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Research Article

Error Bound of Periodic Signals in the Hölder Metric

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We obtain two theorems to determine the error bound between input periodic signals and processed output signals, whenever signals belong to H_ω-space and as a processor we have taken $(C,1)(E,1)$-mean and generalized an early result of Lal and Yadav in (2001).

1. Introduction

Chandra [1] was first to extend Prössdorf's [2] result to find the degree of approximation of a continuous function using the Nörlund transform. Later on, Mohapatra and Chandra [3] obtained a number of interesting results on the degree of approximation in the Hölder metric using matrix transforms, which generalize all the previous results based on Cesàro and Nörlund transforms. In 1992, Singh [4] introduced H_ω-space in place of H_α-space and obtained several results on the degree of approximation of functions and deduced many previous results based on H_α-spaces. In 1996, Das et al. [5] used $H_{(\alpha,p)}$-space in place of H_α-space and obtained degree of approximation of functions and generalized the results of Mohapatra and Chandra [3]. In 2000, Mittal and Rhoades [6] also obtained the degree of approximation of functions in a normed space and generalized the results of Singh [4] by removing the hypothesis of monotonicity of the rows of the matrix. Singh and Soni [7], and Mittal et al. [8] used the technique of approximation of functions in measuring the errors in the input signals and the processed output signals.

2. Definitions and notations

Let the transforms

A:

$$\lambda_n = \sum_{k=1}^{n} a_{nk} s_k, \tag{2.1}$$

B:

$$\tau_n = \sum_{k=1}^{n} b_{nk} s_k, \tag{2.2}$$

be two regular methods of summability. Then, the A transform of the B transform of a sequence $\{s_n\}$ is given by

$$t_n = \sum_{p=1}^{n} a_{np}\tau_p = \sum_{p=1}^{n}\sum_{k=1}^{n} a_{np} b_{pk} s_k, \tag{2.3}$$

the sequence $\{s_n\}$ is said to be summable t_n to the sum s, if

$$\operatorname*{Lim}_{n\to\infty} t_n = s. \tag{2.4}$$

Let $s(t) \in C_{2\pi}$ be a 2π-periodic analog signal whose Fourier trigonometric expansion be given by

$$s(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right) \equiv \sum_{n=0}^{\infty} A_n(t), \tag{2.5}$$

and let $\{s_n(t)\}$ be the sequence of partial sums of (2.5).

Let the $(E,1)$ and $(C,1)$ transforms for the sequence $\{s_n\}$ be defined by

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k(t), \tag{2.6}$$

$$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k(t), \tag{2.7}$$

respectively.

The product $(C,1)(E,1)$-transform is expressed as the $(C,1)$-transform of $(E,1)$-transform of $\{s_n\}$ and is given by sequence-to-sequence transformation (see, e.g., [9]):

$$t_n(s;t) = \frac{1}{n+1}\sum_{k=0}^{n} E_k^1. \tag{2.8}$$

The sequence $\{s_n\}$ is said to be summable $(C,1)(E,1)$ to the sum s, if

$$\operatorname*{Lim}_{n\to\infty} t_n(s;t) = s. \tag{2.9}$$

2.1. Regularity condition of $(C,1)(E,1)$-method

$$t_n(s;t)=\frac{1}{n+1}\sum_{k=0}^{n}E_k^1=\frac{1}{n+1}\sum_{k=0}^{n}\left\{\frac{1}{2^k}\sum_{\upsilon=0}^{k}\binom{k}{\upsilon}s_k\right\}=\sum_{k=0}^{\infty}C_{n,k}s_k, \tag{2.10}$$

where

$$C_{n,k}=\begin{cases}\dfrac{1}{n+1}2^{-k}\displaystyle\sum_{\upsilon=0}^{k}\binom{k}{\upsilon}, & k\le n\\ 0, & k>n.\end{cases} \tag{2.11}$$

Now,

(i) $\sum_{k=0}^{\infty}|C_{n,k}|=\sum_{k=0}^{n}|(1/(n+1))2^{-k}\sum_{\upsilon=0}^{k}\binom{k}{\upsilon}|=1$,

(ii) $C_{n,k}=(1/(n+1))(1)\to 0$, as $n\to\infty$, for fixed k,

(iii) $\sum_{k=0}^{\infty}C_{n,k}=1$,

thus, $(C,1)(E,1)$-method is regular.

