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Research Article

Ordered Structures and Projections

M. Yazi

Probability and Statistics Department, Faculty of Mathematics, University of Sciences and Technology USTHB, 16111 Algiers, Algeria

Correspondence should be addressed to M. Yazi, mohyazi@gmail.com

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We associate a covering relation to the usual order relation defined in the set of all idempotent endomorphisms (projections) of a finite-dimensional vector space. A characterization is given of it. This characterization makes this order an order verifying the Jordan-Dedekind chain condition. We give also a property for certain finite families of this order. More precisely, the family of parts intervening in the linear representation of diagonalizable endomorphism, that is, the orthogonal families forming a decomposition of the identity endomorphism.

1. Introduction

In this paper, we consider a poset $\mathcal{P}$, there is a set of all idempotent endomorphisms (projections) of a finite-dimensional vector space E endowed with a reflexive, symmetric, and transitive binary relation (denoted $\leq$) ([1, 2]). Given two elements p and q of $\mathcal{P}$, $p \leq q \Leftrightarrow p \circ q = q \circ p = p$ is equivalent to $\operatorname{Im} p \subseteq \operatorname{Im} q$ and $\ker q \subseteq \ker p$. We say that q covers p (denoted $p \prec q$) if $p \leq q$, $p \neq q$, and the interval $]p, q[$ is empty. So an element p of $\mathcal{P}$ is an atom (resp., a coatom) if p coverers $0_{\mathcal{P}}$ (resp., covered by id_E) where $0_{\mathcal{P}} \equiv$ zero endomorphism (resp., $\mathrm{id}_E \equiv$ identity endomorphism) is the least element (resp., the greatest element) of $\mathcal{P}$. When the bounds of two elements p and q exist we denote their meet (resp., their join) by $p \wedge q$ (resp., $p \vee q$) ([3, 4]). We remark that the usual order relation defined in $\mathcal{P}$ between two elements p and q is expressed by relations of inclusion between their kernels and their ranges, then between elements of E_s: set of subspaces of E which is well known that when this later is endowed with the relation of set inclusion, the couple $(E_s, \subseteq)$ is a geometric lattice ([5–7]) where the covering relation is defined for two subspaces F and G by $F \prec G \Leftrightarrow F \subset G$ and $\dim G = \dim F + 1$.

The extension of the notion of covering constitutes my principal motivation in this paper. Indeed, the main result of this paper, given in Section 2, is a characterization of covering relation defined by means of the one from $(E_s, \subseteq)$ ([6, 8]). We use this result to show that $\mathcal{P}$

is a graded poset with the rank function R defined by $R(p) = \dim \operatorname{Im} p$, for all $p \in \mathcal{P}$; and all maximal chains between the same endpoints have the same finite length ([3, 4]). As a final point, when two elements p and q of $\mathcal{P}$ satisfy $p \circ q = q \circ p$ we observe, as we will show in Section 3, that the poset $\mathcal{P}$ possesses some properties, among other things is the covering property ([5, 6]).

2. Covering relation

Proposition 2.1. *If E is of finite dimension, the ordered set $\mathcal{P}$ verifies the ascending chain condition.*

Proof. Let $(p_i)_{i\in\mathbb{N}}$ be an increasing sequence of elements from $\mathcal{P}$. Then $p_1 \leq p_2 \leq \cdots \leq p_i \leq p_{i+1} \leq \cdots$ is equivalent to $\operatorname{Im} p_1 \subseteq \operatorname{Im} p_2 \subseteq \cdots \subseteq \operatorname{Im} p_i \subseteq \operatorname{Im} p_{i+1} \subseteq \cdots$ and $\cdots \subseteq \ker p_{i+1} \subseteq \ker p_i \subseteq \cdots \subseteq \ker p_2 \subseteq \ker p_1$. Since E is of finite dimension, then $(E_s, \subseteq)$ verifies the ascending chain condition, that is, there exists $i_0 \in \mathbb{N}$ such that $\operatorname{Im} p_{i_0} = \operatorname{Im} p_{i_0+1} = \cdots$ for all $i \geq i_0$. Hence, for all $i \geq i_0$, the elements $\ker p_i$ are two by two isomorph. Since they are two by two comparable, they are equal. Hence, $p_i = p_{i_0}$ for all $i \geq i_0$. □

We will prove later that this proposition is a consequence of Proposition 4.1.

Before stating a theorem, we remark that for two elements p and q of $\mathcal{P}$ such that $p < q$, one has this following equivalence: $\operatorname{Im} p \prec \operatorname{Im} q \Leftrightarrow \ker q \prec \ker p$. From this, we have the following theorem.

