Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 287218, 13 pages
doi:10.1155/2008/287218

Research Article

Image of $L^p(\mathbb{R}^n)$ under the Hermite Semigroup

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Received 9 June 2008; Revised 8 October 2008; Accepted 9 December 2008

Recommended by Misha Rudnev

It is shown that the Hermite (polynomial) semigroup $\{e^{-t\mathbb{H}} : t > 0\}$ maps $L^p(\mathbb{R}^n, \rho)$ into the space of holomorphic functions in $L^r(\mathbb{C}^n, V_{t,p/2}^{(r+\epsilon)/2})$ for each $\epsilon > 0$, where ρ is the Gaussian measure, $V_{t,p/2}^{(r+\epsilon)/2}$ is a scaled version of Gaussian measure with $r = p$ if $1 < p < 2$ and $r = p'$ if $2 < p < \infty$ with $1/p + 1/p' = 1$. Conversely if F is a holomorphic function which is in a "slightly" smaller space, namely $L^r(\mathbb{C}^n, V_{t,p/2}^{r/2})$, then it is shown that there is a function $f \in L^p(\mathbb{R}^n, \rho)$ such that $e^{-t\mathbb{H}} f = F$. However, a single necessary and sufficient condition is obtained for the image of $L^2(\mathbb{R}^n, \rho_{p/2})$ under $e^{-t\mathbb{H}}$, $1 < p < \infty$. Further it is shown that if F is a holomorphic function such that $F \in L^1(\mathbb{C}^n, V_{t,p/2}^{1/2})$ or $F \in L_m^{1,p}(\mathbb{R}^{2n})$, then there exists a function $f \in L^p(\mathbb{R}^n, \rho)$ such that $e^{-t\mathbb{H}} f = F$, where $m(x, y) = e^{-x^2/(p-1)e^{4t}+1} e^{-y^2/e^{4t}-1}$ and $1 < p < \infty$.

1. Introduction

It is well known that the Bargmann transform is an isometric isomorphism of $L^2(\mathbb{R}^n)$ onto the Fock space $\mathcal{F}(\mathbb{C}^n)$, which is associated with the realization of the creation and annihilation operators for Bosons in quantum mechanics. We refer to [1, 2] for further results. Similar type of results are shown for semigroups generated by the Laplace-Beltrami operator on compact spaces (see [3, 4]). The image of $L^2(\mathbb{H}^n)$ under the heat kernel transform is studied for the Heisenberg group in [5]. Such type of results are also known for Hermite, special Hermite, Bessel, and Laguerre semigroups, see [6].

Hall in [7] studied the problem of characterizing the image of $L^p(\mathbb{R}^n)$ under the Segal-Bargmann transform. In this paper, we want to study this problem for Hermite semigroup instead of Segal-Bargmann transform. In fact, the idea of extending the classical results involving the standard Laplacian on $\mathbb{R}^n$ or Fourier transform on $\mathbb{R}^n$ to Hermite expansions is not new: to cite a few, summability theorems [8, 9], multipliers [10], Sobolev spaces [11], and Hardy's inequalities [12, 13].

In order to prove that a holomorphic function to be in the image of $L^p(\mathbb{R}^n,\rho)$ under the Segal-Bargmann transform Hall in [7] obtained a necessary condition for the range $1<p\le 2$ and the sufficient condition for $2\le p<\infty$. In this paper, we tried to obtain a single necessary and sufficient condition for a holomorphic function to be in the image of $L^p(\mathbb{R}^n,\rho)$ under Hermite (polynomial) semigroup. Though we were not completely successful, we could prove the following.

If $f\in L^p(\mathbb{R}^n,\rho)$, then $e^{-t\mathbb{H}}f$ is holomorphic and $e^{-t\mathbb{H}}f\in L^r(\mathbb{C}^n,V_{t,p/2}^{(r+\epsilon)/2})$ for every $\epsilon>0$, but we are able to prove the converse only for the holomorphic functions which are in $L^r(\mathbb{C}^n,V_{t,p/2}^{r/2})$ (with $r=p$, if $1<p<2$ and $r=p'$, if $2<p<\infty$ with $1/p+1/p'=1$), where $V_{t,p/2}(z)=(\pi^2/4)^{-n/2}e^{2nt}[((p-1)e^{4t}+1)(e^{4t}-1)]^{-n/2}e^{-2x^2/((p-1)e^{4t}+1)}e^{-2y^2/(e^{4t}-1)}$ and $V_{t,p/2}^s$, the sth power of $V_{t,p/2}$. However, we are able to obtain a single necessary and sufficient condition for the image of $L^2(\mathbb{R}^n,\rho_{p/2})$ under $e^{-t\mathbb{H}}$, $1<p<\infty$, where $\rho_{p/2}(u)=(p\pi/2)^{-n/2}e^{-2/pu^2}$.

Notice that $\mathscr{H}L^r(\mathbb{C}^n,V_{t,p/2}^{r/2})\subset\bigcap_{\epsilon>0}\mathscr{H}L^r(\mathbb{C}^n,V_{t,p/2}^{(r+\epsilon)/2})$ as $V_{t,p/2}^{(r+\epsilon)/2}\le CV_{t,p/2}^{r/2}$, where the constant C depends on ϵ,t,p, and n. We remark that the Gaussian-type density $V_{t,p/2}(z)$ defines a finite measure with total mass e^{2nt} when $n\ge 1$ and $t>0$. Let m be a weight function on $\mathbb{R}^{2n}$ and let $1\le p,q\le\infty$. Then the weighted mixed-norm space $L_m^{p,q}(\mathbb{R}^{2n})$ consists of all (Lebesgue) measurable functions on $\mathbb{R}^{2n}$, such that the norm $\|F\|_{L_m^{p,q}}=(\int_{\mathbb{R}^n}(\int_{\mathbb{R}^n}|F(x,w)|^p m(x,w)^p dx)^{q/p}dw)^{1/q}$ is finite. If $p=\infty$ or $q=\infty$, then the corresponding p-norm is replaced by the essential supremum. In this paper, we also prove that if F is holomorphic and if $F\in L^1(\mathbb{C}^n,V_{t,p/2}^{1/2})$ or $L_m^{1,p}(\mathbb{R}^{2n})$, then there exists a function $f\in L^p(\mathbb{R}^n,\rho)$ such that F is the image of f. Here $m(x,y)=e^{-x^2/((p-1)e^{4t}+1)}e^{-y^2/(e^{4t}-1)}$.

