Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 576571, 9 pages
doi:10.1155/2008/576571

Review Article

On Some Subclasses of Harmonic Functions Defined by Fractional Calculus

R. A. Al-Khal and H. A. Al-Kharsani

Department of Mathematics, Faculty of Science, Girls College, P.O. Box 838, Dammam 31113, Saudi Arabia

Correspondence should be addressed to R. A. Al-Khal, ranaab@hotmail.com

Received 1 July 2008; Accepted 3 November 2008

Recommended by Teodor Bulboaca

The purpose of this paper is to study subclasses of normalized harmonic functions with positive real part using fractional derivative. Sharp estimates for coefficients and distortion theorems are given.

1. Introduction

A continuous function $f = u+iv$ is a complex-valued harmonic function in a complex domain C if both u and v are real harmonic in C. In any simply connected domain $D \subseteq C$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation-preserving in D is that $|g'(z)| < |h'(z)|$ in D, see [1].

Denote by H the class of functions $f = h + \overline{g}$ which are harmonic univalent and orientation-preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Therefore, for $f = h + \overline{g} \in H$, we can express h and g by the following power series expansion:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

Observe that H reduces S, the class of normalized univalent analytic functions, if the coanalytic part of f is zero.

For $f = h + \overline{g}$ given by (1.1) and $n > -1$, Murugusundaramoorthy [2] defined the Ruscheweyh derivative of the harmonic function $f = h + \overline{g}$ in H by

$$D^n f(z) = D^n h(z) + \overline{D^n g(z)}, \tag{1.2}$$

where the Ruscheweh derivative of a power series $f(z) = z + \sum_{n=2}^{n} a_n z^n$ is given by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f. \tag{1.3}$$

The operator $*$ stands for the Hadamard product or convolution of two power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.4}$$

defined by

$$(f*g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.5}$$

In [3], Owa introduced the following definition.

Definition 1.1. Let the function $f(z)$ be analytic in a simply connected domain of the z-plane containing the origin and let $0 \leq \lambda < 1$. The fractional derivative of f of order λ is defined by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^1 \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1), \tag{1.6}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

In [4], Owa gave the relation between the fractional derivative and Ruscheweyh operator for the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ as

$$\begin{aligned} D^{\lambda} f(z) &:= \frac{1}{\Gamma(1+\lambda)} D_z^{\lambda} \left[z^{\lambda-1} f(z) \right], \quad 0 < \lambda < 1, \\ D^0 f(z) &= \lim_{\lambda \to \infty} D^{\lambda} f(z), \\ D^1 f(z) &= \lim_{\lambda \to 1} D^{\lambda} f(z). \end{aligned} \tag{1.7}$$

Using (1.2) and the relation between the fractional derivative and Ruscheweyh operator, we define the fractional derivative of order λ, $0 \leq \lambda < 1$, for the harmonic function $f = h + \overline{g}$ as

$$\begin{aligned} D_z^{\lambda} \left[z^{\lambda-1} f(z) \right] &= D_z^{\lambda} \left[z^{\lambda-1} h(z) \right] + \overline{D_z^{\lambda} \left[z^{\lambda-1} g(z) \right]}, \quad 0 < \lambda < 1, \\ D_z^0 f(z) &= \lim_{\lambda \to 0} D_z^{\lambda} f(z), \\ D_z^1 f(z) &= \lim_{\lambda \to 1} D_z^{\lambda} f(z). \end{aligned} \tag{1.8}$$

Since $D^\lambda f = D^\lambda h + \overline{D^\lambda g}$, it was proved in [1] that the harmonic function $D^\lambda f$ is starlike of order 1/2 if and only if the analytic function $D^\lambda h - D^\lambda g$ is starlike of order 1/2, and it was shown in [4, Theorem 3] that $D^\lambda h - D^\lambda g$ is starlike of order 1/2 if and only if $\operatorname{Re}\{D_z^{\lambda+1}[z^\lambda(h-g)]/D_z^\lambda[z^{\lambda-1}h-g)]\} > (1+\lambda)/2$ for $0<\lambda<1$. Since $\operatorname{Re}\{D_z^{\lambda+1}[z^\lambda(h-g)]/D_z^\lambda[z^{\lambda-1}(h-g)]\} = \operatorname{Re}(\Gamma(2+\lambda)D^{\lambda+1}(h-g)/\Gamma(1+\lambda)D^\lambda(h-g))$, then $D^\lambda h - D^\lambda g$ is starlike of order $(1+\lambda)\Gamma(1+\lambda)/2\Gamma(2+\lambda)$, hence $D^\lambda f = D^\lambda h + \overline{D^\lambda g}$ is starlike of order $(1+\lambda)\Gamma(1+\lambda)/2\Gamma(2+\lambda)$. This means

