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Research Article

On the Existence of Solutions of a Nonlocal Elliptic Equation with a p-Kirchhoff-Type Term

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Questions on the existence of positive solutions for the following class of elliptic problems are studied: $-[M(\|u\|_{1,p}^p)]^{1,p}\Delta_p u = f(x,u)$, in Ω, $u = 0$, on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$ are given functions.

1. Introduction

In this paper, we are concerned with the elliptic problem

$$\begin{gathered} -[M(\|u\|_{1,p}^p)]^{1,p}\Delta_p u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded smooth domain, $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$ are given functions, Δ_p is the p-Laplacian:

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad p > 1, \tag{1.2}$$

and $\|\cdot\|_{1,p}$ is the usual norm

$$\|u\|_{1,p}^p = \int_\Omega |\nabla u|^p \tag{1.3}$$

in the Sobolev space $W_0^{1,p}(\Omega)$.

Such a problem, which will be named p-Kirchhoff problem, is a generalization of the classical stationary Kirchhoff equation

$$-[M(\|u\|^2)]\Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\|u\|^2 = \int_\Omega |\nabla u|^2$ is the usual norm in $H_0^1(\Omega)$.

As it is well known, problem (1.4) is the stationary counterpart of the hyperbolic Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.5}$$

that appeared at the first time in the work of Kirchhoff [1], in 1883. The equation in (1.5) is called Kirchhoff equation and it extends the classical D'Alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations.

The interest of the mathematicians on the so-called nonlocal problems like (1.1) (nonlocal because of the presence of the term $M(\|u\|_{1,p}^p)$, which implies that equations in (1.1) and (1.4) are no longer pointwise equalities) has increased because they represent a variety of relevant physical and engineering situations and requires a nontrivial apparatus to solve them.

Particularly, problem (1.1) presents some combinations that, at least to our knowledge, seem to be new. Indeed, in problem (1.1) appears the nonlocal term $M(\|u\|_{1,p}^p)$ motivated, among other things, by the above physical situations. Furthermore, we have the presence of the p-Laplacian operator that appears in several areas of the science such as astronomy, glaciology, climatology, nonnewtonian fluids, petroleum extraction. Problems that involve these two terms, $M(\|u\|_{1,p}^p)$ and $\Delta_p u$, present several difficulties such as uniqueness, regularity, degeneracy, as we will see throughout this paper.

Beside these considerations, we also consider a case with the presence of a singular term which poses an additional difficulty in our study. Singular elliptic problems arise in chemical heterogeneous catalysts, nonnewtonian fluids, nonlinear heat conduction, among other phenomena.

In case $p = 2$, problem (1.1) has been studied by several authors. See [2–7], and the references therein. Particularly, this work was motivated by [2–4, 6].

We will establish existence results for problem (1.1) by considering several classes of functions M and f.

An outline of this work is as follows:

In Section 2, we recall some properties of the p-Laplacian. In Section 3, we study the case in which f depends only on $x \in \Omega$. This is the M_p -Linear case. In Section 4, we attack problem (1.1) when f is sublinear, that is, $f(u) = u^\alpha$, for some $0 < \alpha < 1$.

In both Sections 3 and 4, we suitably adapt ideas developed in [2, 3, 6].

In Section 5, we analyze the case in which f possesses a singular term. More precisely, f is of the following form:

$$f(x,u) = \frac{h(x)}{u^{\gamma-p+2}} + u^{\alpha-p+2}, \tag{1.6}$$

for $x \in \overline{\Omega}$ and $u > 0$, with $\gamma, \alpha \in (p-2, p-1)$. In this section, we have to use some arguments different from those in [4].

To end this introduction we recall that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (1.1) if

$$[M(\|u\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi = \int_\Omega f(x,u)\phi, \quad \forall \phi \in W_0^{1,p}(\Omega). \tag{1.7}$$

2. Preliminaries on the p-Laplacian

We will briefly expose some properties of the p-Laplacian operator defined by

$$\Delta_p u \equiv \operatorname{div}(\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty. \tag{2.1}$$

First, we consider the problem

$$\begin{aligned} -\Delta_p u &= f(x), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

where $f \in W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, $p > 1$, and the boundary condition will be understood as $u \in W_0^{1,p}(\Omega)$.

The following results holds as a simple consequence of a minimization of a suitable functional.

Theorem 2.1. *If $f \in W^{-1,p'}(\Omega)$, then problem* (2.2) *has only a solution $u \in W_0^{1,p}(\Omega)$ in the weak sense, namely,*

$$[M(\|u\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi = \int_\Omega f(x)\phi, \quad \forall \phi \in W_0^{1,p}(\Omega). \tag{2.3}$$

So, we have defined an operator $(-\Delta_p)^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$, the inverse of $-\Delta_p$, which satisfies the following.

(a) $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is uniformly continuous on bounded sets, where such operator is defined as

$$\begin{aligned} -\Delta_p : W_0^{1,p}(\Omega) &\longrightarrow W^{-1,p'}(\Omega), \\ u &\longmapsto -\Delta_p u, \end{aligned} \tag{2.4}$$

where

$$\langle -\Delta_p u, \phi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi, \quad \forall \phi \in W_0^{1,p}(\Omega). \tag{2.5}$$

Here, we denote by $\langle\ ,\ \rangle$ the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

(b) If $f \in C^0(\overline{\Omega})$, then the weak solution of (2.2) belongs to $C^{1,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$, and the mapping $(-\Delta_p)^{-1} : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ is compact.

Theorem 2.2 (weak comparison principle). *Let $u_1, u_2 \in W_0^{1,p}(\Omega)$ satisfy*

$$\begin{gathered} -\Delta_p u_1 \leq -\Delta_p u_2, \quad \text{in } \Omega, \ (\text{in the weak sense}) \\ u_1 \leq u_2, \quad \text{on } \partial\Omega. \end{gathered} \tag{2.6}$$

Then, $u_1 \leq u_2$ a.e. in Ω.

Theorem 2.3 (a Hopf-type maximum principle). *If $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ and verifies*

$$\begin{gathered} -\Delta_p u \geq 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.7}$$

then $\partial u/\partial \eta < 0$ on $\partial\Omega$, where η is the outward normal to $\partial\Omega$.

Theorem 2.4 (a strong maximum principle). *Assume that $k \in \mathbb{R}$ is a nonnegative number $1 < p \leq 2$ and Ω is a bounded domain $\mathbb{R}^N$. Suppose that $u \in C^1(\overline{\Omega})$ satisfies*

$$-\Delta_p u + ku \geq 0 \quad \text{in } \Omega \ (\text{in the weak sense}), \tag{2.8}$$

$u \geq 0$ and $u \not\equiv 0$ in Ω. Then, $u > 0$ in Ω. The conclusion is still true for all $p > 1$ when $k = 0$.

We now consider the eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.9}$$

where $\lambda \in \mathbb{R}$ is a parameter. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of (2.9) if there exists a function $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, satisfying (2.9) in the weak sense. Such a function is called an eigenfunction of (2.9) associated to the eigenvalue λ.

There exists the first positive eigenvalue λ_1 of problem (2.9) which is characterized as the minimum of the Rayleigh quotient

$$\lambda_1 = \min_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} > 0. \tag{2.10}$$

Moreover, λ_1 is simple (i.e., all the associated first eigenfunctions u are merely constant multiples of each other) and isolated (i.e., there are no eigenvalues less than λ_1 and no eigenvalues in some right reduced neighborhood of λ_1). There is a positive (in Ω) eigenfunction φ_1 corresponding to λ_1.