Theorem 2.2. *If p and q are two elements of such that $p < q$, the two following properties are equivalent:*

(i) $]\operatorname{Im} p, \operatorname{Im} q[= \varnothing$;

(ii) $]p, q[= \varnothing$.

Proof. (i)⇒(ii) Assume that q does not cover p, that is, there exists in $\mathcal{P}$ an element r such that $p < r < q$, or equivalently, $\operatorname{Im} p \subset \operatorname{Im} r \subset \operatorname{Im} q$ and $\ker q \subset \ker r \subset \ker p$. Hence, we have a contradiction.

Conversely, assume that $\operatorname{Im} q$ does not cover $\operatorname{Im} p$, it follows from the precedent remark that $\ker p$ does not cover $\ker q$.

The chains of $(E_s, \subseteq)$ of respective endpoints $(\operatorname{Im} p, \operatorname{Im} q)$ and $(\ker q, \ker p)$ are then at least of length two. There exists then at least a couple of elements $(E_s, \subseteq)$ such that

$$\begin{gathered} \operatorname{Im} p \subset F \subset \operatorname{Im} q, \\ \ker q \subset G \subset \ker p \end{gathered} \tag{2.1}$$

with $\dim G = \dim \ker p - 1$ and $\dim F = \dim \operatorname{Im} p + 1$.

Amongts these couples, we prove that it exists one which is the direct sum of E, that is $F \oplus G = E$. From the relation $\ker q \subset \ker p$, it follows $\ker p = \ker q \oplus (\operatorname{Im} q \cap \ker p)$. Then $E = \operatorname{Im} p \oplus \ker q \oplus (\operatorname{Im} q \cap \ker p)$.

Set $r_1 = \dim \operatorname{Im} p$, $r_2 = \dim \ker p$, $s_1 = \dim \operatorname{Im} q$, $s_2 = \dim \ker q$ with $s_1 + s_2 = r_1 + r_2 = \dim E$, and $(g_1, g_2, \ldots, g_{s_2})$ a basis of $\ker q$, $(g_{s_2+1}, \ldots, g_{r_2})$ a basis of $\operatorname{Im} q \cap \ker p$ and $(f_1, \ldots, f_{r_1})$ a basis of $\operatorname{Im} p$. If we consider the union of the precedent bases, we obtain obviously a base of E. Thus, the subspaces F and G defined as follows: $F = [f_1, \ldots, f_{r_1}, g_{r_2}]$ subspace generated by $(f_1, \ldots, f_{r_1}, g_{r_2})$, $G = [g_1, \ldots, g_{s_2}, g_{s_2+1}, \ldots, g_{r_2-1}]$ subspace generated by $(g_1, \ldots, g_{r_2-1})$ form, in fact, a couple of E_s for which E is a direct sum, that is, $E = F \oplus G$; moreover, verifying the relations $\operatorname{Im} p \subset F \subset \operatorname{Im} q$ and $\ker q \subset G \subset \ker p$. Thus, the existence of the projection on F along G. Hence, we have a contradiction. □

Corollary 2.3. *The elements of $\mathcal{P}$ for which the image is a line are atoms. Dually, the elements of $\mathcal{P}$ for which the image is a hyperplane are coatoms.*

3. Covering property

Several ordered sets having the property of lattice satisfy the covering property (e.g., $(E_s, \subseteq)$). In the following result, we prove that one also finds it in $\mathcal{P}$ when two arbitrary elements commute.

Proposition 3.1. *Let r and p be two elements of $\mathcal{P}$. If r is an atom such that $p \wedge r = r \wedge p = 0$, then the covering property is verified, that is, $p \prec p \vee r$.*

Dually, if r is a coatom, then $p \wedge r \prec p$.

Proof. It is well known that $p \vee r$ and $p \wedge r$ exist, and we have the following:

$$\begin{gathered} p \vee r = p + r - p \circ r, \quad \text{with } \operatorname{Im}(p \vee r) = \operatorname{Im} p + \operatorname{Im} r, \quad \ker(p \vee r) = \ker p \cap \operatorname{Im} r, \\ p \wedge r = p \circ r, \quad \text{with } \operatorname{Im}(p \wedge r) = \operatorname{Im} p \cap \operatorname{Im} r, \quad \ker(p \wedge r) = \ker p + \ker r. \end{gathered} \tag{3.1}$$

If r is an atom, the subspaces $\operatorname{Im} r$ (resp., $\ker r$) represent in $(E_s, \subseteq)$ an atom (line) (resp., a coatom is a hyperplane in $(E_s, \subseteq)$). So, the assumption $p \wedge r = 0$ implies

$$\ker p + \ker r = E, \tag{3.2}$$

$$\operatorname{Im} p \cap \operatorname{Im} r = \{0\} \tag{3.3}$$

Hence, in $(E_s, \subseteq)$ the equality (3.2) implies $\operatorname{Im} p \prec \operatorname{Im} p + \operatorname{Im} r$ or equivalently $\operatorname{Im} p \prec \operatorname{Im}(p \vee r)$. Furthermore, the equality (3.3) implies $\ker p \cap \ker r \prec \ker p$ or equivalently $\ker(p \vee r) \prec \ker p$. Then from Theorem 2.2 $p \prec p \vee r$.

