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Research Article

Bipartite Toughness and k-Factors in Bipartite Graphs

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We define a new invariant $t^B(G)$ in bipartite graphs that is analogous to the toughness $t(G)$ and we give sufficient conditions in term of $t^B(G)$ for the existence of k-factors in bipartite graphs. We also show that these results are sharp.

1. Introduction

Toughness, like connectivity, is an important invariant in graphs. There has been extensive work on toughness (see the survey in [1]) since Chvátal introduced the concept in 1973 [2]. The toughness $t(G)$ of a graph G is the minimum value of $|S|/w(G-S)$, where $S \subset V(G)$ is a proper subset of the vertices of G and $w(G-S) > 1$ is the number of connected components after removing S from G. (If G is a complete graph so that $w(G-S)$ is always equal to 1, then $t(G)$ is set to be ∞.) That is, for any integer $k > 1$, G cannot be split into k connected components by removing less than $k \cdot t(G)$ vertices. We also say that G is $t(G)$-tough. Chvátal made a number of conjectures in [2], including the famous 2-tough conjecture saying that every 2-tough graph has a Hamiltonian cycle. Having inspired many interesting results, the 2-tough conjecture itself was showed to be false by Bauer et al. in 2000 [3].

A subgraph H of G is called a *factor* of G if H is a spanning subgraph of G. An important class of factors is *k-factors*, also called *regular degree factors*, where every vertex of G has degree k in H. (Note that a perfect matching is a 1-factor, and a Hamiltonian cycle is a connected 2-factor.) There has been extensive work on the conditions of existence of various factors in graphs. Many results can be found in the latest survey by Plummer [4].

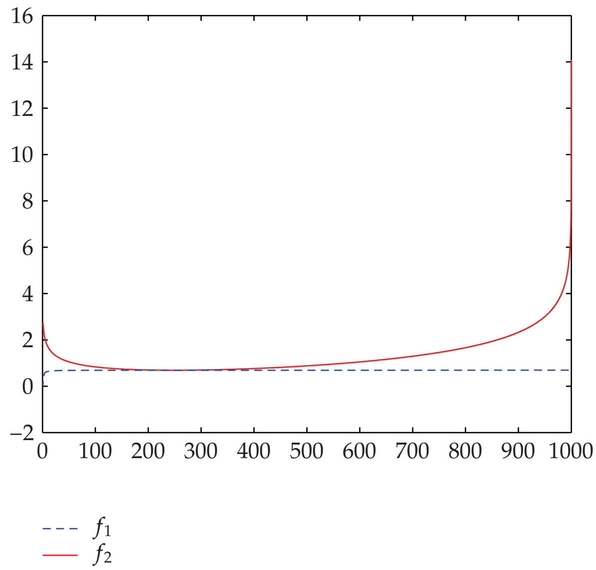

Figure 1: The bound of bipartite toughness in Theorems 1.2 and 1.3, illustrated with $n = 1000$. The x-axis is k and y-axis is $\log(t^B(G))$. The bound is given by f_1 on the left and f_2 on the right of $k = (n+4)/4$.

It is natural to expect that toughness, yet another measure of the connectivity of a graph, ought to relate to the existence of k-factors in graphs. Enomoto et al. [5–7] proved that every k-tough graph contains a k-factor if it satisfies trivial necessary conditions, and there are $(k-\varepsilon)$-tough graphs for any $\varepsilon > 0$ that do not contain a k-factor. Consider a *bipartite graph* $G = (X, Y; E)$, where $X \cup Y = V(G)$ is a partition of $V(G)$ and E is the edge set of G with each edge having one end in X and the other in Y. Katerinis [8] proved that every 1-tough bipartite graph has a 2-factor. Recall that the toughness of a bipartite graph $G = (X, Y; E)$ is at most 1 because the removal of X from G (assuming $|X| \geq |Y|$) results in an independent set Y. Therefore, it is not possible to use toughness to predict the existence of k-factors in balanced bipartite graphs for any $k \geq 3$.

