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Research Article

On New Extensions of Hilbert's Integral Inequality

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It is shown that some new extensions of Hilbert's integral inequality with parameter $\lambda(\lambda > 1/2)$ can be established by introducing a proper weight function. In particular, when $\lambda = 1$, a refinement of Hilbert's integral inequality is obtained. As applications, some new extensions of Widder's inequality and Hardy-Littlewood's inequality are given.

1. Introduction and lemmas

Let $f(x)$, $g(x) \in L^2(0,+\infty)$. It is well known that the inequality of the form

$$\iint_{\alpha}^{\infty} \frac{f(x)g(y)}{x+y-2\alpha}dx\,dy \leq \pi\left\{\int_{\alpha}^{\infty} f^2(x)dx\right\}^{1/2}\left\{\int_{\alpha}^{\infty} g^2(x)dx\right\}^{1/2} \tag{1.1}$$

is called Hilbert's integral inequality, where the coefficient π is the best possible.

In [1], by introducing a parameter λ $(\lambda > 1/2)$, the following extension of (1.1) of the form

$$\iint_{0}^{\infty} \frac{f(x)g(y)}{x^{\lambda}+y^{\lambda}}dx\,dy \leq \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right)\left\{\int_{0}^{\infty} x^{1-\lambda}f^2(x)dx\right\}^{1/2}\left\{\int_{0}^{\infty} x^{1-\lambda}g^2(x)dx\right\}^{1/2} \tag{1.2}$$

was established.

Recently, various improvements and extensions of (1.1) appear in a great deal of papers (see [2]). The aim of this paper is to give some new improvements of (1.1) and (1.2) and then present some important applications.

We now introduce some notations that will be used throughout the paper.

Let $x-\alpha \geq 0,\ y-\alpha \geq 0,\ \lambda > 1/2$, and let $c(x-\alpha)$ be an integrable function in $[\alpha, +\infty)$. Define $E(x,y) = 1 - c(x-\alpha) + c(y-\alpha)$ such that $E(x,y) \geq 0$ for $(x,y) \in (\alpha,+\infty)\times(\alpha,+\infty)$. We also define

$$J_1 = \iint_{\alpha}^{\infty} \frac{f^2(x)}{(x-\alpha)^{\lambda} + (y-\alpha)^{\lambda}} \left(\frac{x-\alpha}{y-\alpha}\right)^{1/2} E(x,y)\,dx\,dy,$$
$$J_2 = \iint_{\alpha}^{\infty} \frac{f^2(y)}{(x-\alpha)^{\lambda} + (y-\alpha)^{\lambda}} \left(\frac{y-\alpha}{x-\alpha}\right)^{1/2} E(x,y)\,dx\,dy. \tag{1.3}$$

Lemma 1.1. *With the above-mentioned assumptions, one has*

$$J_1 J_2 = \left(\frac{\pi}{\lambda \sin(\pi/2\lambda)}\right)^2 \left\{ \left(\int_{\alpha}^{\infty} (x-\alpha)^{1-\lambda} f^2(x)\,dx\right)^2 - \left(\int_{\alpha}^{\infty} k(x) f^2(x)\,dx\right)^2 \right\}, \tag{1.4}$$

where the weight function $k(x)$ is defined by

$$k(x) = (x-\alpha)^{1-\lambda} \left\{ \left(\frac{\lambda \sin(\pi/2\lambda)}{\pi}\right) \int_0^{\infty} \frac{c\left((x-\alpha)t\right) t^{-1/2}}{1 + t^{\lambda}}\,dt - c(x-\alpha) \right\}. \tag{1.5}$$

Proof. By using the substitution $t = (y-\alpha)/(x-\alpha)$, it is easy to deduce that

$$\begin{aligned}
J_1 &= \int_{\alpha}^{\infty} \left\{ \int_{\alpha}^{\infty} \frac{1}{(x-\alpha)^{\lambda}\left(1 + ((y-\alpha)/(x-\alpha))^{\lambda}\right)} \left(\frac{x-\alpha}{y-\alpha}\right)^{1/2} E(x,y)\,dy \right\} f^2(x)\,dx \\
&= \int_{\alpha}^{\infty} \left\{ \int_0^{\infty} \frac{1}{1+t^{\lambda}} \left(\frac{1}{t}\right)^{1/2} \left(1 - c(x-\alpha) + c((x-\alpha)t)\right) dt \right\} (x-\alpha)^{1-\lambda} f^2(x)\,dx \\
&= \int_{\alpha}^{\infty} \left\{ \frac{\pi}{\lambda\sin(\pi/2\lambda)} + \int_0^{\infty} \frac{c((x-\alpha)t)}{1+t^{\lambda}} \left(\frac{1}{t}\right)^{1/2} dt - \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right) c(x-\alpha) \right\} (x-\alpha)^{1-\lambda} f^2(x)\,dx \\
&= \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right) \left\{ \int_{\alpha}^{\infty} (x-\alpha)^{1-\lambda} f^2(x)\,dx + \int_{\alpha}^{\infty} k(x) f^2(x)\,dx \right\},
\end{aligned} \tag{1.6}$$