The advantage of taking $L^p(\mathbb{R}^n,\rho)$ instead of $L^p(\mathbb{R}^n)$ has been nicely explained by Hall in [7]. We also wish to point out the following interesting fact, namely, if $f\in L^p(\mathbb{R}^n)$, then the pointwise estimate of $e^{-t\mathbb{H}}f$ is given by $|e^{-t\mathbb{H}}f(x+iy)|\le C_{t,p,n}e^{(-1/2)\tanh 2t\,x^2}e^{(1/2)\coth 2t\,y^2}$ which is independent of p (except the constant factor), which does not help in the current problem. Here the constant $C_{t,p,n}=e^{-nt}(2\pi)^{-n/2}(\sinh 2t)^{-n/2}(2\pi/q\coth 2t)^{n/2q}$, with $1/p+1/q=1$. However, if $f\in L^p(\mathbb{R}^n,\rho)$, we get pointwise bounds as in Theorem 3.1. Further, we found the semigroup associated with Hermite polynomials to be more suitable for this problem rather than the semigroup associated with the Hermite functions. This is mainly because questions about L^p structure for $p\ne 2$ do depend on the measures used in the particular setup, while questions about L^2 structure do not. In Section 2, we discuss the Hermite (polynomial) semigroup and discuss the image of $L^2(\mathbb{R}^n,\rho)$ under the Hermite semigroup. In Section 3, we prove our main results.

2. Hermite (polynomial) semigroup

Let $\widetilde{\mathbb{H}_k}(x)=(-1)^k e^{x^2}(d^k/dx^k)(e^{-x^2})$ denote the Hermite polynomial. For a multi-index $\alpha\in\mathbb{N}^n$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, define $\widetilde{\mathbb{H}_\alpha}(x)=\widetilde{\mathbb{H}_{\alpha_1}}(x_1)\widetilde{\mathbb{H}_{\alpha_2}}(x_2)\cdots\widetilde{\mathbb{H}_{\alpha_n}}(x_n)$ and $\mathbb{H}_\alpha(x)=(2^\alpha\alpha!)^{-n/2}\widetilde{\mathbb{H}_\alpha}(x)$. This collection $\{\mathbb{H}_\alpha(x):\alpha\in\mathbb{N}^n\}$ forms an orthonormal basis for $L^2(\mathbb{R}^n,\rho)$, where $\rho(x)dx=\pi^{-n/2}e^{-x^2}dx$. Further, the functions $\mathbb{H}_\alpha$ are eigenfunctions of the operator $\mathbb{H}=-\Delta+2\sum_{j=1}^n x_j(\partial/\partial x_j)+nI$ with eigenvalues $2|\alpha|+n$. For $f\in L^2(\mathbb{R}^n,\rho)$, consider $f=\sum_{\alpha\in\mathbb{N}^n}\langle f,\mathbb{H}_\alpha\rangle\mathbb{H}_\alpha$, where the sum converges to f in $L^2(\mathbb{R}^n,\rho)$. For each $k\in\mathbb{N}$, let $\mathbb{Q}_k$ denote the orthogonal projection of $L^2(\mathbb{R}^n,\rho)$ onto the eigenspace spanned by $\{\mathbb{H}_\alpha:|\alpha|=k\}$. Then the spectral decomposition of $\mathbb{H}$ can be written as $\mathbb{H}f=\sum_{k=0}^\infty(2k+n)\mathbb{Q}_k f$. The operator $\mathbb{H}$ defines a semigroup, called the Hermite polynomial semigroup, denoted by $e^{-t\mathbb{H}}$, for each

$t > 0$ using the expansion

$$e^{-t\mathbb{H}} f = \sum_{k=0}^{\infty} e^{-(2k+n)t} \mathbb{Q}_k f. \tag{2.1}$$

Before stating our theorem, we will state the following identity which will be repeatedly used in this paper:

$$\frac{1}{(\coth 2t + 1 - 2/p)\sinh^2 2t} + 1 - \coth 2t = \frac{2}{(p-1)e^{4t} + 1}. \tag{2.2}$$

In fact, the left-hand side of (2.2) can be rewritten as

$$\frac{1}{(\cosh 2t + \sinh 2t - (2/p)\sinh 2t)\sinh 2t} + \frac{\sinh 2t - \cosh 2t}{\sinh 2t}. \tag{2.3}$$

By using the exponential formula for $\sinh 2t$ and $\cosh 2t$, the right-hand side of (2.2) can be obtained by straightforward simplification.

We state here Mehler's formula for Hermite functions

$$\Phi_\alpha(x) = \prod_{i=1}^{n} \left(2^{\alpha_i} \alpha_i! \pi^{1/2}\right)^{-1/2} (-1)^{\alpha_i} \frac{d^{\alpha_i}}{dx_i^{\alpha_i}} \left(e^{-x_i^2}\right) e^{x_i^2/2}, \tag{2.4}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ which will lead to a formula involving Hermite polynomials (for the proof, we refer to [14] or [6]).

Mehler's formula

For all $w \in \mathbb{C}$ with$|w| < 1$, one has

$$\sum_{\alpha \in \mathbb{N}^n} \Phi_\alpha(x)\Phi_\alpha(y) w^{|\alpha|} = \pi^{-n/2} \left(1 - w^2\right)^{-n/2} e^{(-1/2)((1+w^2)/(1-w^2))(x^2+y^2) + (2w/(1-w^2)) x\cdot y}, \tag{2.5}$$

for all $x, y \in \mathbb{R}^n$. Here $x^2 = \sum_{i=1}^n x_i^2$ and $x \cdot y = \sum_{i=1}^n x_i \cdot y_i$. But

$$\sum_{\alpha \in \mathbb{N}^n} \Phi_\alpha(x)\Phi_\alpha(y) w^{|\alpha|} = \pi^{-n/2} e^{-x^2/2} e^{-y^2/2} \sum_{\alpha \in \mathbb{N}^n} \mathbb{H}_\alpha(x)\mathbb{H}_\alpha(y) w^{|\alpha|} \tag{2.6}$$

from which it follows that

$$\sum_{\alpha \in \mathbb{N}^n} \mathbb{H}_\alpha(x)\mathbb{H}_\alpha(y) w^{|\alpha|} = \pi^{n/2} e^{x^2/2} e^{y^2/2} \sum_{\alpha \in \mathbb{N}^n} \Phi_\alpha(x)\Phi_\alpha(y) w^{|\alpha|}. \tag{2.7}$$

