages
doi:10.1155/2007/97250

Research Article

Integral Transforms of Fourier Cosine and Sine Generalized Convolution Type

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Received 5 April 2007; Accepted 27 August 2007

Recommended by Piotr Mikusinski

Integral transforms of the form $f(x) \mapsto g(x) = (1 - d^2/dx^2)\{\int_0^\infty k_1(y)[f(|x+y-1|) + f(|x-y+1|) - f(x+y+1) - f(|x-y-1|)]dy + \int_0^\infty k_2(y)[f(x+y) + f(|x-y|)]dy\}$ from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$, $(1 \leq p \leq 2, p^{-1} + q^{-1} = 1)$ are studied. Watson's and Plancherel's theorems are obtained.

1. Introduction

Let F_c be the Fourier cosine transform [1]

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos xy f(y)dy, \tag{1.1}$$

and let F_s be the Fourier sine transform [1]

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin xy f(y)dy. \tag{1.2}$$

In 1941, Churchill introduced the convolution of two functions f and g for the Fourier cosine transform

$$\left(f \underset{F_c}{*} g\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(x+y) + g(|x-y|)]dy, \quad x > 0, \tag{1.3}$$

and proved the following factorization equality [2]:

$$F_c\left(f \underset{F_c}{*} g\right)(y) = (F_c f)(y)(F_c g)(y). \tag{1.4}$$

Using the factorization property (1.4), one can easily solve the integral equation with the Toeplitz-plus-Hankel kernel

$$f(x)+\int_0^\infty [k_1(x+y)+k_2(|x-y|)]f(y)dy=g(x) \tag{1.5}$$

in case the Toeplitz kernel $k_2(x)$ and the Hankel kernel $k_1(x)$ are the same [3, 4]. The general case is still open.

The convolution of two functions f and g with the weight function $\gamma(y)=\sin y$ for the Fourier sine transform was introduced by Kakichev in [5]

$$\begin{aligned}\Big(f \overset{\gamma}{\underset{F_s}{*}} g\Big)(x) = \frac{1}{2\sqrt{2\pi}}\int_0^\infty f(u)[&\operatorname{sign}(x+u-1)g(|x+u-1|)+\operatorname{sign}(x-u+1)g(|x-u+1|)\\ &-g(x+u+1)-\operatorname{sign}(x-u-1)g(|x-u-1|)]du, \quad x>0,\end{aligned} \tag{1.6}$$

where the following factorization property has been established:

$$F_s\Big(f \overset{\gamma}{\underset{F_s}{*}} g\Big)(y) = \sin y(F_s f)(y)(F_s g)(y). \tag{1.7}$$

Further properties of this convolution have been studied in [6].

Churchill was also the first author who introduced the generalized convolution for two different integral transforms. Namely, in 1941, he defined the generalized convolution of two functions f and g for the Fourier sine and cosine transforms

$$\Big(f \underset{1}{*} g\Big)(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(u)[g(|x-u|)-g(x+u)]du, \quad x>0, \tag{1.8}$$

and proved the following factorization identity [7]:

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)\cdot(F_c g)(y). \tag{1.9}$$

It is easy to see that the integral equation with the Toeplitz-plus-Hankel kernel (1.5) can be written in the form

$$f(x)+\sqrt{2\pi}\Big(f \underset{F_c}{*} h_1\Big)(x)+\sqrt{2\pi}\Big(f \underset{1}{*} h_2\Big)(x) = g(x), \tag{1.10}$$

where $h_1=(1/2)(k_1+k_2)$ and $h_2=(1/2)(k_2-k_1)$. So studying generalized convolutions may shed light on how to solve the integral equation with the Toeplitz-plus-Hankel kernel (1.5) in closed form.