where $k(x)$ is a function defined by (1.5).

Similarly, we have

$$J_2 = \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right) \left\{ \int_{\alpha}^{\infty} (x-\alpha)^{1-\lambda} f^2(x)\,dx - \int_{\alpha}^{\infty} k(x) f^2(x)\,dx \right\}. \tag{1.7}$$

From the above equations involving J_1 and J_2, (1.4) holds true. □

Lemma 1.2. *Let $x - \alpha \geq 0$ and $\lambda > 1/2$. Then*

$$\int_0^{\infty} \frac{t^{-1/2}}{\left(1+t^{\lambda}\right)\left(1 + (x-\alpha)^{\lambda} t^{\lambda}\right)}\,dt = \begin{cases} \left(\dfrac{\pi}{\lambda\sin(\pi/2\lambda)}\right) \dfrac{(x-\alpha)^{\lambda-1/2} - 1}{(x-\alpha)^{\lambda} - 1}, & x-\alpha \neq 1, \\[2ex] \left(\dfrac{\pi}{\lambda\sin(\pi/2\lambda)}\right)\left(1 - \dfrac{1}{2\lambda}\right), & x-\alpha = 1. \end{cases} \tag{1.8}$$

Proof. The case $x-\alpha \neq 1$ was studied in [3], or can be obtained by using [4]. Next, consider the case $x-\alpha = 1$. By the definition and properties of beta function, it is easy to deduce that

$$\int_0^\infty \frac{t^{-1/2}}{(1+t^\lambda)^2}dt = \frac{1}{\lambda}B\left(\frac{1}{2\lambda}, 2-\frac{1}{2\lambda}\right) = \frac{1}{\lambda}\left(1-\frac{1}{2\lambda}\right)B\left(\frac{1}{2\lambda}, 1-\frac{1}{2\lambda}\right) = \left(1-\frac{1}{2\lambda}\right)\frac{\pi}{\lambda\sin(\pi/2\lambda)}. \tag{1.9}$$

□

2. Theorem and its corollary

Theorem 2.1. *Let $f(x)$ and $g(x)$ be two real functions such that $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)dx < +\infty$ and $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} g^2(x)dx < +\infty$, where $\lambda > 1/2$. Then*

$$\begin{aligned}
&\left(\iint_\alpha^\infty \frac{f(x)g(y)}{(x-\alpha)^\lambda + (y-\alpha)^\lambda}dx\,dy\right)^4 \\
&\quad\leq \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right)^4 \left\{\left(\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)dx\right)^2 - \left(\int_\alpha^\infty \omega_\lambda(x) f^2(x)dx\right)^2\right\} \\
&\qquad\times \left\{\left(\int_\alpha^\infty (x-\alpha)^{1-\lambda} g^2(x)dx\right)^2 - \left(\int_\alpha^\infty \omega_\lambda(x) g^2(x)dx\right)^2\right\},
\end{aligned} \tag{2.1}$$

where the weight function $\omega_\lambda(x)$ is defined by

$$\omega_\lambda(x) = \begin{cases} (x-\alpha)^{1-\lambda}\left\{\dfrac{(x-\alpha)^{\lambda-1/2}-1}{(x-\alpha)^\lambda - 1} - \dfrac{1}{1+(x-\alpha)^\lambda}\right\}, & x-\alpha \neq 1, \\ \dfrac{1}{2} - \dfrac{1}{2\lambda}, & x-\alpha = 1. \end{cases} \tag{2.2}$$

Proof. First, assume $f = g$. Let $F(x,y) = f(x)f(y)/((x-\alpha)^\lambda + (y-\alpha)^\lambda)$, $E(x,y) = 1 - c(x-\alpha) + c(y-\alpha)$.