$$\operatorname{Re}\frac{DD^\lambda f}{D^\lambda f} > \frac{(1+\lambda)\Gamma(1+\lambda)}{2\Gamma(2+\lambda)} \Longrightarrow \operatorname{Re}\frac{D_z^{\lambda+1}\left[z^\lambda f\right]}{D_z^\lambda\left[z^{\lambda-1} f\right]} > \frac{(1+\lambda)}{2}. \tag{1.9}$$

Recently, Owa and Srivastava [5] studied the linear Ω^λ defined by operator

$$\Omega^\lambda f(z) := \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z) \quad (0\le\lambda<1), \tag{1.10}$$

where f is normalized and analytic function on U.

It is easily seen that

$$\Omega^0 f = f, \qquad \Omega^1 f = zf'. \tag{1.11}$$

Analogously, we studied the linear operator Ω^λ defined on the harmonic function $f = h + \overline{g}$ by

$$\Omega^\lambda f(z) = \Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}, \tag{1.12}$$

where

$$\begin{aligned}\Omega^\lambda h(z) &:= \Gamma(2-\lambda)z^\lambda D_z^\lambda h(z) = \sum_{n=0}^{\infty}\frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}a_{n+1}z^{n+1}, \quad a_1 = 1,\\ \Omega^\lambda g(z) &:= \Gamma(2-\lambda)z^\lambda D_z^\lambda g(z) = \sum_{n=0}^{\infty}\frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)}b_{n+1}z^{n+1}, \quad b_1 = 0.\end{aligned} \tag{1.13}$$

We will define subclasses of normalized harmonic functions obtained by the Hadamard product and using the fractional derivative.

2. Main results

Let h and g be analytic in U. Let P_H stand for harmonic functions $f = h + \overline{g}$ so that $\operatorname{Re} f > 0$, $z\in U$ and $f(0) = 1$.

If the function $f_z + \overline{f_z} = h' + \overline{g'}$ belongs to P_H for the analytic and normalized functions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n, \tag{2.1}$$

then the class of functions $f = h + \overline{g}$ is denoted by $\widetilde{P}_H^0$ [6].

The function

$$t_\alpha(z) = z + \frac{1}{1+\alpha}z^2 + \cdots + \frac{1}{1+(n-1)\alpha}z^n + \cdots \tag{2.2}$$

is analytic on U when α is a complex number different from $-1, -(1/2), -(1/3), \ldots$. For $\Omega^\lambda f \in \widetilde{P}_H^0$, we denote by $\widetilde{P}_H^{\lambda,0}(\alpha)$ the class of functions defined by

$$\Omega^\lambda F = \Omega^\lambda f * \left(t_\alpha + \overline{t_\alpha}\right). \tag{2.3}$$

Therefore,

$$\begin{aligned} \Omega^\lambda F &= \Omega^\lambda H + \overline{\Omega^\lambda g} \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} a_n z^n + \sum_{n=2}^{\infty} \overline{\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} b_n z^n} \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n}, \quad z \in U \end{aligned} \tag{2.4}$$

is in $\widetilde{P}_H^{\lambda,0}(\alpha)$. Conversely, if $\Omega^\lambda F$ is in the form (2.4), with a_n, b_n being the coefficients of $\Omega^\lambda f \in \widetilde{P}_H^0$, then $\Omega^\lambda F = \widetilde{P}_H^{\lambda,0}(\alpha)$.

Note that $\widetilde{P}_H^{0,0}(\alpha) \equiv \widetilde{P}_H^0(\alpha)$ [7] and $\widetilde{P}_H^{0,0}(0) \equiv \widetilde{P}_H^0$.

Theorem 2.1. *If $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$, then there exists $\Omega^\lambda f \in \widetilde{P}_H^0$ so that*

$$\alpha\left[z\left(\Omega^\lambda F\right)_z(z) + \overline{z}\left(\Omega^\lambda F\right)_{\overline{z}}(z)\right] + (1-\alpha)\Omega^\lambda F(z) = \Omega^\lambda f(z). \tag{2.5}$$

Conversely, for any function f such that $\Omega^\lambda f \in \widetilde{P}_H^0$, there exists $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$ satisfying (2.5).