For more informations on the p-Laplacian the reader may consult [8–11].

3. The M_p-linear case

This section is devoted to the study of the so-called M_p-Linear problem

$$-[M(\|u\|_{1,p}^{p})]^{p-1}\Delta_p u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

where $p > 1$, $f \in W^{-1,p'}(\Omega)$, p' is the conjugate exponent of p, that is, $p' = p/(p-1)$ and $W^{-1,p'}(\Omega)$ is the topological dual of $W_0^{1,p}(\Omega)$.

The next result is an adaptation of some ideas contained in [12, 13] (which were used for another nonlocal problem) for problem (3.1).

Theorem 3.1. *For each $0 \neq f \in W^{-1,p'}(\Omega)$, problem* (3.1) *possesses as many solutions as the folloing equation:*

$$M(t)t^{1/p} = \|\omega\|_{1,p}, \quad t > 0, \tag{3.2}$$

where $\omega \in W_0^{1,p}(\Omega)$ is the only solution of problem (2.2).

Proof. First, let us suppose that $u \in W_0^{1,p}(\Omega)$ is a solution of (3.1), where $0 \neq f \in W^{-1,p'}(\Omega)$ is fixed. Hence

$$\operatorname{div}(|\nabla(M(\|u\|_{1,p}^{p})u)|^{p-2}\nabla(M(\|u\|_{1,p}^{p})u)) = f \tag{3.3}$$

and so $\omega = M(\|u\|_{1,p}^{p})u$ is the solution of

$$-\Delta_p\omega = f \quad \text{in } \Omega,$$
$$\omega = 0 \quad \text{on } \partial\Omega, \tag{3.4}$$

and observe that $\|\omega\|_{1,p} = M(\|u\|_{1,p}^{p})\|u\|_{1,p}$ from which we conclude that $t = \|u\|_{1,p}^{p}$ is a solution of (3.2).

Conversely, let ω be the solution of (3.4) and suppose that $t > 0$ is a solution of (3.2). Define

$$u = t^{1/p}\frac{\omega}{\|\omega\|_{1,p}}, \tag{3.5}$$

and so $\|u\|_{1,p} = t^{1/p}$. A straightforward calculation shows that

$$-[M(\|u\|_{1,p}^{p})]^{p-1}\Delta_p u = -\left[\frac{M(t)t^{1/p}}{\|\omega\|_{1,p}}\right]^{p-1}\Delta_p\omega \quad \text{in } \Omega, \tag{3.6}$$

and, because $t > 0$ is a solution of $M(t)t^{1/p} = \|\omega\|_{1,p}$, one has $(M(t)t^{1/p}/\|\omega\|_{1,p}) = 1$. Hence,

$$-\left[M\left(\|u\|_{1,p}^{p}\right)\right]^{p-1}\Delta_p u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{3.7}$$

which concludes the proof of the theorem. □

Remark 3.2. Let us suppose that $M : \mathbb{R}^{+} \to \mathbb{R}^{+}$ is a continuous function. So,

$$\lim_{t\to 0^{+}} M(t)t^{1/p} = 0. \tag{3.8}$$

Assume, in addition, that there are $t_0, m_0 > 0$ such that $M(t) \geq m_0 > 0$ for $t \geq t_0$. In this case,

$$\lim_{t\to +\infty} M(t)t^{1/p} = +\infty. \tag{3.9}$$

Because of the intermediate value theorem, for $\|\omega\|_{1,p}$, there is $t > 0$ such that

$$M(t)t^{1/p} = \|\omega\|_{1,p}. \tag{3.10}$$

In particular, if $M(t)t^{1/p}$ is an increasing function for $t > 0$, problem (3.1) possesses only a solution for each $0 \neq f \in W^{-1,p'}(\Omega)$. As an example, we take $M(t) = e^{-t}$ for $t \geq 0$. Thus, problem (3.1) possesses as many solutions as the following equation:

$$e^{-t}t^{1/p} = \|\omega\|_{1,p}. \tag{3.11}$$

A simple exercise shows that the function $g(t) = e^{-t}t^{1/p}$ attains its maximum in $1/p$ and $g(1/p) = 1/e^{1/p}\sqrt[p]{p}$. Consequently, if $\|\omega\|_{1,p} > 1/e^{1/p}\sqrt[p]{p}$, problem (3.1) does not possess any solution; if $\|\omega\|_{1,p} = 1/e^{1/p}\sqrt[p]{p}$, problem (3.1) possesses exactly one solution; and if $0 < \|\omega\|_{1,p} < 1/e^{1/p}\sqrt[p]{p}$, problem (3.1) possesses exactly two solutions.

As we see, the presence of the term $[M(\|u\|_{1,p}^{p})]^{p-1}$ produces great difference between problems (2.2) and (3.1).

Remark 3.3. Let us consider the case $f = 0$, that is,

$$-\left[M\left(\|u\|_{1,p}^{p}\right)\right]^{p-1}\Delta_p u = 0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{3.12}$$

If $M(t) > 0$ for all $t \geq 0$, problem (3.12) possesses only the null solution. If $M(t_0) = 0$, for some $t_0 > 0$, problem (3.12) possesses infinitely many solutions. Indeed, if

$u \in W_0^{1,p}(\Omega)$, $u \neq 0$, we have that

$$v = t_0^{1/p} \frac{u}{\|u\|_{1,p}} \tag{3.13}$$

is a solution of (3.12).

4. On a sublinear problem

In this section, we are going to study the sublinear problem

$$\begin{aligned} -[M(\|u\|_{1,p}^{p})]^{p-1} \Delta_p u = u^{\alpha} \quad &\text{in } \Omega, \\ u > 0 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where α satisfies $0 < \alpha < p - 1$ with $p > 1$.

Such a kind of problem belongs to a class of problems known as sublinear whose prototype is

$$\begin{aligned} \Delta u = u^{\alpha} \quad &\text{in } \Omega, \\ u > 0 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

with $0 < \alpha < 1$ (note that, in this case, $p = 2$), which has been vastly studied. See [14–17]. For the nonlocal problem, with $p = 2$, we cite [2, 3, 6, 7], and the references therein.

In particular, Díaz and Saá [16] study the problem

$$\begin{aligned} \Delta_p u = u^{\alpha} \quad &\text{in } \Omega, \\ u > 0 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{4.3}$$

$1 < p < \infty$, $0 < \alpha < p - 1$, and show that it possesses only a positive solution $u \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$. Indeed, such a solution also belongs to $W_{\mathrm{loc}}^{2,2}(\Omega)$. Our next result describes what happens with the nonlocal sublinear problem (4.1).

Theorem 4.1. *Suppose that $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function satisfying $M(t) > 0$ for all $t \geq 0$. Then problem* (4.1) *has at least as many solutions as the equation*

$$[M(t)]^{p-1} \cdot t^{(p-1-\alpha)/p} = \|v\|_{1,p}^{p-1-\alpha}, \quad t > 0, \tag{4.4}$$

where v is the solution of (4.1).