If $f \in L^2(\mathbb{R}^n, \rho)$, then

$$\mathbb{Q}_k f(u) = \sum_{|\alpha|=k} \langle f, \mathbb{H}_\alpha \rangle \, \mathbb{H}_\alpha(u) = \sum_{|\alpha|=k} \left\{ \int_{\mathbb{R}^n} f(x)\mathbb{H}_\alpha(x)\rho(x)dx \right\} \mathbb{H}_\alpha(u). \tag{2.8}$$

Thus, it follows that

$$\begin{aligned} e^{-t\mathbb{H}} f(x) &= \int_{\mathbb{R}^n} \mathbf{K}_t(x,u) f(u)\rho(u)du \\ &= \int_{\mathbb{R}^n} \mathbf{L}_t(x,u) f(u)du, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \mathbf{L}_t(x,u) &= \pi^{-n/2} \sum_{\alpha\in\mathbb{N}^n} e^{-(2|\alpha|+n)t} \mathbb{H}_\alpha(x)\mathbb{H}_\alpha(u) e^{-u^2} \\ &= \pi^{-n/2} e^{-u^2} e^{-nt} \sum_{\alpha\in\mathbb{N}^n} \left(e^{-2t}\right)^{|\alpha|} \mathbb{H}_\alpha(x)\mathbb{H}_\alpha(u). \end{aligned} \tag{2.10}$$

Notice that, $\mathbf{K}_t$ is the kernel of $e^{-t\mathbb{H}}$. However, $\mathbf{L}_t$ is used for computational purpose. Then by using (2.7), we can write

$$\mathbf{L}_t(x,u) = (2\pi)^{-n/2}(\sinh 2t)^{-n/2} e^{-1/2(\coth 2t-1)x^2} e^{-1/2(\coth 2t+1)u^2} e^{x\cdot u/\sinh 2t}. \tag{2.11}$$

Since $\mathbf{K}_t(x,u)$ extends to an entire function $\mathbf{K}_t(z,u)$ for $z \in \mathbb{C}^n$, $F(z) = e^{-t\mathbb{H}} f(z)$ can also be extended to $\mathbb{C}^n$ as an entire function, where $z = x + iy$. This can be verified by using Morera's theorem.

Remark 2.1. The map $e^{-t\mathbb{H}}$ is one-one . Let $e^{-t\mathbb{H}} f = 0$ for $f \in L^2 \cap L^p(\mathbb{R}^n, \rho)$. Then $e^{-t\mathbb{H}} f(-i\xi) = 0\ \forall \xi \in \mathbb{R}^n$.

Let $\widehat{g}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(x) e^{-ix\cdot\xi} dx$ denote the Fourier transform of g. Then it follows from (2.11) that

$$e^{-t\mathbb{H}} f(-i\xi) = (\sinh 2t)^{-n/2} e^{(1/2)(\coth 2t-1)\xi^2} \widehat{G}\left(\frac{\xi}{\sinh 2t}\right) = 0, \tag{2.12}$$

where $G(u) = f(u) e^{(-1/2)(\coth 2t-1)u^2}$ forcing $\widehat{G}(\xi/\sinh 2t) = 0$. Then by uniqueness of Fourier transform $G = 0$, which in turn implies $f = 0$, showing that $e^{-t\mathbb{H}}$ is one-one. (In fact, the above proof shows that $e^{-t\mathbb{H}}$ is injective on $L^2(\mathbb{R}^n, \rho)$. However, in general, one can show that $e^{-t\mathbb{H}}$ is injective on the L^p space $(1 < p < \infty)$ using the fact that the Fourier transform is injective on the L^p space).

We should call $e^{-t\mathbb{H}} f$ *Hermite Bargmann transform*. Hereafter, we should write $\mathscr{H}L^r(\mathbb{C}^n, \alpha(z))$ for the class of holomorphic functions in $L^r(\mathbb{C}^n, \alpha(z))$.

Theorem 2.2. *Fix $t > 0$ and let $1 < p < \infty$. Then the Hermite polynomial semigroup $e^{-t\mathbb{H}}$ is an isometric isomorphism of $L^2(\mathbb{R}^n, \rho_{p/2})$ onto the space of holomorphic functions $\mathscr{H}L^2(\mathbb{C}^n, V_{t,p/2})$.*

Proof. Let $f \in L^2(\mathbb{R}^n, \rho_{p/2})$. Then $F(z) = e^{-t\mathbb{H}} f(z)$ is given by

$$\begin{aligned} F(z) &= (2\pi)^{-n/2}(\sinh 2t)^{-n/2} e^{(-1/2)(\coth 2t-1)z^2} \int_{\mathbb{R}^n} e^{(-1/2)(\coth 2t+1)u^2} e^{(z\cdot u/\sinh 2t)} f(u)\,du \\ &= (2\pi)^{-n/2}(\sinh 2t)^{-n/2} e^{(-1/2)(\coth 2t-1)z^2} \int_{\mathbb{R}^n} e^{(-1/2)(\coth 2t+1)u^2} e^{(-i(-y+ix)\cdot u)/\sinh 2t} f(u)\,du. \end{aligned} \tag{2.13}$$

Put $g(u) = f(u) e^{(-1/2)(\coth 2t+1)u^2} e^{x\cdot u/\sinh 2t}$. Then

$$\begin{aligned} &\int_{\mathbb{R}^n} |F(z)|^2 e^{(\coth 2t-1)(x^2-y^2)} dy \\ &\quad = (\sinh 2t)^{-n} \int_{\mathbb{R}^n} \left| \widehat{g}\left(\frac{-y}{\sinh 2t} \right) \right|^2 dy \\ &\quad = \int_{\mathbb{R}^n} |\widehat{g}(y)|^2 dy \quad \text{(by applying change of variables)} \\ &\quad = \int_{\mathbb{R}^n} |g(y)|^2 du \quad \text{(using Plancherel formula)} \\ &\quad = \int_{\mathbb{R}^n} |f(u)|^2 e^{-(\coth 2t+1)u^2} e^{2x\cdot u/\sinh 2t} du \\ &\quad = \left(\frac{p\pi}{2}\right)^{n/2} \int_{\mathbb{R}^n} |f(u)|^2 e^{-(\coth 2t+1-2/p)u^2} e^{2x\cdot u/\sinh 2t} \rho_{p/2(u)}\,du. \end{aligned} \tag{2.14}$$