1.1. Bipartite toughness

In this paper, we introduce *bipartite toughness*, which is analogous to the concept of toughness but reflects the bipartition of $V(G)$. The bipartite toughness $t^B(G)$ of a bipartite graph $G = (X, Y; E)$ is the minimum value of $|S|/w(G-S)$, where S is a proper subset of X or Y and $w(G-S) > 1$ is the number of connected components after removing S from G. We set $t^B(G) = \infty$ for complete bipartite graphs, just like $t(G) = \infty$ for complete graphs.

A bipartite graph can have a regular degree factor only if $|X| = |Y|$. Therefore, in the rest of the paper, we consider only a *balanced bipartite graph* with $|X| = |Y| = n$. For a subset S of $V(G)$, we use $N(S)$ to denote the set of vertices adjacent to at least one vertex in S. For two disjoint subsets S and T of $V(G)$, we use $e_G(S, T)$ to stand for the number of edges having one end in S and the other in T. Other terminologies and notations used in this paper follow [9] and other references.

Bipartite toughness $t^B(G)$ measures the connectivity of a bipartite graph better than toughness $t(G)$ does. In contrast to toughness $t(G)$ that is at most 1 in a bipartite graph, $t^B(G)$ can be arbitrarily big. For example, in a complete bipartite graph with one edge deleted,

$t(G) = O(n)$, which approaches to ∞, is just like $t(G) = O(n)$ in a complete graph with one edge deleted. Interestingly, $t^B(G)$ a better invariant to predict the existence of k-factors in balanced bipartite graphs, for any k. Furthermore, by their definitions, calculating $t^B(G)$ in a bipartite graph is easier than calculating $t(G)$ since one is a subtask of the other.

1.2. Our results

Let $G = (X, Y; E)$ be a balanced bipartite graph with $|X| = |Y| = n$ and $1 \leq k \leq n$ be an integer. In this paper, we prove the following three theorems.

Theorem 1.1. *Let* $m = \lfloor (n-1)/2 \rfloor$. *If* $t^B(G) > m/(m+2)$, *then G has a 1-factor.*

Theorem 1.2. *For* $k \geq 2$ *and* $n \geq 4k-4$, *if* $t^B(G) > f_1 = (2k-1)(n-1)/(kn+1)$, *then G has a k-factor.*

Theorem 1.3. *For* $n \leq 4k-4$, *if* $t^B(G) > f_2 = (n-1)/(2\sqrt{kn+1} - 2k+1)$, *then G has a k-factor.*

These theorems together give a sharp bound of $t^B(G)$ for G to have a k-factor, for $k = 1, \ldots, n$. (See Figure 1. Note that $m/(m-2) = f_1$ when $k = 1$ and n is odd; and $f_1 = f_2$ when $n = 4k-4$.)

The bound of $t^B(G)$ is sharp in the following senses.

(a) For Theorem 1.1, let $m = \lfloor (n-1)/2 \rfloor$ and construct a balanced bipartite graph $G = (X, Y; E)$ as follows. Let $X = S \cup P$ and $Y = T \cup Q$, where $|P| = |T| = n - m$, $|S| = |Q| = m$, and $|X| = |Y| = n$. Let E be comprised of all possible edges between X and Q and all possible edges between S and Y. If n is even, then we add into E an edge between P and T. Here, $|S| + e_G(X-S, T) - |T| = -1$ so that by Lemma 2.1 below, G has no 1-factor. On the other hand, it is not hard to verify that $t^B(G) = m/(m+2)$ in this construction of G. Therefore, $m/(m+2)$ is a sharp bound.