Then the following holds:

$$\iint_\alpha^\infty F(x,y)dx\,dy = \iint_\alpha^\infty F(x,y)E(x,y)dx\,dy. \tag{2.3}$$

In fact, it is obvious that

$$\begin{aligned}
&\iint_\alpha^\infty F(x,y)E(x,y)dx\,dy \\
&\quad= \iint_\alpha^\infty F(x,y)dx\,dy - \iint_\alpha^\infty F(x,y)c(x-\alpha)dx\,dy + \iint_\alpha^\infty F(x,y)c(y-\alpha)dx\,dy.
\end{aligned} \tag{2.4}$$

We need only to show that

$$\iint_\alpha^\infty F(x,y)c(x-\alpha)dx\,dy = \iint_\alpha^\infty F(x,y)c(y-\alpha)dx\,dy. \tag{2.5}$$

Let $\varphi(x) = \int_{\alpha}^{\infty} f(t)/((x-\alpha)^{\lambda} + (t-\alpha)^{\lambda})dt$. Then

$$\begin{aligned}
&\iint_{\alpha}^{\infty} F(x,y)c(x-\alpha)dx\,dy \\
&\quad= \int_{\alpha}^{\infty}\left(\int_{\alpha}^{\infty}\frac{f(y)}{(x-\alpha)^{\lambda}+(y-\alpha)^{\lambda}}dy\right)f(x)c(x-\alpha)dx \\
&\quad= \int_{\alpha}^{\infty}\left(\int_{\alpha}^{\infty}\frac{f(t)}{(x-\alpha)^{\lambda}+(t-\alpha)^{\lambda}}dt\right)f(x)c(x-\alpha)dx = \int_{\alpha}^{\infty}\varphi(x)f(x)c(x-\alpha)dx \\
&\quad= \int_{\alpha}^{\infty}\varphi(y)f(y)c(y-\alpha)dy = \int_{\alpha}^{\infty}\left(\int_{\alpha}^{\infty}\frac{f(t)}{(y-\alpha)^{\lambda}+(t-\alpha)^{\lambda}}dt\right)f(y)c(y-\alpha)dy \\
&\quad= \int_{\alpha}^{\infty}\left(\int_{\alpha}^{\infty}\frac{f(x)}{(y-\alpha)^{\lambda}+(x-\alpha)^{\lambda}}dx\right)f(y)c(y-\alpha)dy = \iint_{\alpha}^{\infty}F(x,y)c(y-\alpha)dx\,dy.
\end{aligned} \tag{2.6}$$

Noting that $E(x,y) \geq 0$ and applying Schwarz's inequality, we have

$$\begin{aligned}
\left(\iint_{\alpha}^{\infty}F(x,y)dx\,dy\right)^2 &= \left(\iint_{\alpha}^{\infty}F(x,y)E(x,y)dx\,dy\right)^2 \\
&= \left(\iint_{\alpha}^{\infty}\left\{\frac{f(x)}{((x-\alpha)^{\lambda}+(y-\alpha)^{\lambda})^{1/2}}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/4}(E(x,y))^{1/2}\right\}\right. \\
&\qquad \left.\times\left\{\frac{f(y)}{((x-\alpha)^{\lambda}+(y-\alpha)^{\lambda})^{1/2}}\left(\frac{y-\alpha}{x-\alpha}\right)^{1/4}(E(x,y))^{1/2}\right\}dx\,dy\right)^2 \\
&\leq \iint_{\alpha}^{\infty}\left\{\frac{f^2(x)}{(x-\alpha)^{\lambda}+(y-\alpha)^{\lambda}}\left(\frac{x-\alpha}{y-\alpha}\right)^{1/2}E(x,y)\right\}dx\,dy \\
&\qquad \times\iint_{\alpha}^{\infty}\left\{\frac{f^2(y)}{(x-\alpha)^{\lambda}+(y-\alpha)^{\lambda}}\left(\frac{y-\alpha}{x-\alpha}\right)^{1/2}E(x,y)\right\}dx\,dy = J_1J_2.
\end{aligned} \tag{2.7}$$

It follows from (1.4) that

$$\left(\iint_{\alpha}^{\infty}F(x,y)dx\,dy\right)^2 \leq \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right)^2\left\{\left(\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}f^2(x)dx\right)^2 - \left(\int_{\alpha}^{\infty}k(x)f^2(x)dx\right)^2\right\}, \tag{2.8}$$

where the weight function $k(x)$ is defined by (1.5).