Proof. Let $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$. If $\Omega^\lambda f \in \widetilde{P}_H^0$, then since

$$\alpha z t_\alpha'(z) + (1-\alpha)t_\alpha(z) = t_0(z) \tag{2.6}$$

as $\Omega^\lambda F = \Omega^\lambda f * (t_\alpha + \overline{t_\alpha})$, we obtain that

$$\Omega^\lambda f(z) = \alpha\left[\Omega^\lambda f(z) * \left(z t_\alpha' + \overline{z t_\alpha'}\right)\right] + (1-\alpha)\left[\Omega^\lambda f(z) * \left(t_\alpha + \overline{t_\alpha}\right)\right]. \tag{2.7}$$

Therefore,

$$\Omega^\lambda f(z) = \alpha\left[z\left(\Omega^\lambda F\right)_z(z) + \overline{z}\left(\Omega^\lambda F\right)_{\overline{z}}(z)\right] + (1-\alpha)\Omega^\lambda F(z). \tag{2.8}$$

Conversely, for $\Omega^\lambda f \in \widetilde{P}_H^0$, from (2.1), (2.2), and (2.5),

$$\begin{aligned} z &+ \sum_{n=2}^{\infty} \frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n + \sum_{n=2}^{\infty} \overline{\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} b_n z^n} \\ &= z + \sum_{n=2}^{\infty} [1+(n-1)\alpha] A_n z^n + \sum_{n=2}^{\infty} \overline{[1+(n-1)\alpha] B_n z^n}, \end{aligned} \tag{2.9}$$

where

$$A_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} a_n, \qquad B_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} b_n. \tag{2.10}$$

Therefore,

$$\Omega^\lambda F = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n} = \Omega^\lambda f * \left[t_\alpha(z) + \overline{t_\alpha}(z)\right]. \tag{2.11}$$

□

Theorem 2.2. *A function $\Omega^\lambda F$ of the form* (2.4) *belongs to $\widetilde{P}_H^{\lambda,0}(\alpha)$, if and only if*

$$\operatorname{Re}\left\{z(\Omega^\lambda H(z))'' + \overline{\alpha}(\Omega^\lambda G(z))'' + (\Omega^\lambda H(z))' + (\Omega^\lambda G(z))'\right\} > 0, \quad z \in U. \tag{2.12}$$

Proof. If $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$, then from Theorem 2.1,

$$\alpha\left[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}\right] + (1-\alpha)\left[\Omega^\lambda H + \overline{\Omega^\lambda G}\right] = \Omega^\lambda h + \overline{\Omega^\lambda g} \in \widetilde{P}_H^0 \tag{2.13}$$

and $(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'} \in P_H$. Hence

$$\begin{aligned} 0 < \operatorname{Re}&\left\{(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'}\right\} \\ &\times \operatorname{Re}\left\{\alpha z(\Omega^\lambda H)'' + \alpha(\Omega^\lambda H)' + (1-\alpha)(\Omega^\lambda H)' + \overline{\alpha} z(\Omega^\lambda G)'' + \overline{\alpha}(\Omega^\lambda G)' + (1-\overline{\alpha})(\Omega^\lambda G)'\right\} \\ &\times \operatorname{Re}\left\{z\left(\alpha(\Omega^\lambda H)'' + \overline{\alpha}(\Omega^\lambda G)''\right) + (\Omega^\lambda H)' + (\Omega^\lambda G)'\right\}. \end{aligned} \tag{2.14}$$

Conversely, if the function $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G}$ of the form (2.4) satisfies (2.10), then by Theorem 2.1 $(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'} \in P_H$ and the following function holds:

$$\Omega^\lambda f = \Omega^\lambda h + \overline{\Omega^\lambda g} = \alpha\left[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}\right] + (1-\alpha)\left[\Omega^\lambda H + \overline{\Omega^\lambda G}\right] \in \widetilde{P}_H^0. \tag{2.15}$$