Proof. We adapt for problem (4.1) the ideas developed in [3, 6]. Let us suppose that $t > 0$ is a solution of the (4.4) and set $\gamma = t^{1/p}/\|v\|_{1,p}$ which implies $\|\gamma v\|_{1,p} = t^{1/p}$. So, a simple calculation shows that

$$[M(\|\gamma v\|_{1,p}^{p})]^{p-1} = \gamma^{\alpha-(p-1)}. \tag{4.5}$$

Define $u = \gamma v$ and let us show that such a function u is a solution of problem (4.1). Indeed, this follows from the calculation below:

$$\begin{aligned}
-[M(\|u\|_{1,p}^{p})]^{p-1}\Delta_p u &= -[M(\|\gamma v\|_{1,p}^{p})]^{p-1}\Delta_p(\gamma v)\\
&= -[M(\|\gamma v\|_{1,p}^{p})]^{p-1}\operatorname{div}(|\nabla(\gamma v)|^{p-2}\nabla(\gamma v))\\
&= -[M(\|\gamma v\|_{1,p}^{p})]^{p-1}\operatorname{div}(\gamma^{p-2}\gamma|\nabla v|^{p-2}\nabla v)\\
&= [M(\|\gamma v\|_{1,p}^{p})]^{p-1}\gamma^{p-1}(-\Delta_p v)\\
&= \gamma^{\alpha-(p-1)}\gamma^{p-1}(-\Delta_p v)\\
&= \gamma^{\alpha}v^{\alpha} = (\gamma v)^{\alpha} = u^{\alpha} \quad \text{in } \Omega.
\end{aligned} \tag{4.6}$$

Hence, u is a solution of problem (4.1). □

Remark 4.2. Remark 3.2, mutatis mutandis, remains valid for problem (4.1).

5. A singular problem via the Galerkin method

In this section, we will study problem

$$\begin{gathered}
-[M(\|u\|_{1,p}^{p})]^{p-1}\Delta_p u = \frac{h(x)}{u^{\gamma-p+2}} + u^{\alpha-p+2} \quad \text{in } \Omega,\\
u > 0 \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{5.1}$$

which is a singular perturbation of problem (4.1), where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2 \le p < N$ and h is a suitable function defined in Ω.

We will attack problem (5.1) by using the Galerkin method which rests heavily on the following result, which is a variant of the well-known Brouwer fixed point theorem, whose proof may be found in Lions [18].

Proposition 5.1. *Suppose $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function such that $\langle F(\xi), \xi\rangle \ge 0$ with $|\xi| = r$, for some $r > 0$, where $\langle\cdot,\cdot\rangle$ denotes the usual norm in $\mathbb{R}^m$. Then, there is $\xi_0 \in \overline{B}_r(0)$ such that $F(\xi_0) = 0$.*

We will suppose that $M : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

(M_1) there are $m_0 > 0$ and $\theta_1 > 0$ such that $M(t) \ge m_0$ if $t \ge \theta_1$;

(M_2) $\theta_2 = \sup\{t > 0; M(t) = 0\} > 0$.

If $M(t) \ge m_0 > 0$ for all $t \ge 0$, condition (M_2) is vacuous.

Theorem 5.2. *Let $h : \overline{\Omega} \to \mathbb{R}$ be a continuous function with $h > 0$ on $\overline{\Omega}, \gamma, \alpha \in (p-2, p-1)$, $2 \leq p < N$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying (M_1)-(M_2). Then problem* (5.1) *possesses a positive solution.*

We will split the proof of this theorem in several lemmas. First, for each fixed $\epsilon > 0$, we will consider the following problem:

$$\begin{gathered} -\left[M\left(\|u\|_{1,p}^{p}\right)\right]^{p-1}\Delta_p u = \frac{h(x)}{(\epsilon+u)^{\gamma-p+2}} + u^{\alpha-p+2}, \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{5.2}$$

In what follows, throughout Section 5, we are always supposing that M, h, α, and γ enjoy assumptions of Theorem 5.2

Lemma 5.3. *For each fixed $\epsilon > 0$, problem* (5.2) *possesses a solution u_ϵ.*

Proof. Let us consider the problem

$$\begin{gathered} -\left[M\left(\|u\|_{1,p}^{p}\right)\right]^{p-1}\Delta_p u = \frac{h(x)}{(\epsilon+|u|)^{\gamma-p+2}} + |u|^{\alpha-p+2}, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{5.3}$$

where $M^+ : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$M^+(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \theta_2, \\ M(t), & \text{if } t > \theta_2. \end{cases} \tag{5.4}$$

We now consider $\mathbb{B} = \{e_1, e_2, e_3, \ldots\}$ a Schauder's basis for $W_0^{1,p}(\Omega)$ (we recall that a Schauder's basis for a Banach space X is a sequence (e_n) such that to each $x \in X$ there exists a unique sequence of scalars (α_n) for which the partial sums of $\sum \alpha_n e_n$ converge to x in the norm of X). For more informations on Schauder's basis see [19] and the references therein.

For each $m \in \mathbb{N}$, let $\mathbb{B}_m = \operatorname{span}\{e_1, e_2, \ldots\}$ be the finite dimensional vector space spanned by the functions $e_1, \ldots, e_m$. So, each $u \in \mathbb{B}_m$ is written as $u = \sum_{j=1}^{m} \xi_j e_j$. We will use on $\mathbb{B}_m$ the norm

$$\|u\|_m = \sum_{j=1}^{m} |\xi_j|. \tag{5.5}$$

We note that $(\mathbb{B}_m, \|\cdot\|_m)$ and $(\mathbb{R}^m, |\cdot|_s)$ are isomorphic through the following map:

$$\begin{gathered} T : (\mathbb{B}, \|\cdot\|_m) \longrightarrow (\mathbb{R}, |\cdot|_s), \\ u = \sum_{j=1}^{m} \xi_j e_j \longmapsto T(u) = \xi = (\xi_1, \xi_2, \ldots, \xi_m), \end{gathered} \tag{5.6}$$

with $\|u\|_m = |\xi|_s = |T(u)|_s$ where $|\xi|_s = \sum_{j=1}^{m} |\xi_j|$.

Since, for each $m \in \mathbb{N}$, $\mathbb{B}_m$ is a finite dimensional vector space the norms $\|\cdot\|_m$ and $\|\cdot\|_{1,p}$, induced from $W_0^{1,p}(\Omega)$ on $\mathbb{B}_m$, are equivalent to each other and so, there exist positive constants $c(m)$ and $k(m)$ such that

$$c(m)\|u\|_m \le \|u\|_{1,p} \le k(m)\|u\|_m. \tag{5.7}$$

We now consider, for each $m \in \mathbb{N}$, the following application

$$\begin{gathered} F : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \\ \xi \longmapsto F(\xi) = (F_1(\xi), F_2(\xi), \dots, F_m(\xi)), \end{gathered} \tag{5.8}$$

where

$$F_j(\xi) = [M^+(\|u\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla e_j - \int_\Omega \frac{h(x)e_j}{(\epsilon + |u|)^{\gamma-p+2}} - \int_\Omega |u|^{\alpha-p+2} e_j, \tag{5.9}$$

$j = 1, \dots, m$.