Using a similar reasoning as above we obtain $p \wedge r \prec p$. □

Proposition 3.2. *Let p, q, s be three elements of $\mathcal{P}$ such that $p \circ q = q \circ p$. If $s \prec p$ and $s \prec q$, $s = p \wedge q$. Dually, if $p \prec s$ and $q \prec s$, $s = p \vee q$.*

Proof. From Theorem 2.2, $s \prec p$ and $s \prec q$ imply $\operatorname{Im} s \prec \operatorname{Im} p$, $\ker p \prec \ker s$ and $\operatorname{Im} s \prec \operatorname{Im} q$, $\ker q \prec \ker s$. So, in $(E_s, \subseteq)$ we have $\operatorname{Im} s = \operatorname{Im} p \cap \operatorname{Im} q$ and $\ker s = \ker p + \ker q$. We have then $s = p \wedge q$. In a similar reasoning we can prove that $s = p \vee q$ if $p \prec s$ and $q \prec s$. □

4. Rank function

Theorem 2.2 allows us to define in $\mathcal{P}$ a rank function $\mathcal{R}$ defined by $\mathcal{R}(p) = \dim \operatorname{Im} p$, $p \in \mathcal{P}$. We can easily prove that

$$\begin{aligned} &\mathcal{R}(0_{\mathcal{P}}) = 0 = \dim 0_{\mathcal{P}} \\ &p \neq q,\ p \leq q \text{ imply } \mathcal{R}(p) < \mathcal{R}(q), \quad p, q \in \mathcal{P} \text{ (strict isotonicity)} \\ &p \prec q \text{ implies } \mathcal{R}(q) = \mathcal{R}(p) + 1. \end{aligned} \tag{4.1}$$

Proposition 4.1. *$\mathcal{P}$ is of finite length and of length equal to the dimension of E.*

Proof. Let a maximal chain of $\mathcal{P}$: $0_{\mathcal{P}} \prec p_1 \prec p_2 \prec \cdots \prec p_i \prec p_{i+1} \prec \cdots \prec \mathrm{id}_E$. By applying $\mathcal{R}$ to this sequence, we get $0 < \mathcal{R}(p_1) = 1 < 2 = \mathcal{R}(p_2) < \cdots < \dim E$, which proves that the length of $\mathcal{P}$ is equal to the dimension of E. □

$\mathcal{P}$ being of finite length, it verifies the ascending and descending chain condition.

Corollary 4.2. *The poset $\mathcal{P}$ with the rank function $\mathcal{R}$ is of finite length, it verifies then the Jordan-chain condition, that is, all maximal chain between the same endpoints have the same finite length.*

Corollary 4.3. *If p and q are two elements of $\mathcal{P}$ such that $p \circ q = q \circ p$, $\mathcal{R}(p \vee q) = \mathcal{R}(p) + \mathcal{R}(q)$.*

Proof. This follows from a property of the dimension of an endomorphism of a finite-dimensional vector space. □

It is well known that when p, q, and r are three elements commuting two by two, each one of them commutes with the supremum and infimum of the two others. We propose to generalize this result in the theorem below.

Theorem 4.4. *If $p_1, p_2, \ldots, p_m$ is a family of m elements of $\mathcal{P}$ commuting two by two, that is, $p_i \circ p_j = p_j \circ p_i$ $i \neq j$, $1 \leq i, j \leq n$, then*

$$\bigvee_{i=1}^{m} p_i = \sum_{i=1}^{m} p_i - \sum_{i \neq j} p_i \circ p_j + \sum_{i \neq j \neq k} p_i \circ p_j \circ p_k - \cdots + (-1)^{m-1} p_1 \circ p_2 \circ \cdots \circ p_m. \tag{4.2}$$

In particular, if this family is orthogonal, then $\vee_{i=1}^{m} p_i = \sum_{i=1}^{m} p_i$.