Multiplying both sides by $e^{-x^2/(\coth 2t+1-2/p)\sinh^2 2t}$ and integrating with respect to x, we get

$$\begin{aligned} &\int_{\mathbb{C}^n} |F(z)|^2 e^{(\coth 2t-1)(x^2-y^2)} e^{-x^2/(\coth 2t+1-(2/p))\sinh^2 2t} dy\,dx \\ &\quad = \left(\frac{p\pi}{2}\right)^{n/2} \int_{\mathbb{R}^n} |f(u)|^2 e^{-(\coth 2t+1-2/p)u^2} \int_{\mathbb{R}^n} e^{-x^2/((\coth 2t+1-2/p)\sinh^2 t)+2x\cdot y/\sinh 2t} dx\, \rho_{p/2}(u)\,du \\ &\quad = \left(\frac{p\pi}{2}\right)^{n/2} \int_{\mathbb{R}^n} |f(u)|^2 \int_{\mathbb{R}^n} e^{-[(x/\sqrt{(\coth 2t+1-2/p)}\sinh 2t)-u\sqrt{(\coth 2t+1-2/p)}]^2} dx\, \rho_{p/2(u)}\,du. \end{aligned} \tag{2.15}$$

Let $\mathscr{K}_{t,p,n} = (p\pi^2/2)^{-n/2}\left[(\coth 2t + 1 - 2/p)\sinh^2 2t\right]^{-n/2}$. Then it follows that

$$\mathscr{K}_{t,p,n} \int_{\mathbb{C}^n} |F(z)|^2 e^{(\coth 2t-1)(x^2-y^2)} e^{-x^2/((\coth 2t+1-(2/p))\sinh^2 2t)} dx\,dy = \|f\|^2_{L^2(\mathbb{R}^n,\rho_{p/2})}. \tag{2.16}$$

By using (2.2), the left-hand side of the above equation can be written as

$$\left(\frac{p\pi^2}{2}\right)^{-n/2}\left[\left(\coth 2t+1-\frac{2}{p}\right)\sinh^2 2t\right]^{-n/2}\int_{\mathbb{C}^n}|F(z)|^2 e^{-2x^2/((p-1)e^{4t}+1)}e^{-2y^2/(e^{4t}-1)}\,dz$$
$$=\left(\frac{\pi^2}{4}\right)^{-n/2}e^{2nt}\left[((p-1)e^{4t}+1)(e^{4t}-1)\right]^{-n/2}\int_{\mathbb{C}^n}|F(z)|^2 e^{-2x^2/((p-1)e^{4t}+1)}e^{-2y^2/(e^{4t}-1)}\,dz, \tag{2.17}$$

which implies that $e^{-t\mathbb{H}}$ is an isometry from the space $L^2(\mathbb{R}^n,\rho_{p/2})$ into $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$. In the above, $dz=dx\,dy,\ z=x+iy$.

It remains to show that $e^{-t\mathbb{H}}$ defines an onto map. Since $e^{-t\mathbb{H}}$ is an isometry, the range is closed in $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$. It is enough to show that the range is dense in $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$. By using exponential formula for $\sinh 2t$, $\cosh 2t$, and $\coth 2t$, $e^{-t\mathbb{H}}$ can be written as

$$e^{-t\mathbb{H}}f(z)=(2\pi)^{-n/2}(\sinh 2t)^{-n/2}\int_{\mathbb{R}^n}e^{-(e^{-t}z-e^{t}u)^2/2\sinh 2t}f(u)du. \tag{2.18}$$

It can be easily seen that $e^{-t\mathbb{H}}$ will take real variable polynomials in $L^2(\mathbb{R}^n,\rho_{p/2})$ to holomorphic polynomials in $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$. On the other hand, if we take a holomorphic polynomial in $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$, it can be expressed as an image of a real variable polynomial in $L^2(\mathbb{R}^n,\rho_{p/2})$ under $e^{-t\mathbb{H}}$. In fact, suppose, for instance, if $n=1$, the first one can show that

$$e^{-t\mathbb{H}}u^m(z)=C\sum_{k=0}^{m}(-1)^k m_{c_k}(e^{-t}z)^{m-k}w_k, \tag{2.19}$$

for fixed $m\ge 0$, where $C=-(2\pi)^{-n/2}(\sinh 2t)^{-n/2}e^{-(m+1)t}$, $w_k=\int_{\mathbb{R}^n}e^{(-1/2\sinh 2t)y^2}y^k dy$. Now if we take $F(z)=\sum_{i=0}^{m}c_iz^i$, an mth degree holomorphic polynomial in $\mathscr{H}L^2(\mathbb{C},V_{t,p/2})$, we wish to choose an mth degree real variable polynomial $f(x)=\sum_{i=0}^{m}a_iu^i$ in $L^2(\mathbb{R},\rho_{p/2})$ such that $e^{-t\mathbb{H}}f=F$. This leads to the determination of the coefficients a_i such that $e^{-t\mathbb{H}}(\sum_{i=0}^{m}a_iu^i)=\sum_{i=0}^{m}c_iz^i$. Using (2.19) and comparing the coefficients of z^k on both sides for $0\le k\le m$, one ends up with a matrix equation $UX=Y$ with U an upper triangular matrix with $U_{ii}=c_0e^{-(i+1)}w_0$, where $c_0=-(2\pi)^{-n/2}(\sinh 2t)^{-n/2}$, $X=(a_0,a_1,\dots,a_m)^t$, $Y=(c_0,c_1,\dots,c_m)^t$. Since $w_0\ne 0$, $\det U\ne 0$ which in turn gives a unique solution for $a_0,a_1,\dots,a_m$. Thus every holomorphic polynomial in $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$ is an image of a real variable polynomial in $L^2(\mathbb{R}^n,\rho_{p/2})$. This idea can be appropriately extended for higher dimensions also. It remains to show that the set of all holomorphic polynomials are dense in $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$, which will force the image of $e^{-t\mathbb{H}}$ to be dense in $\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$. Toward this end, we show that any $F\in\mathscr{H}L^2(\mathbb{C}^n,V_{t,p/2})$ which is orthogonal to all holomorphic polynomials vanishes identically. In particular, F is orthogonal to all monomials z^α, $\alpha\in\mathbb{N}^n$. Now consider the following Fock spaces $\mathcal{F}_{s(t)}(\mathbb{C}^n)$, defined as the space of all entire functions G for which

$$\|G\|^2_{\mathcal{F}_{s(t)}}=\int_{\mathbb{C}^n}|G(z)|^2e^{-s(t)|z|^2}dz \tag{2.20}$$

are finite. It is easy to see that

$$F \in \mathscr{H}L^2(\mathbb{C}^n, V_{t,p/2}) \Longleftrightarrow G(z) = F(z)e^{w(t)z^2} \in \mathcal{F}_{s(t)}, \tag{2.21}$$

where $s(t) = 1/((p-1)e^{4t}+1) + 1/(e^{4t}-1)(> 0)$, $w(t) = (1/2)[1/(e^{4t}-1) - 1/((p-1)e^{4t}+1)]$ The assumption that F is orthogonal to all z^α leads to the condition that $G(z) = F(z)e^{w(t)z^2}$ is orthogonal to all $z^\alpha e^{w(t)z^2}$ in $\mathcal{F}_{s(t)}$. The Taylor expansion $F(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ leads to

$$G(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha e^{w(t)z^2}. \tag{2.22}$$

Since G is orthogonal to all $z^\alpha e^{w(t)z^2}$, we get $a_\alpha = 0$ for all α and so $F = 0$, thus proving our assertion. □

In particular, when $p = 2$, we obtain the following result in which the function $f \in L^2(\mathbb{R}^n, \rho)$ and the measure ρ are independent of t, but $e^{-t\mathbb{H}} f \in L^2(\mathbb{C}^n, V_t)$, where $V_t = V_{t,1}$, which depends on t (see also [15]).