A simple calculation leads us to

$$\begin{aligned} \langle F(\xi), \xi \rangle &= [M^+(\|u\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla u - \int_\Omega \frac{h(x)u}{(\epsilon + |u|)^{\gamma-p+2}} - \int_\Omega |u|^{\alpha-p+2} u, \\ &= [M^+(\|u\|_{1,p}^p)]^{p-1} \|u\|_{1,p}^p - \int_\Omega \frac{h(x)u}{(\epsilon + |u|)^{\gamma-p+2}} - \int_\Omega |u|^{\alpha-p+2} u. \end{aligned} \tag{5.10}$$

We now note that

$$\int_\Omega \frac{h(x)u}{(\epsilon + |u|)^{\gamma-p+2}} \le \|h\|_\infty \int_\Omega \frac{|u|}{\epsilon^{\gamma-p+2}} \le C_\epsilon \|u\|_{1,p}, \tag{5.11}$$

and because $p - 2 < \alpha < p - 1$, we have $1 < \alpha - p + 3 < 2$, and so

$$\int_\Omega |u|^{\alpha-p+2} u \le \int_\Omega |u|^{\alpha-p+3} \le C\|u\|_{1,p}^{\alpha-p+3}. \tag{5.12}$$

Hence,

$$\langle F(\xi), \xi \rangle \ge [M(\|u\|_{1,p}^p)]^{p-1} \|u\|_{1,p}^p - C_\epsilon \|u\|_{1,p} - C\|u\|_{1,p}^{\alpha-p+3}, \tag{5.13}$$

and from (M_1), for $\|u\|_{1,p}^p \ge \theta_1$, we have $M^+(\|u\|_{1,p}^p) = M(\|u\|_{1,p}^p) \ge m_0 > 0$. Thus,

$$\begin{aligned} \langle F(\xi), \xi \rangle &\ge m_0^{p-1} \|u\|_{1,p}^p - C_\epsilon \|u\|_{1,p} - C\|u\|_{1,p}^{\alpha-p+3} \\ &\ge m_0^{p-1} [c(m)]^p \|u\|_m^p - C_\epsilon k(m) \|u\|_m - [k(m)]^{\alpha-p+3} C\|u\|_m^{\alpha-p+3} \\ &\ge m_0^{p-1} C(m) |\xi|_s^p - N_\epsilon(m) |\xi|_s - Q(m) |\xi|_s^{\alpha-p+3}. \end{aligned} \tag{5.14}$$

We now fix $\rho_m > 0$ and for $\xi \in \mathbb{R}^m$, $|\xi|_s = \rho_m$ and noting that $1 < \alpha - p + 3 < 2 \leq p$, we obtain

$$\langle F(\xi), \xi \rangle \geq m_0^{p-1} C(m) \rho_m^p - N_\epsilon(m) \rho_m - Q(m) \rho_m^{\alpha-p+3}. \tag{5.15}$$

So, if ρ_m is large enough, we have

$$\langle F(\xi), \xi \rangle > 0 \quad \text{if } |\xi|_s = \rho_m, \tag{5.16}$$

which implies

$$\langle F(\xi), \xi \rangle > 0 \quad \text{if } |\xi| = \rho_m^*, \tag{5.17}$$

where $\rho_m^* > 0$, because $|\cdot|$ and $|\cdot|_s$ are equivalent in $\mathbb{R}^m$.

From Proposition 5.1 there is $\xi^m \in \mathbb{R}^m$, with $|\xi| \leq \rho_m^*$, such that $F(\xi^m) = 0$ and note that $|\xi^m|_s \leq \rho_m$ for some ρ_m is large enough.

Through the isometric identification of $(\mathbb{R}^m, |\cdot|_s)$ with $(\mathbb{B}_m, \|\cdot\|_m)$, we find $(u_m) \subset \mathbb{B}_m$, $\|u_m\|_m \leq \rho_m$, that is, $\|u_m\|_{1,p} \leq \overline{\rho}_m$, such that

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u_m|^{p-2} \nabla u_m \nabla e_j - \int_\Omega \frac{h(x) e_j}{(\epsilon + |u_m|)^{\gamma-p+2}} - \int_\Omega |u_m|^{\alpha-p+2} e_j = 0, \tag{5.18}$$

for each $j = 1, \ldots, m$, which gives us

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u_m|^{p-2} \nabla u_m \nabla \omega - \int_\Omega \frac{h(x) \omega}{(\epsilon + |u_m|)^{\gamma-p+2}} - \int_\Omega |u_m|^{\alpha-p+2} \omega = 0, \tag{5.19}$$

for all $\omega \in \mathbb{B}_m$. Since $u_m \in \mathbb{B}_m$,

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1} \|u_m\|_{1,p}^p - \int_\Omega \frac{h(x) u_m}{(\epsilon + |u_m|)^{\gamma-p+2}} - \int_\Omega |u_m|^{\alpha-p+2} u_m = 0. \tag{5.20}$$

Because

$$\begin{gathered} \int_\Omega \frac{h(x) u_m}{(\epsilon + |u_m|)^{\gamma-p+2}} \leq \frac{\|u\|_\infty}{\epsilon^{\gamma-p+2}} \int_\Omega |u_m| \leq C_\epsilon \|u_m\|_{1,p}, \\ \int_\Omega |u_m|^{\alpha-p+2} u_m \leq C \|u_m\|_{1,p}^{\alpha-p+3}, \end{gathered} \tag{5.21}$$

we obtain

$$[M(\|u_m\|_{1,p}^p)]^{p-1} \|u_m\|_{1,p}^p \leq C_\epsilon \|u_m\|_{1,p} + C \|u_m\|_{1,p}^{\alpha-p+3}. \tag{5.22}$$

Let us show that the sequence $(\|u_m\|_{1,p})$ is bounded. Indeed, suppose, on the contrary, that $(\|u_m\|_{1,p})$ is not bounded. So, up to a subsequence, we may suppose that $\|u_m\|_{1,p} \to +\infty$. From condition (M_1) and inequality (5.22) we obtain

$$0 < m_0^{p-1}\|u_m\|_{1,p}^{p} \leq C_\epsilon\|u_m\|_{1,p} + C\|u_m\|_{1,p}^{\alpha-p+3}, \tag{5.23}$$

which give us

$$0 < m_0^{p-1} \leq \frac{C_\epsilon}{\|u_m\|_{1,p}^{p-1}} + \frac{C}{\|u_m\|_{1,p}^{2p-\alpha-3}}. \tag{5.24}$$

Since $\|u_m\|_{1,p} \to +\infty$ we arrive in

$$0 < m_0^{p-1} \leq 0, \tag{5.25}$$

which is impossible. Thus, $(\|u_m\|_{1,p})$ is bounded. Consequently, up to a subsequence,

$$\begin{gathered} \|u_m\|_{1,p}^{p} \longrightarrow t_0, \\ u_m \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \\ u_m \longrightarrow u \quad \text{in } L^q(\Omega),\ 1 \leq q < p^*, \\ u_m(x) \longrightarrow u(x) \quad \text{a.e. } \Omega. \end{gathered} \tag{5.26}$$

And in view of continuity of M

$$[M^+(\|u_m\|_{1,p}^{p})]^{p-1} \longrightarrow [M^+(t_0)]^{p-1}. \tag{5.27}$$

We now fix $l \leq m$ and so $\mathbb{B}_l \subset \mathbb{B}_m$. For $\varphi \in \mathbb{B}_l$ we have

$$[M^+(\|u_m\|_{1,p}^{p})]^{p-1}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla\varphi = \int_\Omega \frac{h(x)\varphi}{(\epsilon + |u_m|)^{\gamma-p+2}} + \int_\Omega |u_m|^{\alpha-p+2}\varphi, \tag{5.28}$$

for all $\varphi \in \mathbb{B}_l$.