Singh [4] defined the space H_ω by

$$H_\omega=\{s(t)\in C_{2\pi}:|s(t_1)-s(t_2)|\le K\omega(|t_1-t_2|)\}, \tag{2.12}$$

and the norm $\|\cdot\|_{\omega^*}$ by

$$\|s\|_{\omega^*}=\|s\|_c+\operatorname*{Sup}_{t_1,t_2}\{\Delta^{\omega^*}s(t_1,t_2)\}, \tag{2.13}$$

where

$$\begin{gathered}\|s\|_c=\operatorname*{Sup}_{0\le t\le 2\pi}|s(t)|,\\ \Delta^{\omega^*}s(t_1,t_2)=\frac{|s(t_1)-s(t_2)|}{\omega^*(|t_1-t_2|)},\quad t_1\ne t_2,\end{gathered} \tag{2.14}$$

and choosing $\Delta^0 s(t_1,t_2)=0$, $\omega(t)$ and $\omega^*(t)$ being increasing signals of t. If $\omega(|t_1-t_2|)\le A|t_1-t_2|^\alpha$ and $\omega^*(|t_1-t_2|)\le K|t_1-t_2|^\beta$, $0\le\beta<\alpha\le 1$, A and K being positive constants, then the space

$$H_\alpha=\{s(t)\in C_{2\pi}:|s(t_1)-s(t_2)|\le K|t_1-t_2|^\alpha,\ 0<\alpha\le 1\} \tag{2.15}$$

is Banach space [2] and the metric induced by the norm $\|\cdot\|_\alpha$ on H_α is said to be Hölder metric.

We write

$$\phi_{t_1}(t)=s(t_1+t)+s(t_1-t)-2s(t_1), \tag{2.16}$$

$$K_n(t)=\sin(n+1)\frac{t}{2}\sum_{k=0}^{n}\binom{n}{k}\sin\left(k+\frac{1}{2}\right)t. \tag{2.17}$$

3. Known result

Lal and Yadav [10] established the following theorem to estimate the error between the input signal $s(t)$ and the signal obtained after passing through the $(C,1)(E,1)$-transform.

Theorem A. *If a function* $s : R \to R$ *is* 2π*-periodic and belonging to class* $\mathrm{Lip}\,\alpha$, $0 < \alpha \leq 1$, *then the degree of approximation by* $(C,1)(E,1)$ *means of its Fourier series is given by*

$$\left\| t_n(s;t_1) - s(t_1) \right\|_\infty = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1 \\ O\left(\dfrac{\log n}{n}\right), & \alpha = 1. \end{cases} \tag{3.1}$$

4. Main result

The object of this paper is to generalize the above result under much more general assumptions. We will measure the error between the input signal $s(t)$ and the processed output signal $t_n(s;t) = (1/(n+1))\sum_{k=1}^{n} E_k^1(t)$, by establishing the following theorems.

Theorem 4.1. *Let* $\omega(t)$ *defined in* (2.12) *be such that*

$$\int_t^\pi \frac{\omega(u)}{u^2}du = O\{H(t)\}, \quad H(t) \geq 0, \tag{4.1}$$

$$\int_0^t H(u)du = O\{tH(t)\}, \quad \text{as } t \longrightarrow 0^+, \tag{4.2}$$

then, for $0 \leq \beta < \eta \leq 1$ *and* $s \in H_\omega$, *we have*

$$\left\| t_n(s;t_1) - s \right\|_{\omega^*} = O\left\{ \left((n+1)^{-1} H\left(\frac{\pi}{n+1} \right) \right)^{1-\beta/\eta} \right\}. \tag{4.3}$$

Theorem 4.2. *Let* $\omega(t)$ *defined in* (2.12) *and for* $0 \leq \beta < \eta \leq 1$ *and* $s \in H_\omega$, *we have*

$$\left\| t_n(s;t_1) - s \right\|_{\omega^*} = O\left\{ \left(\omega\left(\frac{\pi}{n} + 1 \right) \right)^{1-\beta/\eta} + \left((n+1)^{-1} \sum_{k=1}^{n+1} \omega\left(\frac{1}{k+1} \right) \right)^{1-\beta/\eta} \right\}. \tag{4.4}$$

5. Lemmas

We will use following lemmas.