In 1998, Kakichev and Thao proposed a constructive method for defining a generalized convolution for three arbitrary integral transforms (see [8]). For example, for the Fourier cosine and Fourier sine transforms, the following convolution has been introduced in [9]:

$$\Big(f \underset{2}{*} g\Big)(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(u)[\operatorname{sign}(u-x)g(|u-x|)+g(u+x)]du, \quad x>0. \tag{1.11}$$

For this convolution, the following factorization equality holds [9]:

$$F_c\Big(f \underset{2}{*} g\Big)(y) = (F_s f)(y)(F_s g)(y). \tag{1.12}$$

Another generalized convolution with a weight function $\gamma(y) = \sin y$ for the Fourier cosine and sine transforms has been studied in [10]

$$\begin{aligned}\Big(f \overset{\gamma}{\underset{2}{*}} g\Big)(x) = \frac{1}{2\sqrt{2\pi}}\int_0^\infty f(u)[&g(|x+u-1|)+g(|x-u+1|)\\ &-g(x+u+1)-g(|x-u-1|)]du, \quad x>0.\end{aligned} \tag{1.13}$$

It satisfies the factorization property [10]

$$F_c\Big(f \overset{\gamma}{\underset{2}{*}} g\Big)(y) = \sin y(F_s f)(y)(F_c g)(y). \tag{1.14}$$

In any convolution of two functions f and g, if we fix one function, say g, as the kernel, and allow the other function f vary in a certain function space, we will get an integral transform $f \mapsto f*g$. The most famous integral transforms constructed by that way are the Watson transforms that are related to the Mellin convolution and the Mellin transform [11]

$$f(x) \longmapsto g(x) = \int_0^\infty k(xy)f(y)dy. \tag{1.15}$$

Recently, a class of integral transforms that is related to the generalized convolution (1.11) has been introduced and investigated in [12]. In this paper, we will consider a class of integral transform which has a connection with the generalized convolution (1.13), namely, the transforms of the form

$$\begin{aligned}f(x) \longmapsto g(x) = \left(1-\frac{d^2}{dx^2}\right)\Big\{\int_0^\infty k_1(y)[&f(|x+y-1|)+f(|x-y+1|)\\ &-f(x+y+1)-f(|x-y-1|)]dy\\ &+\int_0^\infty k_2(y)[f(x+y)+f(|x-y|)]dy\Big\}, \quad x>0.\end{aligned} \tag{1.16}$$

We show that under certain conditions on the kernels k_1 and k_2, transform (1.16) admits an inverse of similar form. We find conditions on the kernels k_1 and k_2 when transform (1.16) defines a bounded operator from $L_p(\mathbb{R}_+)$ to $L_q(\mathbb{R}_+)$ ($1 \le p \le 2$, $p^{-1}+q^{-1}=1$). Moreover, Watson- and Plancherel-type theorems for transforms (1.16) in $L_2(\mathbb{R}_+)$ are also obtained.

2. A Watson-type theorem

Lemma 2.1. *Let $f,g \in L_2(\mathbb{R}_+)$. Then for any $x>0$, the following identity holds:*

$$\begin{aligned}&\int_0^\infty f(u)[g(|x+u-1|)+g(|x-u+1|)-g(x+u+1)-g(|x-u-1|)]du\\ &\quad = 2\sqrt{2\pi}F_c(\sin y(F_s f)(y)(F_c g)(y))(x).\end{aligned} \tag{2.1}$$

Proof. Let f_1 be the odd extension of f from $\mathbb{R}_+$ to $\mathbb{R}$ and g_1 the even extension of g from $\mathbb{R}_+$ to $\mathbb{R}$. Then let Ff_1 is an odd function while Fg_1 is an even function, where F is the Fourier integral transform

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy} f(y)dy. \tag{2.2}$$

On $\mathbb{R}_+$, we have $Ff_1 = -iF_s f$ and $Fg_1 = F_c g$.