We now remark that

$$\begin{gathered} \left|\frac{h(x)\varphi}{(\epsilon + |u_m|)^{\gamma-p+2}}\right| \leq \frac{C}{\epsilon^{\gamma-p+2}}|\varphi| \in L^1(\Omega), \\ \frac{h(x)\varphi}{(\epsilon + |u_m(x)|)^{\gamma-p+2}} \longrightarrow \frac{h(x)\varphi}{(\epsilon + |u(x)|)^{\gamma-p+2}} \quad \text{a.e. } \Omega. \end{gathered} \tag{5.29}$$

By the Lebesgue dominated convergence theorem

$$\int_\Omega \frac{h(x)\varphi}{(\epsilon+|u|)^{\gamma-p+2}} \longrightarrow \int_\Omega \frac{h(x)\varphi}{(\epsilon+|u|)^{\gamma-p+2}}. \tag{5.30}$$

Furthermore, because $u_m \to u \in L^p(\Omega)$, we have

$$|u_m|^{\alpha-p+2} \longrightarrow |u|^{\alpha-p+2} \quad \text{in } L^{p/(\alpha-p+2)}(\Omega), \tag{5.31}$$

and because Ω is a bounded domain and $p < p/(\alpha-p+2)$ we have $L^{p/(\alpha-p+2)}(\Omega) \subset L^p(\Omega)$ and so

$$|u_m|^{\alpha-p+2} \longrightarrow |u|^{\alpha-p+2} \quad \text{in } L^p(\Omega). \tag{5.32}$$

Consequently,

$$|u_m(x)|^{\alpha-p+2} \longrightarrow |u(x)|^{\alpha-p+2} \quad \text{a.e. in } \Omega, \tag{5.33}$$

and there is $g \in L^p(\Omega)$ such that

$$|u_m(x)|^{\alpha-p+2} \le g(x) \quad \text{a.e. in } \Omega \quad \forall m \in \mathbb{N}. \tag{5.34}$$

Thus,

$$||u_m|^{\alpha-p+2}\varphi| \le g|\varphi| \in L^1(\Omega), \tag{5.35}$$

and using again the Lebesgue dominated convergence theorem

$$\int_\Omega |u_m|^{\alpha-p+2}\varphi \longrightarrow \int_\Omega |u|^{\alpha-p+2}\varphi \quad \forall\varphi \in \mathbb{B}_l. \tag{5.36}$$

We will now consider the following claim, whose proof will be postponed to Section 6.

Claim 1. The following convergence holds

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla\varphi \longrightarrow [M^+(t_0)]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi, \tag{5.37}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

In the inequality (5.28) we make $m \to \infty$ and we use (5.27), (5.30), and (5.36) to obtain

$$[M^+(t_0)]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi = \int_\Omega \frac{h(x)\varphi}{(\epsilon+|u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2}, \tag{5.38}$$

for all $\varphi \in \mathbb{B}_l$.

Since $l \in \mathbb{N}$ is arbitrary, the above inequality holds true for all $\varphi \in W_0^{1,p}(\Omega)$. Because of this $M^+(t_0) > 0$ and so $M^+(t_0) = M(t_0)$, which implies

$$[M(t_0)]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi = \int_\Omega \frac{h(x)\varphi}{(\epsilon+|u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2}, \tag{5.39}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Taking u as a test function in the inequality (5.39), we get

$$[M(t_0)]^{p-1}\|u\|_{1,p}^p = \int_\Omega \frac{h(x)u}{(\epsilon+|u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2}u. \tag{5.40}$$

We now take $\varphi = u_m$ in (5.28) to obtain

$$[M(\|u_m\|_{1,p}^p)]^{p-1}\|u_m\|_{1,p}^p = \int_\Omega \frac{h(x)u_m}{(\epsilon+|u_m|)^{\gamma-p+2}} + \int_\Omega |u_m|^{\alpha-p+2}u_m. \tag{5.41}$$

Since

$$\left|\frac{h(x)u_m}{(\epsilon+|u_m|)^{\gamma-p+2}}\right| \le \frac{C}{\epsilon^{\gamma-p+2}}|u_m| \tag{5.42}$$

and because $u_m \to u \in L^q(\Omega)$, $1 \le q < p^*$, it follows that $|u_m(x)| \to |u(x)|$ a.e. in Ω and there is $g \in L^1(\Omega)$ such that $|u_m| \le g$ a.e. in Ω for all $m \in \mathbb{N}$. Consequently,

$$\left|\frac{h(x)u_m}{(\epsilon+|u_m|)^{\gamma-p+2}}\right| \le \frac{C}{\epsilon^{\gamma-p+2}}|u_m| \le \frac{C}{\epsilon^{\gamma-p+2}}g \in L^1(\Omega). \tag{5.43}$$

Reasoning as before, we obtain

$$\begin{gathered}\int_\Omega \frac{h(x)u_m}{(\epsilon+|u_m|)^{\gamma-p+2}} \longrightarrow \int_\Omega \frac{h(x)u}{(\epsilon+|u|)^{\gamma-p+2}},\\ \int_\Omega |u_m|^{\alpha-p+2}u_m \longrightarrow \int_\Omega |u|^{\alpha-p+2}u.\end{gathered} \tag{5.44}$$

Taking limits on both sides of the equality (5.41) and using $\|u_m\|_{1,p}^p \to t_0$, (5.27), (5.44), we obtain

$$[M^+(t_0)]^{p-1} t_0 = \int_\Omega \frac{h(x)u}{(\epsilon + |u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2} u. \tag{5.45}$$

Comparing this last equality with the one in (5.40) we obtain $t_0 = \|u\|_{1,p}^p$ because $M^+(t_0) > 0$. Then, from (5.39)

$$[M(\|u\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi = \int_\Omega \frac{h(x)\varphi}{(\epsilon + |u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2} \varphi \tag{5.46}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

This shows that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the auxiliary problem (5.2) which is positive by virtue of the maximum principle. This proves the lemma. □

In what follows, for each $n \in \mathbb{N}$, we set $\epsilon = 1/n$ and $u_{1/n} = u_n$ where $u_{1/n}$ is the solution obtained in the last lemma.

Lemma 5.4. *There exists $\delta > 0$ such that $[M(\|u_n\|_{1,p}^p)]^{p-1} \geq \delta > 0$, for all $n \in \mathbb{N}$.*

Proof. We will reason by contradiction. Suppose that

$$\liminf\, [M(\|u_n\|_{1,p}^p)]^{p-1} = 0. \tag{5.47}$$

If it is the case, the sequence $(\|u_n\|_{1,p}^p)$ is bounded because, on the contrary, we would have $\|u_n\|_{1,p}^p > \theta_1$, perhaps for a subsequence, and so

$$[M(\|u_n\|_{1,p}^p)]^{p-1} \geq m_0^{p-1} > 0, \tag{5.48}$$

and this would imply

$$\liminf\, [M(\|u_n\|_{1,p}^p)]^{p-1} \geq m_0^{p-1} > 0, \tag{5.49}$$

which is impossible in view of (5.47). Therefore, up to subsequences,

$$\begin{aligned} \|u_n\|_{1,p}^p &\longrightarrow \theta_0, \\ u_n &\rightharpoonup \quad \text{in } W_0^{1,p}(\Omega), \\ u_n(x) &\longrightarrow u(x) \quad \text{a.e. in } \Omega. \end{aligned} \tag{5.50}$$

Since M is continuous, we have

$$0 = \liminf\, [M(\|u_n\|_{1,p}^p)]^{p-1} = \lim\, [M(\|u_n\|_{1,p}^p)]^{p-1} = [M(\theta_0)]^{p-1}. \tag{5.51}$$

We now note that

$$\begin{aligned}\frac{h(x)}{(1/n+t)^{\gamma-p+2}}+t^{\alpha-p+2} &\geq \frac{h(x)}{(1+t)^{\gamma-p+2}}+t^{\alpha-p+2}\\ &\geq \frac{C}{(1+t)^{\gamma-p+2}}+t^{\alpha-p+2}\\ &\geq \widetilde{C}\left[\frac{1}{(1+t)^{\gamma-p+2}}+t^{\alpha-p+2}\right]\\ &\geq C_0>0,\end{aligned} \tag{5.52}$$

for all $x \in \overline{\Omega}$ and $t \geq 0$, because the function $t \mapsto 1/(1+t)^{\gamma-p+2}+t^{\alpha-p+2},\ t \geq 0$, attains a positive minimum.