Corollary 2.3. *A holomorphic function F on $\mathbb{C}^n$ belongs to $\mathscr{H}L^2(\mathbb{C}^n, V_t)$ if and only if $F(z) = e^{-t\mathbb{H}} f(z)$ for some $f \in L^2(\mathbb{R}^n, \rho)$, where $V_t(z) = (\pi^2/2)^{-n/2}(\sinh 4t)^{-n/2} e^{-2x^2/(e^{4t}+1)} e^{-2y^2/(e^{4t}-1)}$. Moreover, one has the equality of norms*

$$\|F(z)\|_{\mathscr{H}L^2(\mathbb{C}^n,V_t)} = \|f\|_{L^2(\mathbb{R}^n,\rho)}, \tag{2.23}$$

whenever $F = e^{-t\mathbb{H}} f$.

3. The main results

First, we will obtain a pointwise bound for Hermite Bargmann transform of a function $f \in L^p(\mathbb{R}^n, \rho)$. From here onward, in order to define $e^{-t\mathbb{H}}$ on $L^p(\mathbb{R}^n, \rho)$, $1 < p < \infty$, we will first define $e^{-t\mathbb{H}}$ on the class of functions $L^2 \cap L^p(\mathbb{R}^n, \rho)$. Then using standard density argument, we will define $e^{-t\mathbb{H}}$ on $L^p(\mathbb{R}^n, \rho)$.

Theorem 3.1. *Fix $t > 0$ and let $1 < p < \infty$. Then for all $f \in L^p(\mathbb{R}^n, \rho)$, one has*

$$\left|e^{-t\mathbb{H}} f(x+iy)\right| \le K_{t,p,n} \|f\|_{L^p(\mathbb{R}^n,\rho)} e^{x^2/((p-1)e^{4t}+1)} e^{y^2/(e^{4t}-1)}. \tag{3.1}$$

Proof. We have $e^{-t\mathbb{H}} f(z) = (2\pi)^{-n/2}(\sinh 2t)^{-n} e^{(-1/2)(\coth 2t-1)z^2} \int_{\mathbb{R}^n} e^{(-1/2)(\coth 2t-1)u^2} e^{z\cdot u/\sinh 2t} f(u)e^{-u^2} du$, for $f \in L^2 \cap L^p(\mathbb{R}^n, \rho)$. Let $h(u) = e^{(-1/2)(\coth 2t-1)u^2} e^{z\cdot u/\sinh 2t}$, $C_t(x,y) = (2\sinh 2t)^{-n/2} e^{(-1/2)(\coth 2t-1)(x^2-y^2)}$. As $f \in L^p(\mathbb{R}^n, \rho)$ and $h \in L^{p'}(\mathbb{R}^n, \rho)$ by applying Hölder's inequality, it can be shown that

$$\left|e^{-t\mathbb{H}} f(z)\right| \le C_t(x,y) \|f\|_{L^p(\mathbb{R}^n,\rho)} \left\|e^{(-1/2)(\coth 2t-1)u^2} e^{z\cdot u/\sinh 2t}\right\|_{L^{p'}(\mathbb{R}^n,\rho)}. \tag{3.2}$$

Consider

$$
\begin{aligned}
&\left\| e^{(-1/2)(\coth 2t-1)u^2} e^{z\cdot u/\sinh 2t} \right\|_{L^{p'}(\mathbb{R}^n,\rho)}^{p'} \\
&\quad = \int_{\mathbb{R}^n} e^{(-p'/2)(\coth 2t-1)u^2} e^{p' x\cdot u/\sinh 2t} \pi^{-n/2} e^{-u^2} du \\
&\quad = \pi^{-n/2} \int_{\mathbb{R}^n} e^{(-p'/2)[\sqrt{(\coth 2t-1+2/p')}u-(x/\sinh 2t\sqrt{\coth 2t-1+2/p'})]^2} e^{p' x^2/2\sinh^2 2t(\coth 2t-1+2/p')} du \\
&\quad = \left(\frac{2}{p'}\right)^{n/2} e^{p' x^2/2\sinh^2 2t(\coth 2t+1-2/p)} \left(\coth 2t - 1 + \frac{2}{p'}\right)^{-n/2}.
\end{aligned}
\tag{3.3}
$$

Thus

$$
\left| e^{-t\mathbb{H}} f(z) \right| \le K_{t,p,n} \|f\|_{L^p(\mathbb{R}^n,\rho)} e^{(1/2)(\coth 2t-1)y^2} e^{(x^2/2)[1/(\sinh^2 2t(\coth 2t+1-2/p))+1-\coth 2t]}, \tag{3.4}
$$

where $K_{t,p,n}$ is a constant depending on t, p, n. By using (2.2), it follows that

$$
\left| e^{-t\mathbb{H}} f(z) \right| \le K_{t,p,n} \|f\|_{L^p(\mathbb{R}^n,\rho)} e^{x^2/((p-1)e^{4t}+1)} e^{y^2/(e^{4t}-1)}, \quad z = x + iy. \tag{3.5}
$$

Since $L^2 \cap L^p(\mathbb{R}^n, \rho)$ is dense in $L^p(\mathbb{R}^n, \rho)$, $1 < p < \infty$, the result follows. □

The next theorem follows from Theorem 3.1, by a straightforward computation.