Let $c(x) = 1/(1+x^{\lambda})$, where $x \geq \alpha$ and $\lambda > 1/2$. It is obvious that $E(x,y) \geq 0$. By Lemma 1.2, it is easy to deduce that

$$\begin{aligned}
k(x) &= (x-\alpha)^{1-\lambda}\left\{\left(\frac{\lambda\sin(\pi/2\lambda)}{\pi}\right)\int_0^{\infty}\frac{c((x-\alpha)t)t^{-1/2}}{1+t^{\lambda}}dt - c(x-\alpha)\right\} \\
&= (x-\alpha)^{1-\lambda}\left\{\left(\frac{\lambda\sin(\pi/2\lambda)}{\pi}\right)\int_0^{\infty}\frac{t^{-1/2}}{(1+t^{\lambda})(1+(x-\alpha)^{\lambda}t^{\lambda})}dt - c(x-\alpha)\right\} = \omega_{\lambda}(x).
\end{aligned} \tag{2.9}$$

Substitute $k(x) = \omega_\lambda(x)$ into (2.8) to obtain

$$\left(\iint_\alpha^\infty F(x,y)dx\,dy\right)^2 \le \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right)^2 \left\{\left(\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)dx\right)^2 - \left(\int_\alpha^\infty \omega_\lambda(x) f^2(x)dx\right)^2\right\}. \tag{2.10}$$

Next, consider the case $f \neq g$. By Schwarz's inequality, we have

$$\begin{aligned}
&\left(\iint_\alpha^\infty \frac{f(x)\,g(y)}{(x-\alpha)^\lambda + (y-\alpha)^\lambda} dx\,dy\right)^4 \\
&= \left\{\left(\int_0^1 \left(\int_\alpha^\infty t^{(x-\alpha)^\lambda - 1/2} f(x)dx \int_\alpha^\infty t^{(y-\alpha)^\lambda - 1/2} g(y)dy\right)dt\right)^2\right\}^2 \\
&\le \left\{\int_0^1 \left(\int_\alpha^\infty t^{(x-\alpha)^\lambda - 1/2} f(x)dx\right)^2 dt\right\}^2 \left\{\int_0^1 \left(\int_\alpha^\infty t^{(y-\alpha)^\lambda - 1/2} g(y)dy\right)^2 dt\right\}^2 \\
&= \left\{\iint_\alpha^\infty \frac{f(x)\,f(y)}{(x-\alpha)^\lambda + (y-\alpha)^\lambda} dx\,dy\right\}^2 \left\{\iint_\alpha^\infty \frac{g(x)\,g(y)}{(x-\alpha)^\lambda + (y-\alpha)^\lambda} dx\,dy\right\}^2.
\end{aligned} \tag{2.11}$$

Based on (2.10), it follows from (2.11) that the inequality (2.1) is valid at once. Theorem is proved. □

The special case $\lambda = 1$ in Theorem 2.1 yields the following Hilbert's integral inequality.

Corollary 2.2. *If* $0 < \int_\alpha^\infty f^2(x)dx < +\infty$ *and* $0 < \int_\alpha^\infty g^2(x)dx < +\infty$, *then*

$$\begin{aligned}
\left(\iint_\alpha^\infty \frac{f(x)g(y)}{x+y-2\alpha} dx\,dy\right)^4 \le\ & \pi^4 \left\{\left(\int_\alpha^\infty f^2(x)dx\right)^2 - \left(\int_\alpha^\infty \omega_1(x) f^2(x)dx\right)^2\right\} \\
& \times \left\{\left(\int_\alpha^\infty g^2(x)dx\right)^2 - \left(\int_\alpha^\infty \omega_1(x) g^2(x)dx\right)^2\right\},
\end{aligned} \tag{2.12}$$

where the weight function $\omega_1(x)$ *is defined by*

$$\omega_1(x) = \frac{1}{\sqrt{x-\alpha}+1} - \frac{1}{x-\alpha+1}. \tag{2.13}$$

Proof. It follows directly from the proof of Theorem 2.1 and so the details are omitted. □

Remark 2.3. By setting $f = g$ in Corollary 2.2, (2.12) yields

$$\left(\iint_\alpha^\infty \frac{f(x)f(y)}{x+y-2\alpha} dx\,dy\right)^2 \le \pi^2 \left\{\left(\int_\alpha^\infty f^2(x)dx\right)^2 - \left(\int_\alpha^\infty \omega_1(x) f^2(x)dx\right)^2\right\}, \tag{2.14}$$

where the weight function $\omega_1(x)$ is defined by (2.13).