Since

$$\begin{gathered}-[M(\|u_n\|_{1,p}^{p})]^{p-1} = \frac{h(x)}{(1/n+u_n)^{\gamma-p+2}}+u_n^{\alpha-p+2} \geq C_0>0, \text{ in } \Omega,\\ u_n=0, \text{ on } \partial\Omega.\end{gathered} \tag{5.53}$$

Taking $\varphi \in C_0^1(\overline{\Omega}),\ \varphi>0$ in Ω, as a test function in the last equality, we obtain

$$[M(\|u_n\|_{1,p}^{p})]^{p-1}\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi \geq C_0\int_{\Omega}\varphi>0, \tag{5.54}$$

and so $M(\|u_n\|_{1,p}^{p})]^{p-1}>0$. Taking limit on the above expression we get

$$0=[M(\theta_0)]^{p-1}\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla\varphi \geq C_0\int_{\Omega}\varphi>0, \tag{5.55}$$

which is impossible. This completes the proof of the lemma. □

Lemma 5.5. *The sequence* $(\|u_n\|_{1,p}^{p})$ *is bounded, where* u_n *is as above.*

Proof. First of all we note that

$$\begin{gathered}[M(\|u_n\|_{1,p}^{p})]^{p-1}\|u_n\|_{1,p}^{p} = \int_{\Omega}\frac{h(x)u_n}{(1/n+u_n)^{\gamma-p+2}}+\int_{\Omega}u_n^{\alpha-p+3},\\ \int_{\Omega}\frac{h(x)u_n}{(1/n+u_n)^{\gamma-p+2}} \leq C_1\|u_n\|_{1,p}^{p-\gamma-1},\\ \int_{\Omega}u_n^{\alpha-p+3} \leq C_2\|u_n\|_{1,p}^{\alpha-p+3},\end{gathered} \tag{5.56}$$

where C_1 and C_2 are positive constants independent of n. Consequently,

$$\delta\|u_n\|_{1,p}^{p} \leq [M(\|u_n\|_{1,p}^{p})]^{p-1}\|u_n\|_{1,p}^{p} \leq C_1\|u_n\|_{1,p}^{p-\gamma-1} + C_2\|u_n\|_{1,p}^{\alpha-p+3}. \tag{5.57}$$

Since $p-\gamma-1<1,\ \alpha-p+3<2\leq p$, we conclude that $(\|u_n\|_{1,p}^{p})$ is bounded. □

As a consequence of the preceding lemma we have

$$0<\delta\leq [M(\|u_n\|_{1,p}^{p})]^{p-1}\leq M_\infty, \quad \forall n\in\mathbb{N}. \tag{5.58}$$

Lemma 5.6. *The sequence* (u_n)*, obtained in the last lemma, converges to a solution of problem* (5.1).

Proof. As (u_n) is a bounded sequence in $W_0^{1,p}(\Omega)$ we have, up to subsequence, that

$$\begin{gathered}\|u_n\|_{1,p}^{p}\longrightarrow t_0>0,\\ u_n\rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega),\\ u_n\longrightarrow \quad \text{in } L^{q}(\Omega),\quad 1\leq q<p^*,\\ u_n(x)\longrightarrow u(x)\quad \text{a.e. } \Omega.\end{gathered} \tag{5.59}$$

From the continuity of M,

$$M(\|u_n\|_{1,p}^{p})\longrightarrow M(t_0). \tag{5.60}$$

Let $\varphi_1>0$ be an eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ associated to first eigenvalue λ_1 and satisfying

$$C_0>\lambda_1 M_\infty^{p-1}\varphi_1^{p-1},\quad \forall x\in\overline{\Omega}, \tag{5.61}$$

where C_0 is the constant obtained in (5.54). Consequently,

$$\begin{aligned}-[M(\|u_n\|_{1,p}^{p})]^{p-1}\Delta_p u_n &= \frac{h(x)}{(1/n+u_n)^{\gamma-p+2}}+u_n^{\alpha-p+2}\quad \text{in } \Omega,\\ &\geq C_0>\lambda_1 M_\infty^{p-1}\varphi_1^{p-1}\quad \text{in } \Omega,\\ u_n=\varphi_1&=0\quad \text{on } \partial\Omega.\end{aligned} \tag{5.62}$$

Since

$$-[M(\|u_n\|_{1,p}^{p})]^{p-1}\Delta_p u_n\geq C_0>\lambda_1 M_\infty^{p-1}\varphi_1^{p-1}\quad \text{in } \Omega, \tag{5.63}$$

it follows that

$$-\Delta_p([M(\|u_n\|_{1,p}^p)]u_n) > -\Delta_p(M_\infty \varphi_1) \quad \text{in } \Omega, \tag{5.64}$$

and as $[M(\|u_n\|_{1,p}^p)]u_n = M_\infty \varphi_1 = 0$ on $\partial\Omega$. By the comparison principle,

$$[M(\|u_n\|_{1,p}^p)]u_n > M_\infty \varphi_1 \quad \text{in } \Omega, \tag{5.65}$$

and from (5.58)

$$u_n(x) > \frac{M_\infty \varphi_1(x)}{M_\infty^{1/(p-1)}} \quad \text{in } \Omega. \tag{5.66}$$

From which we conclude that $u_n(x) \nrightarrow 0$ for each $x \in \Omega$.

Next, we will use the following.

Hardy-Sobolev Inequality. If $u \in W_0^{1,p}(\Omega)$ and $1 < p \leq N$, then $u/\varphi_1^r \in L^r(\Omega)$, where $1/r = 1/p - (1-\tau)/N,\ 0 \leq \tau \leq 1$, and

$$\left\| \frac{u}{\varphi_1^r} \right\|_{L^r} \leq \|\nabla u\|_{L^p}, \tag{5.67}$$

where $C > 0$ is a constant and is an eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ associated to the first eigenvalue λ_1.

Whose proof may be found in [20].

As

$$\left| \frac{h(x)\varphi}{(1/n + u_n)^{\gamma-p+2}} \right| \leq h(x) \frac{|\varphi|}{|u_n|^{\gamma-p+2}}, \tag{5.68}$$

and using (5.66) it follows

$$h(x) \frac{|\varphi|}{|u_n|^{\gamma-p+2}} < \frac{\|h\|_\infty}{C} \frac{|\varphi|}{\varphi_1^{\gamma-p+2}}. \tag{5.69}$$

From the Hardy-Sobolev inequality

$$\frac{\|h\|_\infty}{C} \frac{|\varphi|}{\varphi_1^{\gamma-p+2}} \in L^1(\Omega). \tag{5.70}$$

Furthermore,

$$\frac{h(x)\varphi}{(1/n+u_n(x))^{\gamma-p+2}} \longrightarrow \frac{h(x)\varphi}{u(x)^{\gamma-p+2}} \quad \text{in } \Omega, \tag{5.71}$$

and so, by the Lebesgue dominated convergence theorem

$$\int_\Omega \frac{h(x)\varphi}{(1/n+u_n(x))^{\gamma-p+2}} \longrightarrow \int_\Omega \frac{h(x)\varphi}{u(x)^{\gamma-p+2}}, \quad \forall\varphi \in W_0^{1,p}(\Omega). \tag{5.72}$$