The Parseval identity for the Fourier integral transform yields

$$\begin{aligned}
&\int_0^\infty f(u)[g(|x+u-1|)+g(|x-u+1|)-g(x+u+1)-g(|x-u-1|)]du \\
&\quad=\int_0^\infty f(u)g_1(x-u+1)du-\int_0^\infty f(u)g(x+u+1)du \\
&\qquad-\int_0^\infty f(u)g(x-u-1)du+\int_0^\infty f(u)g_1(x+u-1)du \\
&\quad=\int_{-\infty}^\infty f_1(u)g_1(x-u+1)du-\int_{-\infty}^\infty f_1(u)g_1(x-u-1)du \\
&\quad=\int_{-\infty}^\infty (Ff_1)(u)(Fg_1)(u)e^{i(x+1)u}du-\int_{-\infty}^\infty (Ff_1)(u)(Fg_1)(u)e^{i(x-1)u}du \\
&\quad=\int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)(\cos(x+1)y+i\sin(x+1)y)dy \\
&\qquad-\int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)(\cos(x-1)y+i\sin(x-1)y)dy.
\end{aligned} \tag{2.3}$$

On the other hand, note that $(Ff_1)(y)(Fg_1)(y)\cos(x+1)y$, $(Ff_1)(y)(Fg_1)(y)\cos(x-1)y$ are odd functions in y. Hence, their integrals over $\mathbb{R}$ vanish, and therefore,

$$\begin{aligned}
&\int_0^\infty f(u)[g(|x+u-1|)+g(|x-u+1|)-g(x+u+1)-g(|x-u-1|)]du \\
&\quad=\int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)i\sin(x+1)y\,dy-\int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)i\sin(x-1)y\,dy \\
&\quad=2i\int_{-\infty}^\infty (Ff_1)(y)(Fg_1)(y)\sin y\cos(xy)dy \\
&\quad=2\sqrt{2\pi}F_c(\sin y(F_s f)(y)(F_c g)(y))(x).
\end{aligned} \tag{2.4}$$

This completes the proof. We assumed that all the integrals over $\mathbb{R}$ are interpreted as Cauchy principal value integrals, if necessary. □

Theorem 2.2. *Let* $k_1, k_2 \in L_2(\mathbb{R}_+)$. *Then*

$$|2\sin y(F_s k_1)(y) + (F_c k_2)(y)| = \frac{1}{\sqrt{2\pi}(1+y^2)}, \tag{2.5}$$

is a necessary and sufficient condition to ensure that the integral transform $f \mapsto g$

$$\begin{aligned} g(x) := \left(1 - \frac{d^2}{dx^2}\right)\Bigg\{\int_0^\infty k_1(y)[f(|x+y-1|) + f(|x-y+1|) - f(x+y+1) \\ - f(|x-y-1|)]dy + \int_0^\infty k_2(y)[f(x+y) + f(|x-y|)]dy\Bigg\} \end{aligned} \tag{2.6}$$

is unitary on $L_2(\mathbb{R}_+)$ *and the inverse transformation has the form*

$$\begin{aligned} f(x) = \left(1 - \frac{d^2}{dx^2}\right)\Bigg\{\int_0^\infty k_1(y)[g(|x+y-1|) + g(|x-y+1|) - g(x+y+1) \\ - g(|x-y-1|)]dy + \int_0^\infty k_2(y)[g(x+y) + g(|x-y|)]dy\Bigg\}. \end{aligned} \tag{2.7}$$

Proof

Necessity. Suppose that k_1 and k_2 satisfy condition (2.5). It is well known that $(1+y^2)h(y) \in L_2(\mathbb{R})$, if and only if $(Fh)(x)$, $(d/dx)(Fh)(x)$ and $(d^2/dx^2)(Fh)(x) \in L_2(\mathbb{R})$ ([11, Theorem 68, page 92]). Moreover,

$$\frac{d^2}{dx^2}(Fh)(x) = -F(y^2 h(y))(x). \tag{2.8}$$

In particular, if h is an even or odd function such that $(1+y^2)h(y) \in L_2(\mathbb{R}_+)$, then the following equalities hold:

$$\begin{aligned} \left(1 - \frac{d^2}{dx^2}\right)(F_c h)(x) &= F_c((1+y^2)h(y))(x), \\ \left(1 - \frac{d^2}{dx^2}\right)(F_s h)(x) &= F_s((1+y^2)h(y))(x). \end{aligned} \tag{2.9}$$