Lemma 5.1. *Let* $\phi_{t_1}(t)$ *be defined in* (2.16), *then for* $s \in H_\omega$, *we have*

$$\left| \phi_{t_1}(t) - \phi_{t_2}(t) \right| \leq 4K\omega\left(\left| t_1 - t_2 \right| \right), \tag{5.1}$$

$$\left| \phi_{t_1}(t) - \phi_{t_2}(t) \right| \leq 4K\omega(|t|). \tag{5.2}$$

It is easy to verify.

Lemma 5.2. *Let $K_n(t)$ be defined in* (2.17)*, then*

$$K_n(t) \leq C\left(\frac{2^{n+1}}{t}\right)\cos^n\left(\frac{t}{2}\right)\sin(n+1)\left(\frac{t}{2}\right), \tag{5.3}$$

where "C" is an absolute constant, not necessarily the same at each occurrence.

Proof.

$$\begin{aligned} K_n(t) &= \frac{1}{\sin(t/2)} I.P.\left\{\sum_{k=0}^{n}\binom{n}{k} e^{i(k+1/2)t}\right\} \\ &= \frac{1}{\sin(t/2)} I.P.\left\{e^{it/2}\left(1+e^{it}\right)^n\right\} \\ &= \frac{1}{\sin(t/2)} I.P.\left\{2^n\cos^n\left(\frac{t}{2}\right)e^{i(n+1)t/2}\right\} \\ &\leq C\left(\frac{2^{n+1}}{t}\right)\cos^n\left(\frac{t}{2}\right)\sin(n+1)\left(\frac{t}{n}\right). \end{aligned} \tag{5.4}$$

□

Lemma 5.3.

$$\sum_{k=0}^{n}\left(\frac{1}{t}\right)\cos^k\left(\frac{t}{2}\right)\sin(k+1)\left(\frac{t}{2}\right) \leq \left(\frac{C}{t^2}\right)\left(1-\cos(n+1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right)\right). \tag{5.5}$$

Proof.

$$\begin{aligned} &\sum_{k=0}^{n}\left(\frac{1}{t}\right)\cos^k\left(\frac{t}{2}\right)\sin(k+1)\left(\frac{t}{2}\right) \\ &\quad= \sum_{k=0}^{n}\left(\frac{1}{t}\right) I.P.\left\{e^{i(k+1)t/2}\cos^k\left(\frac{t}{2}\right)\right\} \\ &\quad= \left(\frac{1}{t}\right) I.P.\left\{e^{it/2}\left(\frac{1-e^{i(n+1)t/2}\cos^{n+1}(t/2)}{1-e^{it/2}\cos(t/2)}\right)\right\} \\ &\quad\leq \left(\frac{C}{t^2}\right) I.P.\left\{i - i\cos(n+1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right) + \sin(n+1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right)\right\} \\ &\quad= \left(\frac{C}{t^2}\right)\left(1-\cos(n+1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right)\right). \end{aligned} \tag{5.6}$$

□

Lemma 5.4 (see [9]). *For $0 \leq t \leq 1/n+1$, then*

$$1-\cos(n+1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right) = O\left\{(n+1)^2t^2\right\}. \tag{5.7}$$

Lemma 5.5 (see [6]). *If $\omega(t)$ satisfies conditions* (4.1) *and* (4.2)*, then*

$$\int_0^u t^{-1}\omega(t)dt = O(uH(u)), \quad u \longrightarrow 0^+. \tag{5.8}$$

6. Proof of Theorem 4.1

Proof of Theorem 4.1. Following Zygmund [11], we have

$$s_n(t_1) - s = \frac{1}{2\pi}\int_0^{\pi} \frac{\phi_{t_1}(t)}{\sin(t/2)} \sin\left(n+\frac{1}{2}\right)t\,dt. \tag{6.1}$$

From (2.6) and (2.16), we have

$$E_n^1(t_1) - s = \frac{2^{-n}}{2\pi}\int_0^{\pi} \phi_{t_1}(t)K_n(t)dt. \tag{6.2}$$

Using Lemma 5.2, we have

$$E_n^1(t_1) - s \le C\frac{2^{-(n+1)}}{\pi}\int_0^{\pi} \frac{\phi_{t_1}(t)}{t}2^{n+1}\cos^n\left(\frac{t}{2}\right)\sin(n+1)\left(\frac{t}{2}\right)dt. \tag{6.3}$$