Then by Theorem 2.1, $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$. □

Theorem 2.3. *$\widetilde{P}_H^{\lambda,0}(\alpha)$ is convex and compact.*

Proof. Let $\Omega^\lambda F_1 = \Omega^\lambda H_1 + \overline{\Omega^\lambda G_1}$, $\Omega^\lambda F_2 = \Omega^\lambda H_2 + \overline{\Omega^\lambda G_2} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and let $\mu \in [0,1]$. Then

$$\begin{aligned}
&\operatorname{Re}\{z[\alpha(\mu(\Omega^\lambda H_1(z))'' + (1-\mu)(\Omega^\lambda H_2(z))'') + \overline{\alpha}(\Omega^\lambda G_1(z))'' + (1-\mu)(\Omega^\lambda G_2(z))'')] \\
&\quad + \mu[(\Omega^\lambda H_1(z))' + (\Omega^\lambda G_1(z))'] + (1-\mu)[(\Omega^\lambda H_2(z))' + (\Omega^\lambda G_2(z))']\} \\
&\quad = \mu \operatorname{Re}\{z[\alpha(\Omega^\lambda H_1(z))'' + \overline{\alpha}(\Omega^\lambda G_1(z))''] + (\Omega^\lambda H_1(z))' + (\Omega^\lambda G_1(z))'\} \\
&\qquad + (1-\mu)\operatorname{Re}\{z[\alpha(\Omega^\lambda H_2(z))'' + \overline{\alpha}(\Omega^\lambda G_2(z))''] + (\Omega^\lambda H_2(z))' + (\Omega^\lambda G_2(z))'\} \\
&\quad > 0.
\end{aligned} \tag{2.16}$$

Hence from Theorem 2.2, $\mu\Omega^\lambda F_1 + (1-\mu)\Omega^\lambda F_2 \in \widetilde{P}_H^{\lambda,0}(\alpha)$. Therefore, $\widetilde{P}_H^{\lambda,0}(\alpha)$ is convex.

Now, let $\Omega^\lambda F_n = \Omega^\lambda H_n + \overline{\Omega^\lambda G_n} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and let $\Omega^\lambda F_n \to \Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G}$. By Theorem 2.2,

$$\alpha[z(\Omega^\lambda H_n)' + \overline{z(\Omega^\lambda G_n)'}] + (1-\alpha)[\Omega^\lambda H_n + \overline{\Omega^\lambda G_n}] \in \widetilde{P}_H^0. \tag{2.17}$$

Since $\widetilde{P}_H^0$ is compact, see [6],

$$\alpha[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}] + (1-\alpha)[\Omega^\lambda H + \overline{\Omega^\lambda G}] \in \widetilde{P}_H^0. \tag{2.18}$$

Hence by Theorem 2.1, $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$, therefore $\widetilde{P}_H^{\lambda,0}(\alpha)$ is compact. □

Theorem 2.4. *If $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and $|z| = r < 1$, then*

$$\begin{aligned}
-r + 2\operatorname{In}(1+r) &\le \operatorname{Re}\{\alpha[z(\Omega^\lambda H_n)' + \overline{z(\Omega^\lambda G_n)'}] + (1-\alpha)[\Omega^\lambda H_n + \overline{\Omega^\lambda G_n}]\} \\
&\le -r - 2\operatorname{In}(1-r).
\end{aligned} \tag{2.19}$$

Equality is obtained for the function (2.3) *where*

$$\Omega^\lambda f = 2z + \ln(1-z) - 3\overline{z} - 3\ln(1-\overline{z}), \quad z \in U. \tag{2.20}$$

Proof. From Theorem 2.1, if $\Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$, then there exists $\Omega^\lambda f = \Omega^\lambda h + \overline{\Omega^\lambda g} \in \widetilde{P}_H^0$ so that

$$\alpha[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}] + (1-\alpha)[\Omega^\lambda H + \overline{\Omega^\lambda G}] = \Omega^\lambda f. \tag{2.21}$$

Since by [5, Proposition 2.2]

$$-r + 2\ln(1+r) \le \operatorname{Re}(\Omega^\lambda f) \le -r - 2\ln(1-r), \tag{2.22}$$

this completes the proof. □

Theorem 2.5. *If* $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ *and* $\operatorname{Re}\alpha > 0$, *then there exists* $\Omega^\lambda f \in \widetilde{P}_H^0$ *so that*

$$\Omega^\lambda F = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-2}(\Omega^\lambda f)(z\zeta)d\zeta, \quad z \in U. \tag{2.23}$$