Using the factorization equalities for convolutions (1.3), (1.6), we have

$$\begin{aligned} g(x) &= \left(1 - \frac{d^2}{dx^2}\right)F_c(2\sqrt{2\pi}\sin y(F_s k_1)(y)(F_c f)(y) + \sqrt{2\pi}(F_c k_2)(y)(F_c f)(y))(x) \\ &= F_c(\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y))(x). \end{aligned} \tag{2.10}$$

By virtue of the Parseval equalities for the Fourier cosine and sine transforms $\|f\|_{L_2(\mathbb{R}_+)} = \|F_c f\|_{L_2(\mathbb{R}_+)} = \|F_s f\|_{L_2(\mathbb{R}_+)}$ and noting that k_1 and k_2 satisfy condition (2.5), we have

$$\begin{aligned}\|g\|_{L_2(R_+)} &= \left\|\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y)\right\|_{L_2(R_+)} \\ &= \|F_c f\|_{L_2(R_+)} = \|f\|_{L_2(R_+)}.\end{aligned} \tag{2.11}$$

It follows that the transformation (2.6) is unitary.

On the other hand, in view of condition (2.5), $\sqrt{2\pi}(1+y^2)(2\sin y(F_s\, k_1)(y) + (F_c\, k_2)(y))$ is bounded, hence $\sqrt{2\pi}(1+y^2)(2\sin y(F_s\, k_1)(y) + (F_c\, k_2)(y))(F_c g)(y) \in L_2(\mathbb{R}_+)$. We have

$$\begin{aligned} g(x) &= F_c(\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y))(x) \\ &\Longleftrightarrow (F_c g)(y) = \sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y) \\ &\Longleftrightarrow (F_c f)(y) = \sqrt{2\pi}(1+y^2)(2\sin y(F_s\, k_1\,)(y) + (F_c\, k_2\,)(y))(F_c g)(y). \end{aligned} \tag{2.12}$$

Using formula (2.9), we obtain

$$\begin{aligned} f(x) &= F_c(\sqrt{2\pi}(1+y^2)(2\sin y(F_s\, k_1\,)(y) + (F_c\, k_2\,)(y))(F_c g)(y))(x) \\ &= \left(1 - \frac{d^2}{dx^2}\right) F_c(2\sqrt{2\pi}\sin y F_s k_1(y)(F_c g)(y) + \sqrt{2\pi}(F_c k_2)(y)(F_c g)(y)) \\ &= \left(1 - \frac{d^2}{dx^2}\right)\left\{\int_0^\infty k_1\,(y)[g(|x+y-1|) + g(|x-y+1|) - g(x+y+1) \right. \\ &\qquad \left. - g(|x-y-1|)]dy + \int_0^\infty k_2\,(y)[g(x+y) + g(|x-y|)]dy\right\}. \end{aligned} \tag{2.13}$$

Therefore, the transformation (2.6) is unitary on $L_2(\mathbb{R}_+)$ and the inverse transformation has the form (2.7).

Sufficiency. If transform (2.6) is unitary, then the Parseval identities for the Fourier cosine and sine transforms yield

$$\begin{aligned}\|g\|_{L_2(R_+)} &= \left\|\sqrt{2\pi}(1+y^2)(2\sin y(F_s k_1)(y) + (F_c k_2)(y))(F_c f)(y)\right\|_{L_2(R_+)} \\ &= \|F_c f\|_{L_2(R_+)} = \|f\|_{L_2(R_+)}.\end{aligned} \tag{2.14}$$

The middle equality is possible if and only if k_1 and k_2 satisfy condition (2.5). This completes the proof of the theorem. □