Theorem 3.2. *Fix $t > 0$ and let $1 < p \le 2$. If $f \in L^p(\mathbb{R}^n, \rho)$, then $e^{-t\mathbb{H}} f \in \mathscr{H}L^p(\mathbb{C}^n, V_{t,p/2}^{(p+\epsilon)/2})$ for every fixed $\epsilon > 0$. In particular $e^{-t\mathbb{H}} f \in \bigcap_{\epsilon>0} \mathscr{H}L^p(\mathbb{C}^n, V_{t,p/2}^{(p+\epsilon)/2})$.*

Remark 3.3. The above theorem is valid for $1 < p < \infty$. We will see in Theorem 3.8 that Theorems 3.2 and 3.7 can be put together in a general form. *At this point, we thank one of the referees for suggesting us this general form, namely, Theorem 3.8.*

Theorem 3.4. *If F is holomorphic and $F \in L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$, where $1 < p \le 2$ and t is a fixed number greater than zero, then there exists a unique function $f \in L^p(\mathbb{R}^n, \rho)$ such that $e^{-t\mathbb{H}} f = F$.*

Proof. Let $Gf = e^{-t\mathbb{H}} f$. Then it follows from Theorem 2.2, that G is an isometric isomorphism from $L^2(\mathbb{R}^n, \rho_{p/2})$ onto $\mathscr{H}L^2(\mathbb{C}^n, V_{t,p/2})$. G can be explicitly written as $Gf(z) = \int_{\mathbb{R}^n} (\mathbb{L}_t(z,u)/\rho_{p/2}(u)) f(u) \rho_{p/2(u)} du$, where $\mathbb{L}_t$ is given in (2.11). Let $G^{*,p}$ denote the adjoint of G, where G is an operator from Hilbert space $L^2(\mathbb{R}^n, \rho_{p/2})$ into the Hilbert space $L^2(\mathbb{C}^n, V_{t,p/2})$. Note that, $G^{*,p}F$ will coincide with $G^{-1}F$ if F is a holomorphic function on $\mathbb{C}^n$. Thus we can write

$$
G^{*,p} F(u) = \int_{\mathbb{C}^n} \frac{\overline{\mathbb{L}_t(z,u)}}{\rho_{p/2}} F(z) V_{t,p/2} dx\, dy. \tag{3.6}
$$

In order to change the measures $V_{t,p/2}$ and $\rho_{p/2}$ into Lebesgue measure, construct a map $\overline{G^{*,p}}$: $L^2(\mathbb{C}^n, dxdy) \to L^2(\mathbb{R}^n, du)$ defined by

$$\overline{G^{*,p}}F(u) = \rho_{p/2}^{1/2} G^{*,p}(V_{t,p/2}^{-1/2}F(z)). \tag{3.7}$$

An explicit computation shows that $\overline{G^{*,p}}$ can be written as

$$\begin{aligned}\overline{G^{*,p}}F(u) = C\int_{\mathbb{C}^n} & e^{ix\cdot y(\coth 2t-1)} e^{-iy\cdot u/\sinh 2t} \\ & \times e^{\{-1/2(\coth 2t+1-2/p)[u-(x/\sinh 2t(\coth 2t+1-2/p))]^2\}} F(z)dx\,dy.\end{aligned} \tag{3.8}$$

It can be easily verified that $\overline{G^{*,p}}$ defines a bounded linear map of $L^1(\mathbb{C}^n, dx\,dy)$ into $L^1(\mathbb{R}^n, du)$. By applying interpolation theorem ([16], M. Riesz convexity theorem), it follows that $\overline{G^{*,p}}$ is a bounded map of $L^q(\mathbb{C}^n, dx\,dy)$ into $L^q(\mathbb{R}^n, du)$ for q satisfying $1 \le q \le 2$. In particular if we take $p = q$, then $\overline{G^{*,p}}$ will be bounded from $L^p(\mathbb{C}^n, dx\,dy)$ into $L^p(\mathbb{R}^n, du)$. Again, we wish to change Lebesgue measure on $\mathbb{R}^n$ to ρ, and Lebesgue measure on $\mathbb{C}^n$ to $V_{t,p/2}^{p/2}$. Toward this end, we define

$$\overline{\overline{G^{*,p}}} : L^p(\mathbb{C}^n, V_{t,p/2}^{p/2}) \longrightarrow L^p(\mathbb{R}^n, \rho) \quad \text{by} \quad \overline{\overline{G^{*,p}}}F(u) = \rho^{-1/p}(u)\overline{G^{*,p}}(V_{t,p/2}^{1/2}F(z)). \tag{3.9}$$

Note that, $\rho(u)^{-1/p} = ce^{u^2/p} = c'\rho_{p/2}^{-1/2}$ for some constants c and c'. Then it follows from (3.7) that $\overline{\overline{G^{*,p}}}F(u) = CG^{*,p}F(u)$. This shows that $G^{*,p}$ is bounded from $L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$ into $L^p(\mathbb{R}^n, \rho)$.

We claim that if F is in the holomorphic subspace of $L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$, then $GG^{*,p}F = F$.

Let $P_s(z)$ be the "polydisk" of radius s, centered at z, namely, $P_s(z) = \{w \in \mathbb{C}^n : |w_k - z_k| < s,\ k = 1, 2, \dots, n$, and $w_k = u_k + iv_k$, $z_k = x_k + iy_k\}$ (see [17]). Then $F(z)$ can be written as

$$F(z) = (\pi s^2)^{-n} \int_{\mathbb{C}^n} \chi_{P_s(z)} \frac{1}{\alpha(w)} F(w)\alpha(w)du\,dv, \tag{3.10}$$

where $\chi_{P_s(z)}$ denotes the characteristic function on $P_s(z)$.

Define $\alpha(w) = e^{-pu^2/((p-1)e^{4t}+1)}e^{-pv^2/(e^{4t}-1)}$, $w = u + iv$, $\beta = p' - 1/p'$. By using Hölder's inequality,

$$\begin{aligned}&\int_{\mathbb{C}^n} |F_m(z)|^2 e^{-2x^2/((p-1)e^{4t}+1)} e^{-2y^2/(e^{4t}-1)} dx\,dy \\ &\quad = \int_{\mathbb{C}^n} |F(\lambda_m z)|^2 e^{-2x^2/((p-1)e^{4t}+1)} e^{-2y^2/(e^{4t}-1)} dx\,dy \\ &\quad = C\int_{\mathbb{C}^n} |F(z)|^2 e^{-2x^2/((p-1)e^{4t}+1)} e^{-2y^2/\lambda_m^2(e^{4t}-1)} dx\,dy.\end{aligned} \tag{3.11}$$