(b) For Theorem 1.2, for integers $k \geq 2$ and $r \geq 2$, construct a balanced bipartite graph $G_r = (X, Y; E)$ as follows. Let $X = S \cup P$ and $Y = T \cup Q$, where $|P| = |T| = kr - 1$, $|S| = |Q| = (k-1)r - 1$, and $|X| = |Y| = n = (2k-1)r - 2 \geq 4k - 5$. Let E be comprised of all possible edges between X and Q, all possible edges between S and Y, and a 1-factor between P and T. Here, $k|S| + e_G(X - S, T) - k|T| = -1$ so that by Lemma 2.1 below, G_r has no k-factor. On the other hand, it is not hard to verify that $t^B(G_r) = (n-1)/(n - |S|) = (2k-1)(n-1)/(kn+1) = f_1$ in G_r. Therefore, f_1 is a sharp bound.

(c) For Theorem 1.3. Let $n/4 < k < n$ and $\sqrt{kn+1} = t$ be an integer. Obviously, $n/2 < t < n$. Construct a balanced bipartite graph $G = (X, Y; E)$ as follows. Let $X = S \cup P$ and $Y = T \cup Q$, where $|P| = |T| = t$, $|S| = |Q| = n - t$, and $|X| = |Y| = n$. Let E be comprised of all possible edges between X and Q, all possible edges between S and Y, and a $(2k-t)$-factor between P and T. Then $k|S| + e_G(X - S, T) - k|T| = k(n-t) + (2k-t)t - kt = kn - t^2 = -1$. Again, by Lemma 2.1 below, G has no k-factor. Moreover, it is not hard to verify that $t^B(G) = (n-1)/(2\sqrt{kn+1} - 2k+1)$. Therefore, f_2 is also a sharp bound.

It is also worth to mention that, unlike Enomoto et al.'s well-known result that k-tough graphs have k-factors, in our results the bound of $t^B(G)$ is much smaller than k, in fact less than 2 for most k (see Figure 1). This looks counterintuitive but it is due to

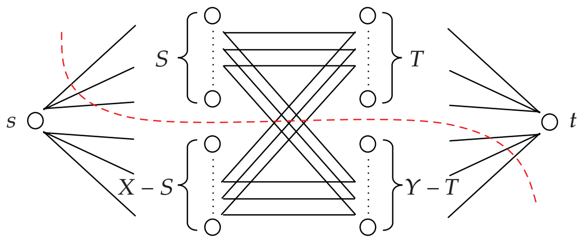

Figure 2: For proof of Lemma 2.1, red dashed line is the minimum cut.

a (not so good) feature of $t^B(G)$. Although $t^B(G)$ can approach to ∞, most time it does not increase significantly with edge connectivity or minimum degree. For example, if $G = (X, Y; E)$, $|X| = |Y| = n$ has minimum degree $\delta(G) = n/2$ (say on vertex $x \in X$), then removing all vertices in X except x would split Y into $n/2$ components. So $t^B(G) \leq 2$ even when $\delta(G)$ is as high as $n/2$.

2. Proofs of the theorems

The following lemma will be needed in the proofs of theorems.

Lemma 2.1. *Let $G = (X, Y; E)$ be a balanced bipartite graph, where $|X| = |Y| = n$, and let $k \geq 1$ be an integer. Then the following three statements are equivalent:*

(i) *G has a k-factor;*

(ii) *G has k edge-disjoint 1-factors;*

(iii) *for any $S \subseteq X$ and $T \subseteq Y$, $k|S| + e_G(X - S, T) - k|T| \geq 0$.*

Proof. (i) and (ii): following the König-Hall theorem [9, Theorem 5.2 and Lemma 5.2], a regular degree bipartite graph has a perfect matching. Therefore, a k-factor of a bipartite graph G can be partitioned into a collection of k edge-disjoint perfect matchings (1-factors). (ii) to (i) is trivial.