Let $h_1, h_2 \in L_2(\mathbb{R}_+)$ satisfy

$$|(F_s h_1)(y)(F_s h_2)(y)| = \frac{1}{(1+y^2)(1+\sin^2 y)}, \tag{2.15}$$

and let k_1, k_2 be defined by

$$k_1(x) = \frac{1}{2\sqrt{2\pi}}(h_1 \overset{\gamma}{\underset{F_s}{*}} h_2)(x), \qquad k_2(x) = \frac{1}{\sqrt{2\pi}}(h_1 \underset{2}{*} h_2)(x). \tag{2.16}$$

Then $k_1, k_2 \in L_2(\mathbb{R}_+)$ and from (1.7) and (1.12), we have

$$\begin{aligned} &\left|2\sin y(F_sk_1)(y) + (F_ck_2)(y)\right| \\ &\quad = \left|\frac{1}{\sqrt{2\pi}}\sin^2 y(F_sh_1)(y)(F_sh_2)(y) + \frac{1}{\sqrt{2\pi}}(F_sh_1)(y)(F_sh_2)(y)\right| \\ &\quad = \left|\frac{1}{\sqrt{2\pi}}(1+\sin^2 y)(F_sh_1)(y)(F_sh_2)(y)\right| = \frac{1}{\sqrt{2\pi}(1+y^2)}. \end{aligned} \tag{2.17}$$

Thus k_1 and k_2 satisfy condition (2.5).

3. A Plancherel-type theorem

In order to examine the Plancherel-type theorem, we will need the following lemma.

Lemma 3.1. *Let f and g be $L_2(\mathbb{R}_+)$ functions, then*

$$\begin{aligned} &\int_0^\infty f(y)[g(|x+y-1|)+g(|x-y+1|)-g(x+y+1)-g(|x-y-1|)]dy \\ &\quad = \int_0^\infty g(y)[f(x+y+1)+\operatorname{sign}(x-y+1)f(|x-y+1|) \\ &\qquad\qquad - \operatorname{sign}(x-y-1)f(|x-y-1|)-\operatorname{sign}(x+y-1)f(|x+y-1|)]dy, \end{aligned} \tag{3.1}$$

$$\int_0^\infty f(y)[g(x+y)+g(|x-y|)]dy = \int_0^\infty g(y)[f(x+y)+f(|x-y|)]dy. \tag{3.2}$$

Proof. Again, let f_1 be the odd extension of f from $\mathbb{R}_+$ to $\mathbb{R}$ and $g_1(x) = g(|x|)$ the even extension of g from $\mathbb{R}_+$ to $\mathbb{R}$. By the Parseval equality, we have

$$\begin{aligned} &\int_0^\infty f(y)[g(|x+y-1|)+g(|x-y+1|)-g(x+y+1)-g(|x-y-1|)]dy \\ &\quad = \int_0^\infty f(y)g(|x+y-1|)dy + \int_0^\infty f(y)g(|x-y+1|)dy \\ &\qquad - \int_0^\infty f(y)g(x+y+1)dy - \int_0^\infty f(y)g(|x-y-1|)dy \\ &\quad = -\int_{-\infty}^0 f_1(y)g_1(x-y-1)dy + \int_0^\infty f_1(y)g_1(x-y+1)dy \\ &\qquad + \int_{-\infty}^0 f_1(y)g_1(x-y+1)dy - \int_0^\infty f_1(y)g_1(x-y-1)dy \\ &\quad = \int_{-\infty}^\infty (Ff_1)(u)(Fg_1)(u)e^{i(x+1)u}du - \int_{-\infty}^\infty (Ff_1)(u)(Fg_1)(u)e^{i(x-1)u}du \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g_1(y) f_1(x-y+1)\,dy - \int_{-\infty}^{\infty} g_1(y) f_1(x-y-1)\,dy \\
&= \int_0^{\infty} g_1(y) f_1(x-y+1)\,dy + \int_0^{\infty} g_1(y) f_1(x+y+1)\,dy \\
&\quad - \int_0^{\infty} g_1(y) f_1(x-y-1)\,dy - \int_0^{\infty} g_1(y) f_1(x+y-1)\,dy \\
&= \int_0^{\infty} g(y)\big[f(x+y+1) + \operatorname{sign}(x-y+1) f(|x-y+1|) \\
&\qquad - \operatorname{sign}(x-y-1) f(|x-y-1|) - \operatorname{sign}(x+y-1) f(|x+y-1|)\big]\,dy.
\end{aligned} \tag{3.3}
$$