Proof. Since

$$t_\alpha(z) = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-1}\frac{z}{1-z\zeta}d\zeta, \quad |\zeta| \le 1,\ \operatorname{Re}\alpha > 0, \tag{2.24}$$

and for $\Omega^\lambda f = \Omega^\lambda h + \overline{\Omega^\lambda g} \in \widetilde{P}_H^0$,

$$(\Omega^\lambda h)(z) * \frac{z}{1-z\zeta} = \frac{(\Omega^\lambda h)(z\zeta)}{\zeta}, \qquad (\Omega^\lambda g)(z) * \frac{z}{1-z\zeta} = \frac{(\Omega^\lambda g)(z\zeta)}{\zeta}, \tag{2.25}$$

we have

$$\begin{aligned}(\Omega^\lambda H)(z) &= (\Omega^\lambda h)(z) * t_\alpha = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-2}(\Omega^\lambda h)(z\zeta)d\zeta,\\ (\Omega^\lambda G)(z) &= (\Omega^\lambda g)(z) * t_\alpha = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-2}(\Omega^\lambda g)(z\zeta)d\zeta.\end{aligned} \tag{2.26}$$

Hence $\Omega^\lambda F$ is type (2.23). □

Theorem 2.6. *If* $\operatorname{Re}\alpha > 0$, *then* $\widetilde{P}_0^{\lambda,0}(\alpha) \subset \widetilde{P}_H^0$.

Proof. Let $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and $\operatorname{Re}\alpha > 0$. Then there exists $\Omega^\lambda f \in \widetilde{P}_H^0$ so that

$$\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} = \Omega^\lambda f * \left(t_\alpha + \overline{t_\alpha}\right) = \left(\Omega^\lambda h * t_\alpha\right) + \left(\Omega^\lambda g * \overline{t_\alpha}\right). \tag{2.27}$$

Hence $0 < \operatorname{Re}\{(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'}\} = \operatorname{Re}\{(\Omega^\lambda h)' + (\Omega^\lambda g)'\}$ and since $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\{(\Omega^\lambda H)' + (\Omega^\lambda G)'\} > 0$ and $\Omega^\lambda H(0) = 0$, $(\Omega^\lambda H)'(0) = 1$, $\Omega^\lambda G(0) = 0$, $(\Omega^\lambda G)'(0) = 0$. And hence $\Omega^\lambda F \in \widetilde{P}_H^0$. □

Theorem 2.7. *Let* $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$. *Then*

(i)

$$\left\| A_n| - |B_n \right\| \le \frac{2\Gamma(n+1)\Gamma(2-\lambda)}{n\Gamma(n+1-\lambda)|(1+(n-1)\alpha|}, \quad n \ge 1, \tag{2.28}$$

(ii) *if* $\Omega^\lambda F$ *is sense-preserving, then*

$$|A_n| \le \frac{2n-1}{n}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}, \quad n=1,2,\ldots,$$
$$|B_n| \le \frac{2n-3}{n}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}, \quad n=2,3,\ldots. \tag{2.29}$$

Proof. By (2.10),

$$\left\| A_n| - |B_n \right\| = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}\left\| a_n| - |b_n \right\|. \tag{2.30}$$

Also, by [6, Theorem 2.3], we have

$$\left\| a_n| - |b_n \right\| \le \frac{2}{n}. \tag{2.31}$$

The required results are obtained.

On the other hand, from (2.10), it is known [6, Corollary 2.5] that

$$|a_n| \le \frac{2n-1}{n}, \qquad |b_n| \le \frac{2n-3}{n}. \tag{2.32}$$

Then we get the coefficient inequalities for $\widetilde{P}_0^{\lambda,0}(\alpha)$. □

Remark 2.8. Taking $\lambda = 0$ in Theorems 2.1–2.7, we get the similar results in [7].

Theorem 2.9. *Let* $\Omega^\lambda F = \Omega^\lambda H = \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ *and sense-preserving in* U, *then for* $|z| = r < 1$,

$$\left|\alpha z(\Omega^\lambda H)' + (1-\alpha)\Omega^\lambda H\right| \le \frac{2r}{1-r} + \ln(1-r),$$
$$\left|\overline{\alpha} z(\Omega^\lambda G)' + (1-\overline{\alpha})\Omega^\lambda G\right| \le \frac{3r-r^2}{1-r} + 3\ln(1-r). \tag{2.33}$$

Proof. From Theorems 2.1 and 2.2, if $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$, then there exists $\Omega^\lambda f = \Omega^\lambda h + \overline{\Omega^\lambda g} \in \widetilde{P}_H^0$ such that

$$\alpha z(\Omega^\lambda H)' + (1-\alpha)\Omega^\lambda H = \Omega^\lambda h,$$
$$\overline{\alpha} z(\Omega^\lambda G)' + (1-\overline{\alpha})\Omega^\lambda G = \Omega^\lambda g. \tag{2.34}$$

By [6, Theorem 3.5], we obtain the results. □

Remark 2.10. Taking $\lambda = 0$ and $\alpha = 0$ in Theorem 2.9, we get [6, Theorem 2.4].