Proof. For proving the equality we use induction on $m \geq 2$. Since by assumption the elements p_i commuting two by two, we have for $m = 2$, $p_1 \vee p_2 = p_1 + p_2 - p_1 \circ p_2$. For $m = 3$,

$$\begin{aligned} p_1 \vee p_2 \vee p_3 &= (p_1 \vee p_2) \vee p_3 \\ &= (p_1 \vee p_2) + p_3 - (p_1 \vee p_2) \circ p_3 \\ &= p_1 + p_2 - p_1 \circ p_2 + p_3 - p_1 \circ p_3 - p_2 \circ p_3 + p_1 \circ p_2 \circ p_3 \\ &= \sum_{i=1}^{3} p_i - \sum_{1 \leq i \neq j \leq 3} p_i \circ p_j + p_1 \circ p_2 \circ p_3. \end{aligned} \tag{4.3}$$

Assume that the equality is verified until the order $m - 1$,

$$\begin{aligned} \bigvee_{i=1}^{m-1} p_i = \sum_{i=1}^{m-1} p_i - \sum_{1 \leq i \neq j \leq m-1} p_i \circ p_j + \sum_{1 \leq i \neq j \neq k \leq m-1} p_i \circ p_j \circ p_k \\ - \cdots + (-1)^{m-2} p_1 \circ p_2 \circ \cdots \circ p_{m-1}. \end{aligned} \tag{4.4}$$

Then

$$\begin{aligned}
\bigvee_{i=1}^{m} p_i &= \left(\bigvee_{i=1}^{m-1} p_i\right) \vee p_m \\
&= \bigvee_{i=1}^{m-1} p_i + p_m - \left(\bigvee_{i=1}^{m-1} p_i\right) \circ p_m \\
&= \sum_{i=1}^{m-1} p_i - \sum_{1\le i\ne j\le m-1} p_i \circ p_j + \sum_{1\le i\ne j\ne k\le m-1} p_i \circ p_j \circ p_k - \cdots + (-1)^{m-2} p_1 \circ p_2 \circ \cdots \circ p_{m-1} \\
&\quad + p_m - \left(\sum_{i=1}^{m-1} p_i\right) \circ p_m + \sum_{1\le i\ne j\le m-1} (p_i \circ p_j) \circ p_m - \sum_{1\le i\ne j\ne k\le m-1} (p_i \circ p_j \circ p_k) \circ p_m \\
&\quad + \cdots + (-1)^{m-1} p_1 \circ p_2 \circ \cdots \circ p_{m-1} \circ p_m \\
&= \sum_{i=1}^{m-1} p_i - \sum_{1\le i\ne j\le m-1} p_i \circ p_j - \left(\sum_{i=1}^{m-1} p_i\right) \circ p_m + \sum_{1\le i\ne j\ne k\le m-1} p_i \circ p_j \circ p_k \\
&\quad + \sum_{1\le i\ne j\le m-1} (p_i \circ p_j) \circ p_m - \sum_{1\le i\ne j\ne k\le m-1} (p_i \circ p_j \circ p_k) \circ p_m + \cdots + (-1)^{m-1} p_1 \circ p_2 \circ \cdots \circ p_{m-1} \circ p_m.
\end{aligned} \tag{4.5}$$

It follows from the combinatorial relation $C_m^s = C_{m-1}^s + C_{m-1}^{s-1}$ for all $s \le m$ that

$$\sum_{1\le i\ne j\ne k\le m-1} p_i \circ p_j \circ p_k + \left(\sum_{1\le i\ne j\le m-1} p_i \circ p_j\right) \circ p_m = \sum_{1\le i\ne j\ne k\le m} p_i \circ p_j \circ p_k. \tag{4.6}$$

Hence we have the desired equality. □

It is clear that if the family $p_1, p_2, \ldots, p_m$ is orthogonal ($p_i \circ p_j = p_j \circ p_i = 0_{\mathcal{P}}$, $i \ne j$), the equality proved before becomes $\vee_{i=1}^m p_i = \sum_{i=1}^m p_i$. It is well known in Linear algebra that if $(p_1, \ldots, p_m)$ is an orthogonal family verifying in addition $\vee_{i=1}^m p_i = \mathrm{id}_E$, then this family characterizes diagonalizable endomorphisms.

5. Conclusion

The consequence of the preceding theorem can constitute a characterization of a diagonalizable endomorphism of a finite dimensional vector space by means of the ordered set of the idempotent endomorphism in the same space. It is thus natural to ask the following question. Can one in the general case for a given order ϑ of finite length satisfying in addition the condition of Jordan-Dedekind identify a finite family $x_1, \ldots, x_p$ such that $\vee x_i = 1_\vartheta$ and $x_i \wedge x_j = 0_\vartheta$ for $i \ne j$, where 1_ϑ and 0_ϑ are the bounds of ϑ. If the answer is affirmative, can one in this case establish a junction between the Linear algebra and the ordered structures so that the identification of a diagonalizable endomorphism is concrete.

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