Then formula (3.1) holds. Formula (3.2) follows easily from formula (1.4)

$$
\begin{aligned}
\int_0^{\infty} f(y)\big[g(x+y) + g(|x-y|)\big]\,dy &= \sqrt{2\pi} F_c\big[(F_c f)(y)(F_c g)(y)\big](x) \\
&= \sqrt{2\pi} F_c\big[(F_c g)(y)(F_c f)(y)\big](x) \\
&= \int_0^{\infty} g(y)\big[f(x+y) + f(|x-y|)\big]\,dy.
\end{aligned} \tag{3.4}
$$

The lemma has been proved. □

Theorem 3.2. *Let k_1, k_2 be functions satisfying condition (2.5) and suppose that $K_1(x) = (1 - d^2/dx^2) k_1(x)$ and $K_2(x) = (1 - d^2/dx^2) k_2(x)$ are locally bounded. Let $f \in L_2(\mathbb{R}_+)$ and for each positive integer N, put*

$$
\begin{aligned}
g_N(x) = \int_0^{\infty} K_1(y)\big[f^N(|x+y-1|) + f^N(|x-y+1|) - f^N(x+y+1) \\
- f^N(|x-y-1|)\big]\,dy + \int_0^{\infty} K_2(y)\big[f^N(x+y) + f^N(|x-y|)\big]\,dy,
\end{aligned} \tag{3.5}
$$

where $f^N = f.\chi_{(0,N)}$, the restriction of f over $(0,N)$. Then

(1) *$g_N \in L_2(\mathbb{R}_+)$ and as $N \to \infty$, g_N converges in $L_2(\mathbb{R}_+)$ norm to a function $g \in L_2(\mathbb{R}_+)$ with $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$;*

(2) *put $g^N = g.\chi_{(0,N)}$, then*

$$
\begin{aligned}
f_N(x) = \int_0^{\infty} K_1(y)\big[g^N(|x+y-1|) + g^N(|x-y+1|) - g^N(x+y+1) \\
- g^N(|x-y-1|)\big]\,dy + \int_0^{\infty} K_2(y)\big[g^N(x+y) + g^N(|x-y|)\big]\,dy
\end{aligned} \tag{3.6}
$$

belongs to $L_2(\mathbb{R}_+)$ and converges in $L_2(\mathbb{R}_+)$ norm to f as $N \to \infty$.

Remark 3.3. Because of the definitions of f^N and g^N, these integrals are over finite intervals and therefore converge.

Proof. Applying the identities (3.1) and (3.2) in Lemma 3.1, we have

$$
\begin{aligned}
g_n(x) = {}& \int_0^\infty K_1(y)[f^N(|x+y-1|)+f^N(|x-y+1|)-f^N(x+y+1)-f^N(|x-y-1|)]dy \\
&+\int_0^\infty K_2(y)[f^N(x+y)+f^N(|x-y|)]dy \\
= {}& \int_0^\infty f^N(y)[K_1(x+y+1)+\operatorname{sign}(x-y+1)K_1(|x-y+1|) \\
&\qquad -\operatorname{sign}(x-y-1)K_1(|x-y-1|)-\operatorname{sign}(x+y-1)K_1(|x+y-1|)]dy \\
&+\int_0^\infty f^N(y)[k_1(x+y)+K_1(|x-y|)]dy \\
= {}& \left(1-\frac{d^2}{dx^2}\right)\left\{\int_0^\infty f^N(u)[k_1(x+u+1)+\operatorname{sign}(x-u+1)k_1(|x-u+1|)\right. \\
&\qquad -\operatorname{sign}(x-u-1)k_1(|x-u-1|) \\
&\qquad -\operatorname{sign}(x+u-1)k_1(|x+u-1|)]du \\
&\left.+\int_0^\infty f^N(y)[k_1(x+y)+k_1(|x-y|)]dy\right\}.
\end{aligned}
\tag{3.7}
$$