(i) and (iii): the equivalence of (i) and (iii) can be deduced from the max-flow min-cut theorem [10, 11]. Convert $G = (X, Y; E)$ into a network by (a) adding a source vertex s with k multiedges between s and each vertex $x \in X$; (b) adding a sink vertex t with k multiedges between t and each vertex $y \in Y$; and (c) orienting each edge into a directed arc going from s to X, from X to Y, or from Y to t (see Figure 2). Clearly, G has a k-factor $\Leftrightarrow$ the network has a kn-flow from s to t $\Leftrightarrow$ any cut in the network that separates s and t contains at least kn forward edges. For any $S \subseteq X$ and $T \subseteq Y$, consider the cut shown in dashed line in Figure 2, we have

$$k|S| + e_G(X - S, T) + k|Y - T| \geq kn = k|T| + k|Y - T|, \tag{2.1}$$

so that

$$k|S| + e_G(X - S, T) - k|T| \geq 0. \tag{2.2}$$

□

Proof of Theorem 1.1 (By contradiction). Suppose G has no k-factor and $n \geq 4k-4$, we will infer that $t^B(G) \leq f_1$. According to Lemma 2.1, there exist $S \subseteq X$ and $T \subseteq Y$ such that $k|S| + e_G(X-S,T) - k|T| < 0$. Let $s = |S|$ and $t = |T|$. Then

$$e_G(X-S,T) \leq kt - ks - 1. \tag{2.3}$$

Obviously, $t > s$. We can further assume that

$$s + t \leq n. \tag{2.4}$$

Because, if $s+t > n$, then we can let $S' = X - S$ and $T' = Y - T$ and have $|S'| + |T'| < n$, $|S'| > |T'|$, and $k|T'| + e_G(S', Y - T') - k|S'| = k|S| + e_G(X-S,T) - k|T|$. By symmetry, this converts to the case of $s + t \leq n$.

We then have two cases to consider.

Case 1.

$$k(t-s) \leq t. \tag{2.5}$$

If $k = 1$, then $w(G-S) \geq t + 1 - (t - s - 1) = s + 2$ by (2.3). By $t > s$ and (2.4), we have $s \leq m$, where $m = \lfloor (n-1)/2 \rfloor$. Thus

$$t^B(G) \leq \frac{|S|}{w(G-S)} \leq \frac{s}{s+2} \leq \frac{m}{m+2}. \tag{2.6}$$

This completes the proof of Theorem 1.1. (Note that when $k = 1$, we have only Case 1 to consider.) □

Proof of Theorem 1.2 (Continue the proof of Theorem 1.1). Now suppose $k \geq 2$, by (2.5), we have $t \leq ks/(k-1)$. Let $T' = T \cap N(X-S)$. Then by (2.3), $|T'| \leq kt - ks - 1$. Let $T'' = (Y - T) \cup T'$. Then $|T''| \leq n - t + (kt - ks - 1) < n$ and $w(G - T'') \geq n - s + 1$. Therefore,

$$t^B(G) \leq \frac{|T''|}{w(G-T'')} \leq \frac{n + (k-1)t - ks - 1}{n - s + 1}. \tag{2.7}$$

Case 1.1. If $n - s \leq ks/(k-1)$, then we have $s \geq (k-1)n/(2k-1)$. By (2.4) and (2.7),

$$\begin{aligned} t^B(G) &\leq \frac{n + (k-1)(n-s) - ks - 1}{n-s+1} = 2k - 1 - \frac{(k-1)n + 2k}{n-s+1} \\ &\leq 2k - 1 - \frac{(k-1)n + 2k}{n - (k-1)n/(2k-1) + 1} = \frac{(2k-1)(n-1)}{kn + 2k - 1} \leq \frac{(2k-1)(n-1)}{kn+1} = f_1. \end{aligned} \tag{2.8}$$

Case 1.2. If $n - s > ks/(k-1)$, then we have $s < (k-1)n/(2k-1)$. By (2.5) and (2.7),

$$t^B(G) \leq \frac{n-1}{n-s+1} < \frac{n-1}{n-(k-1)n/(2k-1)+1} = \frac{(2k-1)(n-1)}{kn+2k-1} \leq \frac{(2k-1)(n-1)}{kn+1} = f_1. \tag{2.9}$$