Since Ω is bounded and $p < p/(\alpha-p+2)$, we have $L^{p/(\alpha-p+2)} \hookrightarrow L^p(\Omega)$ and so

$$u_n^{\alpha-p+2} \longrightarrow u^{\alpha-p+2} \quad \text{in } L^p(\Omega). \tag{5.73}$$

Since Ω is bounded and $p < p/(\alpha-p+2)$, we have $L^{p/(\alpha-p+2)} \hookrightarrow L^p(\Omega)$ and so

$$u_n(x)^{\alpha-p+2} \longrightarrow u(x)^{\alpha-p+2} \quad \text{a.e. in } \Omega, \tag{5.74}$$

and there exists $g \in L^p(\Omega)$ such that

$$u_n^{\alpha-p+2} \le g \quad \text{a.e. in } \Omega, \quad \forall n \in \mathbb{N}. \tag{5.75}$$

Hence,

$$|u_n^{\alpha-p+2}\varphi| \le g|\varphi| \in L^1(\Omega), \tag{5.76}$$

and using again the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega u_n^{\alpha-p+2}\varphi \longrightarrow \int_\Omega u^{\alpha-p+2}\varphi, \tag{5.77}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

As (u_n) is solution of the auxiliary problem

$$[M(\|u_n\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi = \int_\Omega \frac{h(x)\varphi}{(1/n+u_n)^{\gamma-p+2}} + \int_\Omega u_n^{\alpha-p+2}\varphi, \tag{5.78}$$

for all $\varphi \in W_0^{1,p}(\Omega)$, taking limits on both sides of the above expression, using the same reasoning as in the proof of the claim and the convergences in (5.60), (5.72), and (5.77) we obtain

$$[M(t_0)]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega \frac{h(x)\varphi}{u^{\gamma-p+2}} + \int_\Omega u^{\alpha-p+2}, \tag{5.79}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Using u as a test function in the last expression

$$[M(t_0)]^{p-1}\|u\|_{1,p}^{p} = \int_\Omega h(x)u^{p-\gamma-1} + \int_\Omega u^{\alpha-p+3}. \tag{5.80}$$

Taking $\varphi = u_n$ in the expression (5.78), it follows that

$$\left[M\left(\|u_n\|_{1,p}^{p}\right)\right]^{p-1}\|u_n\|_{1,p}^{p} = \int_\Omega \frac{h(x)u_n}{(1/n+u_n)^{\gamma-p+2}} + \int_\Omega u_n^{\alpha-p+3}. \tag{5.81}$$

Noticing that $u_n \rightarrow u$ in $L^q(\Omega)$, $1 \le q < p^*$, and $1 < 1/(p-\gamma-1)$ we have $L^{1/(p-\gamma-1)}(\Omega) \hookrightarrow L^1(\Omega)$ and so

$$\begin{aligned} &u_n \longrightarrow u \quad \text{in } L^{1/(p-\gamma-1)}(\Omega), \\ &u_n(x) \longrightarrow u(x) \quad \text{a.e. in } \Omega, \end{aligned} \tag{5.82}$$

and there exists $\omega \in L^{1/(p-\gamma-1)}(\Omega)$ such that

$$0 \le u_n(x) \le \omega(x) \quad \text{a.e. in } \Omega, \quad \forall n \in \mathbb{N}. \tag{5.83}$$

Thus,

$$\begin{aligned} &\left|\frac{h(x)u_n}{(1/n+u_n)^{\gamma-p+2}}\right| \le \|h\|_\infty u_n^{p-\gamma-1} \in L^1(\Omega), \\ &\frac{h(x)u_n}{(1/n+u_n)^{\gamma-p+2}} \longrightarrow h(x)u(x)^{p-\gamma-1} \quad \text{a.e. in } \Omega. \end{aligned} \tag{5.84}$$

By the Lebesgue dominated convergence theorem

$$\int_\Omega \frac{h(x)u_n}{(1/n+u_n)^{\gamma-p+2}} \longrightarrow \int_\Omega h(x)u(x)^{p-\gamma-1}, \tag{5.85}$$

and because $1 < \alpha - p + 3 < 2$, it follows that

$$\int_\Omega u_n^{\alpha-p+3} \longrightarrow \int_\Omega u^{\alpha-p+3}. \tag{5.86}$$

We now take limits on both sides of (5.81), by using (5.60), (5.85), and (5.86), to obtain

$$[M(t_0)]^{p-1}t_0 = \int_\Omega \frac{h(x)u}{u^{\gamma-p+2}} + \int_\Omega u^{\alpha-p+3}, \tag{5.87}$$

that is,

$$[M(t_0)]^{p-1}t_0 = \int_\Omega h(x)u^{p-\gamma-1} + \int_\Omega u^{\alpha-p+3}. \tag{5.88}$$

Comparing (5.80), (5.88) and using $[M(t_0)]^{p-1} > 0$, we get $\|u\|_{1,p}^p = t_0$. Therefore, from (5.79)

$$[M(\|u\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega \frac{h(x)\varphi}{u^{\gamma-p+2}} + \int_\Omega u^{\alpha-p+2}\varphi, \tag{5.89}$$

for all $\varphi \in W_0^{1,p}(\Omega)$, which shows that u is a weak solution of (5.1). □

6. Proof of the claim

We recall that we have to prove that

$$[M^+(\|u\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla\varphi \longrightarrow [M^+(t_0)]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \tag{6.1}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

First of all, we claim that $(-\Delta_p u_m) \subset (W_0^{1,p}(\Omega))'$ is a bounded sequence. Indeed,

$$|\langle -\Delta_p u_m, v\rangle| = \left|\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla v\right| \le \int_\Omega |\nabla u_m|^{p-1}|\nabla v|, \tag{6.2}$$

and because $u_m \in \mathbb{B}_m \subset W_0^{1,p}(\Omega)$, we have $|\nabla u_m| \in L^p(\Omega)$ and $|\nabla u_m|^{p-1} \in L^{p/(p-1)}(\Omega)$. Applying the Hölder inequality, by using the exponents p and $p/(p-1)$, it follows that

$$|\langle -\Delta_p u_m, v\rangle| \le \left(\int_\Omega |\nabla u_m|^p\right)^{(p-1)/p}\left(\int_\Omega |\nabla v|^p\right)^{1/p} = \|u_m\|_{1,p}^p \|v\|_{1,p}, \tag{6.3}$$

which implies

$$\| -\Delta_p u_m\|_{(W_0^{1,p}(\Omega))'} \le \|u\|_{1,p}^p \le C_\epsilon, \tag{6.4}$$

where $C_\epsilon > 0$ is a constant, because the boundedness of $(\|u_m\|_{1,p})$ was proved in Lemma 5.5.