From this we can see that

$$|F(z)| \le Ce^{(x+s)^2/((p-1)e^{4t}+1)}e^{(y+s)^2/(e^{4t}-1)}. \tag{3.12}$$

Now define $F_m(z) = F(\lambda_m z)$, where λ_m is an increasing sequence of numbers tending to 1. Then $F_m \in L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$ and F_m will converge to F in the norm of $L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$. Consider

$$\begin{aligned} |F(z)| &\le C\left\| \chi_{P_s(z)} \frac{1}{|\alpha(w)|} \right\|_{L^{p'}(\mathbb{C}^n,\alpha(w))} \|F\|_{L^p(\mathbb{C}^n,\alpha(w))} \\ &\le C \sup_{w\in P_s(z)} \frac{1}{|\alpha(w)|^{\beta}} m(P_s(z))^{1/p'} \|F\|_{L^p(\mathbb{C}^n,\alpha(w))} \\ &= C \sup_{w\in P_s(z)} \frac{1}{|\alpha(w)|^{\beta}} \\ &= C \sup_{w\in P_s(z)} e^{u^2/((p-1)e^{4t}+1)} e^{v^2/(e^{4t}-1)} \quad \text{since } p\beta = 1. \end{aligned} \tag{3.13}$$

Then by using (3.12), we get

$$\begin{aligned} (3.11) &\le C\int_{\mathbb{C}^n} e^{(x+s)^2/((p-1)e^{4t}+1)} e^{(y+s)^2/(e^{4t}-1)} e^{-2x^2/\lambda_m^2((p-1)e^{4t}+1)} e^{-2y^2/\lambda_m^2(e^{4t}-1)} dx\, dy \\ &< \infty \quad \left(\text{as } 2 < \frac{2}{\lambda_m^2} \text{ for each } m\right). \end{aligned} \tag{3.14}$$

This shows that $F_m \in \mathscr{H}L^2(\mathbb{C}^n, V_{t,p/2})$ for each m which in turn implies that $GG^{*,p}F_m = F_m$ for each m. Since $G^{*,p}$ is bounded from $L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$ into $L^p(\mathbb{R}^n, \rho)$, $G^{*,p}F_m$ converges to $G^{*,p}F$. Then $GG^{*,p}F_m = F_m$ will converge uniformly to $GG^{*,p}F$ on compact sets. Since F_m also converges to F in the norm of $L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$, the pointwise limit and L^p limit must coincide, showing $GG^{*,p}F = F$. Then taking $f = G^{*,p}F$ proves our existence assertion. The uniqueness follows from Remark 2.1. □

Remark 3.5. As mentioned in the introduction, $\bigcap_{\epsilon>0}\mathscr{H}L^p(\mathbb{C}^n, V_{t,p/2}^{(p+\epsilon)/2})$ is larger than $\mathscr{H}L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$. In fact, if $f(z) = e^{z^2/((p-1)e^{4t}+1)}$, then $f \in \bigcap_{\epsilon>0}\mathscr{H}L^p(\mathbb{C}^n, V_{t,p/2}^{(p+\epsilon)/2})$ but $f \notin \mathscr{H}L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$. But we are able to show that the transform $e^{-t\mathbb{H}}$ is only onto the functions in $\mathscr{H}L^p(\mathbb{C}^n, V_{t,p/2}^{p/2})$.

The following theorem shows that the image of $L^p(\mathbb{R}^n, \rho)$ under Hermite polynomial semigroup will be contained in $\mathscr{H}L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$ also.

Theorem 3.6. *Fix $t > 0$ and let $1 < p \le 2$. If $f \in L^p(\mathbb{R}^n, \rho)$, then $e^{-t\mathbb{H}}f \in \mathscr{H}L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$, where p' is such that $1/p + 1/p' = 1$.*

Proof. Let $Gf = e^{-t\mathbb{H}}f$. Then it follows from Theorem 2.2 that G is an isometric isomorphism from $L^2(\mathbb{R}^n, \rho_{p/2})$ onto $\mathscr{H}L^2(\mathbb{C}^n, V_{t,p/2})$. In order to change the weighted measure into Lebesgue measures, construct a map $G_p : L^2(\mathbb{R}^n, du) \to L^2(\mathbb{C}^n, dx\,dy)$ defined by $G_pf(z) = V_{t,p/2}^{1/2} G(\rho_{p/2}(u)^{-1/2} f(u))$. An explicit computation shows that G_p can be written as

$$G_pf(x+iy) = C\int_{\mathbb{R}^n} e^{-i(\coth 2t-1)x\cdot y} e^{iy\cdot u/\sinh 2t} e^{\{(-1/2)(\coth 2t+1-2/p)[u-x/\sinh 2t(\coth 2t+1-2/p)]^2\}} f(u)\,du, \tag{3.15}$$

where $z = x + iy$ and C is a constant depending on t, p, n. It can be easily verified that G_p defines a bounded operator from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{C}^n)$. By the interpolation theorem, G_p is also bounded from $L^q(\mathbb{R}^n, du)$ into $L^{q'}(\mathbb{C}^n, dx\,dy)$, for q satisfying $1 \le q \le 2$. In particular we take $p = q$, then G_p will be bounded from $L^p(\mathbb{R}^n, du)$ into $L^{p'}(\mathbb{C}^n, dx\,dy)$. Again, to change measures, we define $\overline{G_p} : L^p(\mathbb{R}^n, \rho(u)) \to L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$ by $\overline{G_p}f(z) = V_{t,p/2}^{-1/2} G_p(\rho(u)^{1/p} f(u))$ we see that the operators G and $\overline{G_p}$ turn out to be the same up to a constant multiple. Thus G is bounded from $L^p(\mathbb{R}^n, \rho(u))$ into $L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$, proving our assertion. □

Using pointwise estimate in Theorem 3.1, one can obtain the following result.

Theorem 3.7. *Fix $t > 0$ and let $2 \le p < \infty$. If $f \in L^p(\mathbb{R}^n, \rho)$, then $e^{-t\mathbb{H}}f \in \mathscr{H}L^{p'}(\mathbb{C}^n, V_{t,p/2}^{(p+\epsilon)/2})$ for any fixed $\epsilon > 0$. In particular $e^{-t\mathbb{H}}f \in \bigcap_{\epsilon>0} \mathscr{H}L^{p'}(\mathbb{C}^n, V_{t,p/2}^{(p+\epsilon)/2})$.*

As mentioned earlier, Theorems 3.2 and 3.7 are special cases of the following theorem.

Theorem 3.8. *Suppose that $f \in L^p(\mathbb{R}^n, \rho)$ and that $1 < p < \infty$, $t > 0$ and $\epsilon > 0$. Then $e^{-t\mathbb{H}}f \in \mathscr{H}L^s(\mathbb{C}^n, V_{t,p/2}^{(s+\epsilon)/2})$ for any $1 \le s < \infty$.*

The proof is simply an application of the pointwise estimate proved in Theorem 3.1. Then one gets Theorem 3.2 by taking $s = p$ and Theorem 3.7 by taking $s = p'$.