Now from (2.8), the $(C,1)$-transform of $(E,1)$-transform is given by

$$\left|t_n(s;t_1) - s\right| \le \frac{C}{n+1}\int_0^{\pi} \frac{\left|\phi_{t_1}(t)\right|}{t}\left|\sum_{k=0}^{n}\cos^k\left(\frac{t}{2}\right)\sin(k+1)\left(\frac{t}{2}\right)\right|dt. \tag{6.4}$$

Setting

$$\begin{aligned} E_n(t_1) = \left|t_n(s;t_1) - s(t_1)\right| &\le \frac{C}{n+1}\int_0^{\pi} \frac{\left|\phi_{t_1}(t)\right|}{t}\left|\sum_{k=0}^{n}\cos^k\left(\frac{t}{2}\right)\sin(k+1)\left(\frac{t}{2}\right)\right|dt, \\ E_n(t_1,t_2) = \left|E_n(t_1) - E_n(t_2)\right| &\le \frac{C}{n+1}\int_0^{\pi} \frac{\left|\phi_{t_1}(t) - \phi_{t_2}(t)\right|}{t}\left|\sum_{k=0}^{n}\cos^k\left(\frac{t}{2}\right)\sin(k+1)\left(\frac{t}{2}\right)\right|dt \\ &= O\left(\frac{1}{n+1}\right)\left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right) = I_1 + I_2, \quad \text{say}, \end{aligned} \tag{6.5}$$

now using (4.1), (4.2), (5.2), and Lemma 5.5, we get

$$I_1 = O(1)\frac{1}{n+1}\int_0^{\pi/n+1} t^{-1}\omega(t)dt = O\left\{(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right\}. \tag{6.6}$$

Again using (5.2), (4.1), and Lemma 5.3, we have

$$\begin{aligned} I_2 &= O(1)\frac{1}{n+1}\int_{\pi/n+1}^{\pi} t^{-2}\omega(t)\left|1 - \cos(n+1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right)\right|dt \\ &= O(1)\frac{1}{n+1}\int_{\pi/n+1}^{\pi} t^{-2}\omega(t)dt \\ &= O\left\{(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right\}. \end{aligned} \tag{6.7}$$

Now from (5.1), Lemmas 5.3 and 5.4, we have

$$\begin{aligned} I_1 &= O(1)\frac{1}{n+1}\int_0^{\pi/n+1}\frac{\omega(|t_1-t_2|)}{t^2}\left|1-\cos(n+1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right)\right|dt \\ &= O(1)\frac{\omega(|t_1-t_2|)}{n+1}\int_0^{\pi/n+1} t^{-2}(n+1)^2t^2dt \\ &= O\{\omega(|t_1-t_2|)\}, \end{aligned} \tag{6.8}$$

$$\begin{aligned} I_2 &= O(1)\frac{\omega(|t_1-t_2|)}{n+1}\int_{\pi/(n+1)}^{\pi} t^{-2}dt \\ &= O\{\omega(|t_1-t_2|)\}. \end{aligned} \tag{6.9}$$

Now noting that

$$I_r = I_r^{1-\beta/\eta}I_r^{\beta/\eta}, \quad r=1,2, \tag{6.10}$$

we have, from (6.6) and (6.8),

$$I_1 = O\left\{(\omega(|t_1-t_2|))^{\beta/\eta}\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\beta/\eta}\right\}, \tag{6.11}$$

and from (6.7) and (6.9), we have

$$I_2 = O\left\{(\omega(|t_1-t_2|))^{\beta/\eta}\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\beta/\eta}\right\}. \tag{6.12}$$

Thus, from (2.13), (6.11) and (6.12), we have

$$\begin{aligned} \sup_{t_1,t_2}\Delta^{\omega^*}|E_n(t_1,t_2)| &= \sup_{t_1,t_2}\frac{|E_n(t_1)-E_n(t_2)|}{\omega^*(|t_1-t_2|)} \\ &= O\left\{(\omega(|t_1-t_2|))^{\beta/\eta}(\omega^*(|t_1-t_2|))^{-1}\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\beta/\eta}\right\}. \end{aligned} \tag{6.13}$$