Remark 2.4. For the case $\lambda = 2$ in Theorem 2.1, (2.1) becomes

$$\left(\iint_{\alpha}^{\infty}\frac{f(x)g(y)}{(x-\alpha)^2+(y-\alpha)^2}dx\,dy\right)^4$$
$$\leq \frac{\pi^4}{4}\left\{\left(\int_{\alpha}^{\infty}(x-\alpha)^{-1}f^2(x)dx\right)^2-\left(\int_{\alpha}^{\infty}\omega_2(x)f^2(x)dx\right)^2\right\} \quad (2.15)$$
$$\times\left\{\left(\int_{\alpha}^{\infty}(x-\alpha)^{-1}g^2(x)dx\right)^2-\left(\int_{\alpha}^{\infty}\omega_2(x)g^2(x)dx\right)^2\right\},$$

where the weight function $\omega_2(x)$ is defined by

$$\omega_2(x)=\begin{cases}(x-\alpha)^{-1}\left\{\dfrac{(x-\alpha)^{3/2}-1}{(x-\alpha)^2-1}-\dfrac{1}{1+(x-\alpha)^2}\right\}, & x-\alpha\neq 1,\\ \dfrac{1}{4}, & x-\alpha=1.\end{cases} \quad (2.16)$$

3. Some applications

As applications, we will give some extensions and refinements of Widder's inequality and Hardy-Littlewood's inequality.

Let $a_n \geq 0$ $(n=0,1,2,\dots)$, $A(x)=\sum_{n=0}^{\infty}a_nx^n$, $A^*(x)=\sum_{n=0}^{\infty}a_nx^n/n!$. Then

$$\int_0^1 A^2(x)dx \leq \pi\int_0^{\infty}\left(e^{-x}A^*(x)\right)^2dx. \quad (3.1)$$

Inequality (3.1) is called Widder's inequality (see [5]).

We will give an extension of (3.1) below.

Theorem 3.1. *Under the above assumptions, if* $f(x)=e^{-(x-\alpha)}A^*(x-\alpha)$, *then*

$$\left(\int_0^1 A^2(x)dx\right)^2 \leq \pi^2\left\{\left(\int_{\alpha}^{\infty}f^2(x)dx\right)^2-\left(\int_{\alpha}^{\infty}\omega_1(x)f^2(x)dx\right)^2\right\}, \quad (3.2)$$

where $\omega_1(x)$ *is defined by* (2.13).

Proof. First, observe that the following holds:

$$\int_0^{\infty}e^{-t}A^*(tx)dt=\int_0^{\infty}e^{-t}\sum_{n=0}^{\infty}\frac{a_n(xt)^n}{n!}dt=\sum_{n=0}^{\infty}\frac{a_nx^n}{n!}\int_0^{\infty}t^ne^{-t}dt=\sum_{n=0}^{\infty}a_nx^n=A(x). \quad (3.3)$$

Let $tx=s-\alpha$. Then we have

$$\int_0^1 A^2(x)dx=\int_0^1\left\{\int_0^{\infty}e^{-t}A^*(tx)dt\right\}^2dx=\int_0^1\left(\int_{\alpha}^{\infty}e^{-(s-\alpha)/x}A^*(s-\alpha)ds\right)^2\frac{1}{x^2}dx$$
$$=\int_1^{\infty}\left(\int_{\alpha}^{\infty}e^{-(s-\alpha)y}A^*(s-\alpha)ds\right)^2dy=\int_0^{\infty}\left(\int_{\alpha}^{\infty}e^{-(s-\alpha)u-(s-\alpha)}A^*(s-\alpha)ds\right)^2du$$
$$=\int_0^{\infty}\left(\int_{\alpha}^{\infty}e^{-(s-\alpha)u}f(s)ds\right)^2du=\iint_{\alpha}^{\infty}\frac{f(s)f(t)}{s+t-2\alpha}ds\,dt, \quad (3.4)$$

where $f(x) = e^{-(x-\alpha)}A^*(x-\alpha)$. Using Remark 2.3, the inequality (3.2) follows from (3.4) at once.

In particular, when $\alpha = 0$, we obtain a refinement of (3.1). □

Corollary 3.2. *With the assumptions as Theorem 3.1, if $f(x) = e^{-x}A^*(x)$, then*

$$\left(\int_0^1 A^2(x)dx\right)^2 \leq \pi^2\left\{\left(\int_0^\infty f^2(x)dx\right)^2 - \left(\int_0^\infty \omega_1(x)f^2(x)dx\right)^2\right\}, \tag{3.5}$$

where $\omega_1(x)$ is defined by (2.13).