Since $W_0^{1,p}(\Omega)$ is a separable Banach space and $(-\Delta u_m)$ is a bounded sequence in $(W_0^{1,p}(\Omega))'$ then, up to a subsequence,

$$-\Delta_p u_m \overset{*}{\rightharpoonup} \chi \quad \text{in } (W_0^{1,p}(\Omega))', \tag{6.5}$$

that is,

$$\langle -\Delta_p u_m, \psi \rangle \longrightarrow \langle \chi, \psi \rangle \quad \forall\, \psi \in W_0^{1,p}(\Omega), \tag{6.6}$$

which is equivalent to

$$\int_\Omega |\nabla u_m|^{p-2} \nabla u_m \nabla \psi \longrightarrow \langle \chi, \psi \rangle \quad \forall\, \psi \in W_0^{1,p}(\Omega). \tag{6.7}$$

Since

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1} \longrightarrow [M^+(t_0)]^{p-1}, \tag{6.8}$$

it follows from (6.7) that

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u_m|^{p-2} \nabla u_m \nabla \psi \longrightarrow [M^+(t_0)]^{p-1} \langle \chi, \psi \rangle, \tag{6.9}$$

for all $\psi \in W_0^{1,p}(\Omega)$. Taking limits on both sides of (5.28) and using the convergences (5.30) and (5.36), we obtain

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1} \int_\Omega |\nabla u_m|^{p-2} \nabla u_m \nabla \psi \longrightarrow \int_\Omega \frac{h(x)\psi}{(\epsilon + |u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2} \psi, \tag{6.10}$$

for all $\psi \in W_0^{1,p}(\Omega)$. So, from (6.9), (6.10) and the uniqueness of the limit

$$[M^+(t_0)]^{p-1} \langle \chi, \psi \rangle = \int_\Omega \frac{h(x)\psi}{(\epsilon + |u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2} \psi, \tag{6.11}$$

for all $\psi \in W_0^{1,p}(\Omega)$.

Using the monotonicity of the operator $(-\Delta_p)$, that is,

$$\langle -\Delta_p \omega - (-\Delta_p v), \omega - v \rangle \geq 0 \quad \forall \omega, v \in W_0^{1,p}(\Omega), \tag{6.12}$$

we have

$$[M^+(\|\omega\|_{1,p}^p)]^{p-1} \int_\Omega (|\nabla \omega|^{p-2} \nabla \omega - |\nabla v|^{p-2} \nabla v, \nabla \omega - \nabla v) dx \geq 0. \tag{6.13}$$

Taking $\omega = u_m$ and $v = \psi$ in the last expression, we obtain

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1} \int_\Omega (|\nabla u_m|^{p-2} \nabla u_m - |\nabla \psi|^{p-2} \nabla \psi, \nabla u_m - \nabla \psi) dx \geq 0, \tag{6.14}$$

and so

$$\begin{aligned}&[M^+(\|u_m\|_{1,p}^p)]^{p-1}\|u_m\|_{1,p}^p - [M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla\psi \\ &-[M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\nabla u_m + [M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla\psi|^p \geq 0.\end{aligned} \tag{6.15}$$

In (5.28) we take $\varphi = u_m$ to obtain

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1}\|u_m\|_{1,p}^p = \int_\Omega \frac{h(x)u_m}{(\epsilon + |u_m|)^{\gamma-p+2}} + \int_\Omega |u_m|^{\alpha-p+2}u_m. \tag{6.16}$$

Consequently,

$$\begin{aligned}&\int_\Omega \frac{h(x)u_m}{(\epsilon + |u_m|)^{\gamma-p+2}} + \int_\Omega |u_m|^{\alpha-p+2}u_m - [M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla\psi \\ &-[M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\nabla u_m + [M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla\psi|^p \geq 0,\end{aligned} \tag{6.17}$$

using the definition of $-\Delta_p$

$$\begin{aligned}&\int_\Omega \frac{h(x)u_m}{(\epsilon + |u_m|)^{\gamma-p+2}} + \int_\Omega |u_m|^{\alpha-p+2}u_m - [M^+(\|u_m\|_{1,p}^p)]^{p-1}\langle -\Delta_p u_m, \psi\rangle \\ &-[M^+(\|u_m\|_{1,p}^p)]^{p-1}\langle -\Delta_p\psi, u_m\rangle + [M^+(\|u_m\|_{1,p}^p)]^{p-1}\langle -\Delta_p\psi, \psi\rangle \geq 0,\end{aligned} \tag{6.18}$$

taking limits as $m \to \infty$ and using (5.44) we have

$$\begin{aligned}&\int_\Omega \frac{h(x)u}{(\epsilon + |u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2}u - [M^+(\|u\|_{1,p}^p)]^{p-1}\langle \chi, \psi\rangle \\ &-[M^+(t_0)]^{p-1}\langle -\Delta_p\psi, u\rangle + [M^+(t_0)]^{p-1}\langle -\Delta_p\psi, \psi\rangle \geq 0.\end{aligned} \tag{6.19}$$

We note that

$$[M^+(t_0)]^{p-1}\langle \chi, \psi\rangle = \int_\Omega \frac{h(x)\psi}{(\epsilon + |u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2}, \tag{6.20}$$

for all $\psi \in \mathbb{B}_l \subset W_0^{1,p}(\Omega)$, and so

$$[M^+(t_0)]^{p-1}\langle \chi, u\rangle = \int_\Omega \frac{h(x)u}{(\epsilon + |u|)^{\gamma-p+2}} + \int_\Omega |u|^{\alpha-p+2}u. \tag{6.21}$$

Therefore,

$$\begin{gathered}[M^{+}(t_0)]^{p-1}\langle \chi, u\rangle - [M^{+}(t_0)]^{p-1}\langle \chi, \psi\rangle \\ -[M^{+}(t_0)]^{p-1}\langle -\Delta_p\psi, u\rangle + [M^{+}(t_0)]^{p-1}\langle -\Delta_p\psi, \psi\rangle \geq 0,\end{gathered} \tag{6.22}$$

that is,

$$\langle \chi - (-\Delta_p\psi), u - \psi\rangle \geq 0. \tag{6.23}$$

Setting

$$\psi = u - \lambda\varphi, \quad \lambda > 0,\ \varphi \in \mathbb{B}_l, \tag{6.24}$$

we obtain

$$\langle \chi, -[-\Delta_p(u - \lambda\varphi)], \varphi\rangle \geq 0, \quad \forall\lambda > 0,\ \varphi \in \mathbb{B}_l, \tag{6.25}$$

taking limits on both sides of the above expression as $\lambda \longrightarrow 0$ and using the continuity of the operator $-\Delta_p$

$$\langle \chi, -(-\Delta_p u), \varphi\rangle \geq 0, \quad \forall\varphi \in \mathbb{B}_l. \tag{6.26}$$

So,

$$\langle \chi - (-\Delta_p u), -\varphi\rangle \geq 0, \tag{6.27}$$

and we obtain

$$\langle \chi - (-\Delta_p u), \varphi\rangle \leq 0, \quad \forall\varphi \in \mathbb{B}_l. \tag{6.28}$$

Hence,

$$\langle \chi - (-\Delta_p u), \varphi\rangle = 0 \quad \forall\varphi \in \mathbb{B}_l, \tag{6.29}$$

which implies $\chi = -\Delta_p u$, because l is arbitrary. This means that the above equality is valid for all $\varphi \in W_0^{1,p}(\Omega)$. From (6.10) and (6.11) we conclude that

$$[M^{+}(\|u_m\|_{1,p}^{p})]^{p-1}\int_{\Omega}|\nabla u_m|^{p-2}\nabla u_m \nabla\psi \longrightarrow [M^{+}(t_0)]^{p-1}\langle -\Delta u, \psi\rangle, \tag{6.30}$$

for all $\psi \in W_0^{1,p}(\Omega)$. Hence,

$$[M^+(\|u_m\|_{1,p}^p)]^{p-1}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla\psi \longrightarrow [M^+(t_0)]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\psi \qquad (6.31)$$

for all $\psi \in W_0^{1,p}(\Omega)$. This concludes the proof of the claim.

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