It is legitimate to interchange the order of integration and differentiation since the integrals are actually over finite intervals. By applying Lemma 3.1 one more time, we obtain

$$
\begin{aligned}
g_N(x) = \left(1-\frac{d^2}{dx^2}\right)&\left\{\int_0^\infty k_1(y)[f^N(|x+y-1|)+f^N(|x-y+1|)\right. \\
&\qquad -f^N(x+y+1)-f^N(|x-y-1|)]dy \\
&\left.+\int_0^\infty k_2(y)[f^N(x+y)+f^N(|x-y|)]dy\right\}.
\end{aligned}
\tag{3.8}
$$

From this and in view of Theorem 2.2, we conclude that $g_N \in L_2(\mathbb{R}_+)$. Let g be the transform of f under the transformation (2.6). Then Theorem 2.2 guarantees that $g \in L_2(\mathbb{R}_+)$, $\|g\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$, and the reciprocal formula (2.7) holds. For $g-g_N$, we have

$$
\begin{aligned}
(g-g_N)(x) = \left(1-\frac{d^2}{dx^2}\right)&\left\{\int_0^\infty k_1(y)[(f-f^N)(|x+y-1|)+(f-f^N)(|x-y+1|)\right. \\
&\qquad -(f-f^N)(x+y+1)-(f-f^N)(|x-y-1|)]dy \\
&\left.+\int_0^\infty k_2(y)[(f-f^N)(x+y)+(f-f^N)(|x-y|)]dy\right\}.
\end{aligned}
\tag{3.9}
$$

Again by Theorem 2.2, $(g-g_N)(x)\in L_2(\mathbb{R}_+)$ and

$$||g-g_N||_{L_2(\mathbb{R}_+)} = ||f-f^N||_{L_2(\mathbb{R}_+)}. \tag{3.10}$$

And since $\|f-f^N\|_{L_2(\mathbb{R}_+)}\to 0$ as $N\to\infty$ then g_N converses in $L_2(\mathbb{R}_+)$ norm to $g\in L_2(\mathbb{R}_+)$.

Similarly, one can obtain the second part of the theorem. □

Theorem 3.4. *Let k_1 and k_2 be functions satisfying condition (2.5) and suppose that $K_1(x)$ and $K_2(x)$ defined as in the previous theorem are bounded on $\mathbb{R}_+$. Let $1\le p\le 2$ and q be its conjugate exponent $1/p+1/q=1$. Then the transformation $f\mapsto g$, where g is defined by*

$$g(x)=\lim_{N\to\infty}\left\{\int_0^\infty K_1(y)\left[f^N(|x+y-1|)+f^N(|x-y+1|)-f^N(x+y+1)\right.\right.$$
$$\left.\left.-f^N(|x-y-1|)\right]dy+\int_0^\infty K_2(y)\left[f^N(x+y)+f^N(|x-y|)\right]dy\right\}, \tag{3.11}$$

is a bounded operator from $L_p(\mathbb{R}_+)$ into $L_q(\mathbb{R}_+)$. Here the limit is understood in $L_q(\mathbb{R}_+)$ norm.

Proof. From the boundedness of K_1 and K_2, it is clear that transformation (3.11) is a bounded operator from $L_1(\mathbb{R}_+)$ into $L_\infty(\mathbb{R}_+)$.

On the other hand, Theorem 3.2 shows that transformation (3.11) defines a bounded operator from $L_2(\mathbb{R}_+)$ into $L_2(\mathbb{R}_+)$. Hence, Riesz's interpolation theorem implies that (3.11) is a bounded operator from $L_p(\mathbb{R}_+)$, $1\le p\le 2$, into $L_q(\mathbb{R}_+)$, where q is the conjugate exponent of p. □

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