Case 2.

$$k(t-s) > t. \tag{2.10}$$

Let d be the unique integer satisfying

$$t \cdot d < k(t-s) \leq (d+1)t. \tag{2.11}$$

By (2.10), $1 \leq d \leq k-1$. By (2.3) and (2.11), there is a vertex $y_0 \in T$ that is adjacent to at most d vertices in $X - S$. Let $T' = Y - \{y_0\}$ so $|T'| = n-1$ and $w(G - T') \geq n - s - d + 1$. By (2.4) and (2.11), we have $s \leq [(k-d)n - 1]/(2k-d)$. Therefore,

$$t^B(G) \leq \frac{n-1}{n-s-d+1} \leq \frac{n-1}{n-((k-d)n-1)/(2k-d)-d+1}. \tag{2.12}$$

Define a function $g(d) = n - [(k-d)n - 1]/(2k-d) - d + 1$. It is easy to verify that, by the assumption of $n \geq 4k-4$, $g(1) \leq g(2)$. Since $g(d)$ is a convex function, it follows that $f(1) \leq f(d)$ for $d > 1$. By (2.12),

$$t^B(G) \leq \frac{n-1}{f(d)} \leq \frac{n-1}{f(1)} = \frac{(2k-1)(n-1)}{(kn+1)} = f_1. \tag{2.13}$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3 (By contradiction). Indeed, we will prove that the result in Theorem 1.3 holds for all $1 \leq k \leq n$. The condition of $n \geq 4k-4$ in Theorem 1.3 is only because that f_2 is not as tight a bound as f_1 when $n < 4k-4$.

Suppose G has no k-factor, we will infer that $t^B(G) \leq f_2$. According to Lemma 2.1, there exist $S \subseteq X$ and $T \subseteq Y$ such that

$$e_G(X - S, T) \leq kt - ks - 1, \tag{2.14}$$

where $s = |S|$ and $t = |T|$. Like in the proof of Theorems 1.1 and 1.2, we can still assume (2.4).

Suppose y_0 is vertex in T that is adjacent to the least number (denoted by d) of vertices in $X-S$. By (2.14), we have $t\cdot d \le kt-ks-1$. Then with (2.4), we further have $s \le [(k-d)n-1]/(2k-d)$. Let $T' = Y - \{y_0\}$, then $|T'| = n-1$ and $w(G-T') \ge n-s-d+1$. Therefore,

$$\begin{aligned} t^B(G) &\le \frac{|T'|}{w(G-T')} \le \frac{n-1}{n-s-d+1} \le \frac{n-1}{n-((k-d)n-1)/(2k-d)-d+1} \\ &= \frac{n-1}{(2k-d)+(kn+1)/(2k-d)-2k+1} \le \frac{n-1}{2\sqrt{kn+1}-2k+1} = f_2. \end{aligned} \tag{2.15}$$

This completes the proof of Theorem 1.3. □

3. Conclusion and future work

We have defined a new invariant in bipartite graphs called bipartite toughness and provided a sharp bound of it for a balanced bipartite graph to have a k-factor, for k from 1 through n. We view this as a big improvement from using toughness to predict k-factors in bipartite graphs, as toughness of a bipartite graph is at most 1 and it cannot predict k-factors for any $k \ge 3$.

There is also research on computational complexity of toughness. In general, recognizing toughness of a graph is NP-hard [12]. Furthermore, 1-tough of graphs is also NP-hard [13], and even 1-tough of bipartite graphs is NP-hard [14] too. Toughness in claw-free ($K_{1,3}$-free) graphs [15], 1-tough in split graphs [14], and toughness in split graphs [16] have been shown in P. In the future, it would be very interesting to determine the complexity of bipartite toughness.

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