As in the case of Theorem 3.4, we prove the following result.

Theorem 3.9. *If F is holomorphic and $F \in L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$, where $2 \le p < \infty$ and t is a fixed number greater than zero, then there exists a unique function $f \in L^p(\mathbb{R}^n, \rho)$ such that $e^{-t\mathbb{H}}f = F$, where p' is such that $1/p + 1/p' = 1$.*

Proof. In the proof of Theorem 3.4, we have noticed that $\overline{G^{*,p}}$ is bounded from $L^1(\mathbb{C}^n, dx\,dy)$ into $L^1(\mathbb{R}^n, du)$. Instead, one can also verify that $\overline{G^{*,p}}$ is bounded from $L^1(\mathbb{C}^n, dx\,dy)$ into $L^\infty(\mathbb{R}^n, du)$. In this case the interpolation theorem will show that $\overline{G^{*,p}}$ will be bounded from $L^{p'}(\mathbb{C}^n, dx\,dy)$ into $L^p(\mathbb{R}^n, du)$. So $\overline{\overline{G^{*,p}}}$ will also be bounded from $L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$ into $L^p(\mathbb{R}^n, \rho)$. In this case also, we can show that $GG^{*,p}F = F$, for $F \in \mathscr{H}L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$. Now, let $f = G^{*,p}F$. The uniqueness follows from Remark 2.1. □

Remark 3.10. As mentioned earlier, we are able to show that the transform $e^{-t\mathbb{H}}$ is only onto the functions in $\mathscr{H}L^{p'}(\mathbb{C}^n, V_{t,p/2}^{p'/2})$, instead of $\bigcap_{\epsilon>0} \mathscr{H}L^{p'}(\mathbb{C}^n, V_{t,p/2}^{(p'+\epsilon)/2})$.

The following theorem gives a sufficient condition for a holomorphic function F to be in the image of $e^{-t\mathbb{H}}$. As we will now see, this condition is a certain type of integrability of F.

Theorem 3.11. *If F is holomorphic and $F \in L^1(\mathbb{C}^n, V_{t,p/2}^{1/2})$, where $1 < p < \infty$ and t is a fixed number greater than zero, then there exists a unique function $f \in L^p(\mathbb{R}^n, \rho)$ such that $e^{-t\mathbb{H}} f = F$.*

Proof. By proceeding as in Theorem 3.4, $G^{*,p}$ can be rewritten as

$$G^{*,p}F(u) = C\iint_{\mathbb{R}^n} e^{2u^2/p} e^{(-1/2)(\coth 2t-1)(x-iy)^2} e^{(-1/2)(\coth 2t+1)u^2} e^{(x-iy)\cdot u/\sinh 2t} \times e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/e^{4t}-1} \left[F(z) e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/e^{4t}-1}\right] dx\, dy. \tag{3.16}$$

Further, by considering L^p norm with respect to the variable u, we can show that

$$\left\| e^{2u^2/p} e^{(-1/2)(\coth 2t-1)(x^2-y^2)} e^{(-1/2)(\coth 2t+1)u^2} e^{x\cdot u/\sinh 2t} e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/(e^{4t}-1)} \right\|_{L^p(\mathbb{R}^n,\rho)} = C, \tag{3.17}$$

where $C = (p/2(\coth 2t + 1) - 1)^{-n/2}$ is independent of x and y. Since $F \in L^1(\mathbb{C}^n, V_{t,p/2}^{1/2})$, we can show by Minkowski's integral inequality that

$$\|G^{*,p}F\|_{L^p(\mathbb{R}^n,\rho)} \le C\int_{\mathbb{C}^n} |F(x+iy)| e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/(e^{4t}-1)} dx\, dy. \tag{3.18}$$

Thus $G^{*,p}$ is bounded from $L^1(\mathbb{C}^n, e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/(e^{4t}-1)})$ into $L^p(\mathbb{R}^n, \rho)$. Again as in Theorem 3.4, by taking an increasing sequence λ_m of real numbers converging to 1, one can show that $GG^{*,p}F = F$, for any holomorphic function F in $L^1(\mathbb{C}^n, e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/(e^{4t}-1)})$. Then taking $f = G^{*,p}F$ proves our existence assertion. The uniqueness follows from Remark 2.1. □

Theorem 3.12. *Fix $t > 0$ and let $1 < p < \infty$. Suppose F is holomorphic and $F \in L_m^{1,p}(\mathbb{R}^{2n})$, where $m(x, y) = e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/(e^{4t}-1)}$ then there exists a unique $f \in L^p(\mathbb{R}^n, \rho)$ with $e^{-t\mathbb{H}} f = F$.*

Proof. We have $G^{*,p}F(u)$ as in (3.16). Then

$$|G^{*,p}F(u)| \le Ce^{u^2/p} \int_{\mathbb{R}^n} e^{(-1/2)(\coth 2t+1-2/p)[u-x/\sinh 2t(\coth 2t+1-2/p)]^2} \times \left(\int_{\mathbb{R}^n} |F(x+iy)| e^{-x^2/((p-1)e^{4t}+1)} e^{-y^2/(e^{4t}-1)} dy \right) dx. \tag{3.19}$$

Then, by applying Minkowski's integral inequality, it follows from hypothesis that $e^{-u^2/p}G^{*,p}F \in L^p(\mathbb{R}^n, du)$, which means that $G^{*,p}F \in L^p(\mathbb{R}^n, \rho(u)du)$. Thus $G^{*,p}$ is bounded from $L_m^{1,p}(\mathbb{R}^{2n})$ into $L^p(\mathbb{R}^n, \rho)$. Again, by showing $GG^{*,p}F = F$, for F holomorphic and $F \in L_m^{1,p}(\mathbb{R}^{2n})$, we obtain the required result. The uniqueness follows from Remark 2.1. □

Acknowledgments

The authors wish to express their sincere gratitude to the referees for spending their valuable time in reading the manuscript very carefully, pointing out many inaccuracies, gaps, and errors and giving them several useful suggestions for improving the earlier version of the manuscript. The authors thank Professor S. Thangavelu for initiating them into this work and giving them valuable suggestions. They also wish to thank Professor B. C. Hall for giving clarifications to their questions related to Segal-Bargmann transform. D. Venku Naidu thanks the Council of Scientific and Industrial Research, India, for the financial support.

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