3. Positive order

We say that the harmonic function $f = h + \overline{g}$ of the form (2.1) is in the class $P_H(\beta)$, $0 \leq \beta < 1$ for $|z| = r$ if $\operatorname{Re} f > \beta$ and $f(0) = 1$.

If the function $f_z + \overline{f_{\overline{z}}} = h' + \overline{g'}$ belongs to $P_H(\beta)$ for the analytic and normalized functions h and g of the form (2.1), then the class of functions $f = h + \overline{g}$ is denoted by $\widehat{P}^0_H(\beta)$.

Denote by $\widehat{P}^{\lambda,0}_H(\beta, \alpha)$ the class of functions defined By (2.3) where $\Omega^\lambda f \in \widehat{P}^0_H(\beta)$.

Many of our results can be rewritten for functions in the class $\widehat{P}^{\lambda,0}_H(\beta, \alpha)$. For instance, see the following theorems.

Theorem 3.1. *If* $\Omega^\lambda F \in \widehat{P}^{\lambda,0}_H(\beta, \alpha)$*, then there exists* $\Omega^\lambda f \in \widehat{P}^0_H(\beta)$ *so that*

$$\alpha\left[z\left(\Omega^\lambda F\right)_z(z) + \overline{z}\left(\Omega^\lambda F\right)_{\overline{z}}(z)\right] + (1 - \alpha)\Omega^\lambda F(z) = \Omega^\lambda f(z). \tag{3.1}$$

Conversely, for any function f *such that* $\Omega^\lambda f \in \widehat{P}^0_H(\beta)$*, there exists* $\Omega^\lambda F \in \widehat{P}^{\lambda,0}_H(\beta, \alpha)$ *satisfying* (3.1).

Theorem 3.2. *A function* $\Omega^\lambda F$ *belongs to* $\widehat{P}^{\lambda,0}_H(\beta, \alpha)$ *if and only if*

$$\operatorname{Re}\left\{z\left(\Omega^\lambda H(z)\right)'' + \overline{\alpha}\left(\Omega^\lambda G(z)\right)'' + \left(\Omega^\lambda H(z)\right)' + \left(\Omega^\lambda G(z)\right)'\right\} > \beta, \quad z \in U. \tag{3.2}$$

Theorem 3.3. *If* $\Omega^\lambda F \in \widehat{P}^{\lambda,0}_H(\beta, \alpha)$ *and* $\operatorname{Re} \alpha > 0$*, then there exists* $\Omega^\lambda f \in \widehat{P}^0_H(\beta)$ *so that*

$$\Omega^\lambda F = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha - 2}\left(\Omega^\lambda f\right)(z\zeta)\,d\zeta, \quad z \in U. \tag{3.3}$$

Theorem 3.4. *If* $\operatorname{Re} \alpha > 0$*, then* $\widehat{P}^{\lambda,0}_H(\beta, \alpha) \subset \widehat{P}^0_H(\beta)$.

References

[1] J. Clunie and T. Sheil-Small, “Harmonic univalent functions,” *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica*, vol. 9, pp. 3–25, 1984.

[2] G. Murugusundaramoorthy, “A class of Ruscheweyh-type harmonic univalent functions with varying arguments,” *Southwest Journal of Pure and Applied Mathematics*, no. 2, pp. 90–95, 2003.

[3] S. Owa, “On the distortion theorems. I,” *Kyungpook Mathematical Journal*, vol. 18, no. 1, pp. 53–59, 1978.

[4] S. Owa, “A remark on new criteria for univalent functions,” *Kyungpook Mathematical Journal*, vol. 21, no. 1, pp. 15–23, 1981.

[5] S. Owa and H. M. Srivastava, “Univalent and starlike generalized hypergeometric functions,” *Canadian Journal of Mathematics*, vol. 39, no. 5, pp. 1057–1077, 1987.

[6] S. Yalçın, M. Öztürk, and M. Yamankaradeniz, “On some subclasses of harmonic functions,” in *Functional Equations and Inequalities*, vol. 518 of *Mathematics and Its Applications*, pp. 325–331, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[7] M. Öztürk, S. Yalçin, and M. Yamankaradeniz, “On harmonic functions constructed by the Hadamard product,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 1, article 9, pp. 1–8, 2002.