Let $f(x) \in L^2(0,1)$. If $a_n = \int_0^1 x^n f(x)dx$, $n = 0, 1, 2, \ldots$, then we have Hardy-Littlewood's inequality (see [6]) of the form

$$\sum_{n=0}^{\infty} a_n^2 \leq \pi\int_0^1 f^2(x)dx, \tag{3.6}$$

where π is the best constant that keeps (3.6) valid. In [7], the inequality (3.6) was extended to the following inequality:

$$\int_0^\infty f^2(x)dx \leq \pi\int_0^1 h^2(x)dx, \tag{3.7}$$

where $f(x) = \int_0^1 t^x h(x)dx$, $x \in [0,+\infty)$.

The inequality (3.7) is called Hardy-Littlewood's integral inequality. Afterwards, the inequality (3.7) was refined into the following form (see [8]):

$$\int_0^\infty f^2(x)dx \leq \pi\int_0^1 t\,h^2(t)dt. \tag{3.8}$$

We will give a new extension of (3.8) here.

Theorem 3.3. *Let $\lambda > 1/2$, $h(t) \in L^2(0,1)$, and $h(t) \neq 0$. Define a function by $f(x) = \int_0^1 t^{(x-\alpha)^\lambda}|h(t)|dt$. If $0 < \int_\alpha^{+\infty}(x-\alpha)^{1-\lambda}f^2(x)dx < +\infty$, then*

$$\left(\int_\alpha^\infty f^2(x)dx\right)^4 \leq \left(\frac{\pi}{\lambda\sin(\pi/2\lambda)}\right)^2\left\{\left(\int_\alpha^\infty (x-\alpha)^{1-\lambda}f^2(x)dx\right)^2 - \left(\int_\alpha^\infty \omega_\lambda(x)f^2(x)dx\right)^2\right\}\left(\int_0^1 t\,h^2(t)dt\right)^2, \tag{3.9}$$

where the weight function $\omega_\lambda(x)$ is defined by (2.2).

Proof. By writing $f^2(x)$ in the form

$$f^2(x) = \int_0^1 f(x)t^{(x-\alpha)^\lambda}|h(t)|dt \tag{3.10}$$

and applying Schwarz's inequality, we obtain

$$\begin{aligned}\left(\int_{\alpha}^{+\infty} f^2(x)dx\right)^4 &= \left\{\int_{\alpha}^{\infty}\left(\int_0^1 f(x)t^{(x-\alpha)^{\lambda}}|h(t)|dt\right)dx\right\}^4 \\ &= \left\{\int_0^1\left(\int_{\alpha}^{+\infty} f(x)t^{(x-\alpha)^{\lambda}-1/2}dx\right)t^{1/2}|h(t)|dt\right\}^4 \\ &\le \left\{\int_0^1\left(\int_{\alpha}^{+\infty} f(x)t^{(x-\alpha)^{\lambda}-1/2}dx\right)^2 dt\int_0^1 t\,h^2(t)dt\right\}^2 \\ &= \left\{\int_0^1\left(\iint_{\alpha}^{+\infty} f(x)f(y)t^{(x-\alpha)^{\lambda}+(y-\alpha)^{\lambda}-1}dx\,dy\right)dt\int_0^1 t\,h^2(t)dt\right\}^2 \\ &= \left(\iint_{\alpha}^{+\infty}\frac{f(x)f(y)}{(x-\alpha)^{\lambda}+(y-\alpha)^{\lambda}}dx\,dy\right)^2\left(\int_0^1 t\,h^2(t)dt\right)^2.\end{aligned} \tag{3.11}$$

By (2.10), the inequality (3.9) follows from (3.11) at once. □

Remark 3.4. By setting $\lambda = 1$ and $\alpha = 0$, we obtain the following refinement of (3.8):

$$\left(\int_0^{\infty} f^2(x)dx\right)^4 \le \pi^2\left\{\left(\int_0^{\infty} f^2(x)dx\right)^2 - \left(\int_0^{\infty}\omega_1(x)f^2(x)dx\right)^2\right\}\left(\int_0^1 t\,h^2(t)dt\right)^2, \tag{3.12}$$

where the weight function $\omega_1(x)$ is defined by (2.13).

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