It is to be noted from (6.6) and (6.7),

$$\|E_n(t_1)\|_c = \max_{0\le t_1\le 2\pi}|t_n(s;t_1)-s| = O\left\{(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right\}. \tag{6.14}$$

Combining (6.13) and (6.14), we get

$$\|t_n(s;t_1)-s\|_{\omega^*} = O\left\{\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right)^{1-\beta/\eta}\right\}. \tag{6.15}$$

This completes the proof of Theorem 4.1. □

Proof of Theorem 4.2. Follows analogously as the proof of Theorem 4.1 with slight changes, so we omit details. □

7. Applications

The following results can easily be derived from the Theorem 4.1. If we put $\omega^*(|t_1 - t_2|) \leq K|t_1 - t_2|^{\beta}$, $\omega(|t_1 - t_2|) \leq A|t_1 - t_2|^{\alpha}$ and replace η by α and set

$$H(u) = \begin{cases} u^{\alpha-1}, & 0 < \alpha < 1 \\ \log\left(\dfrac{1}{u}\right), & \alpha = 1, \end{cases} \tag{7.1}$$

then we get Corollary 7.1.

Corollary 7.1. *If $s \in H_{\alpha}$, $0 \leq \beta < \alpha \leq 1$, then*

$$\|t_n(s;t_1) - s\|_{\beta} = \begin{cases} O(n+1)^{\beta-\alpha}, & 0 < \alpha < 1 \\ O\left(\dfrac{\log(n+1)}{(n+1)}\right)^{1-\beta}, & \alpha = 1. \end{cases} \tag{7.2}$$

If we put $\beta = 0$, then from above corollary, we have Corollary 7.2.

Corollary 7.2. *If $s \in Lip\,\alpha$, $0 < \alpha \leq 1$, then*

$$\|t_n(s;t_1) - s\| = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1 \\ O\left(\dfrac{\log n}{n}\right), & \alpha = 1. \end{cases} \tag{7.3}$$

Hence Theorem 3 is particular case of Theorem 4.1.

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References

[1] P. Chandra, "On the generalised Fejér means in the metric of Hölder space," *Mathematische Nachrichten*, vol. 109, no. 1, pp. 39–45, 1982.

[2] S. Prössdorf, "Zur Konvergenz der Fourierreihen Hölder stetiger Funktionen," *Mathematische Nachrichten*, vol. 69, no. 1, pp. 7–14, 1975.

[3] R. N. Mohapatra and P. Chandra, "Degree of approximation of functions in the Hölder metric," *Acta Mathematica Hungarica*, vol. 41, no. 1-2, pp. 67–76, 1983.

[4] T. Singh, "Degree of approximation to functions in a normed space," *Publicationes Mathematicae Debrecen*, vol. 40, no. 3-4, pp. 261–271, 1992.

[5] G. Das, T. Ghosh, and B. K. Ray, "Degree of approximation of functions by their Fourier series in the generalized Hölder metric," *Proceedings of the Indian Academy of Sciences. Mathematical Sciences*, vol. 106, no. 2, pp. 139–153, 1996.

[6] M. L. Mittal and B. E. Rhoades, "Degree of approximation to functions in a normed space," *Journal of Computational Analysis and Applications*, vol. 2, no. 1, pp. 1–10, 2000.

[7] T. Singh and B. Soni, "Approximation by generalized de la Vallee Poussin operators," *The Mathematics Student*, vol. 74, no. 1–4, pp. 199–206, 2005.

[8] M. L. Mittal, B. E. Rhoades, and V. N. Mishra, “Approximation of signals (functions) belonging to the weighted $W(L_p, \xi(t))$-class by linear operators,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 53538, 10 pages, 2006.

[9] S. Lal and P. N. Singh, “On approximation of Lip $(\xi(t), p)$ function by $(C, 1)(E, 1)$ means of its Fourier series,” *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 9, pp. 1443–1449, 2002.

[10] S. Lal and K. N. Singh Yadav, “On degree of approximation of function belonging to the Lipschitz class by $(C, 1)(E, 1)$ means of its Fourier series,” *Bulletin of the Calcutta Mathematical Society*, vol. 93, no. 3, pp. 191–196, 2001.

[11] A. Zygmund, *Trigonometric Series*, vol. 1, Cambridge University Press, New York, NY, USA, 1959.