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Research Article

Regularity and Green's Relations on a Semigroup of Transformations with Restricted Range

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Let $T(X)$ be the full transformation semigroup on the set X and let $T(X,Y)=\{\alpha\in T(X):X\alpha\subseteq Y\}$. Then $T(X,Y)$ is a sub-semigroup of $T(X)$ determined by a nonempty subset Y of X. In this paper, we give a necessary and sufficient condition for $T(X,Y)$ to be regular. In the case that $T(X,Y)$ is not regular, the largest regular sub-semigroup is obtained and this sub-semigroup is shown to determine the Green's relations on $T(X,Y)$. Also, a class of maximal inverse sub-semigroups of $T(X,Y)$ is obtained.

1. Introduction

Let Y be a nonempty subset of X and let $T(X)$ denote the semigroup of transformations from X into itself. We consider the sub-semigroup of $T(X)$ defined by

$$T(X,Y)=\{\alpha\in T(X):X\alpha\subseteq Y\},\tag{1.1}$$

when $X\alpha$ denotes the range of α. In fact, if $|Y|=1$, then $T(X,Y)$ contains exactly one element (namely, the constant map with range Y) and if $Y=X$, then $T(X,Y)=T(X)$.

In 1975, Symons [1] described all the automorphisms of $T(X,Y)$ and found that the most difficult case occurs when $|Y|=2$. He also determined when $T(X_1,Y_1)$ is isomorphic to $T(X_2,Y_2)$ and, surprisingly, the answer depends on the cardinals $|X_i|$ and $|X_i\setminus Y_i|$, not on $|Y_i|$ for $i=1,2$. Here, we study other algebraic properties of this semigroup. Recall that an element a of a semigroup S is called *regular* if $a=axa$ for some x in S. A semigroup S is *regular* if every element of S is regular. It is already known that $T(X)$ is a regular semigroup (see [[2], page 33]). But $T(X,Y)$ is not regular in general. So, in Section 2, we prove that $T(X,Y)$ is regular if and only if $|Y|=1$ or $Y=X$. We also prove that if $T(X,Y)$ is not regular, the set $F = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$ is the largest regular sub-semigroup of $T(X,Y)$. In Section 3, we characterize the Green's relations on $T(X,Y)$ and find that its $\mathfrak{D}$ and $\mathcal{J}$ relations are surprising, but they reduce to those on $T(X)$ when $Y = X$. And in Section 4, we give a class of maximal inverse sub-semigroups of $T(X,Y)$, of the form $F_a = \{\alpha \in F : a\alpha = a$ and α is injective on $X \setminus a\alpha^{-1}\}$. When $Y = X$, the set $F_a = \{\alpha \in T(X) : a\alpha = a$ and α is injective on $X \setminus a\alpha^{-1}\}$ is a class of maximal inverse sub-semigroups of $T(X)$ given in [4].

Note that throughout the paper, we write functions on the right; in particular, this means that for a composition $\alpha\beta$, α is applied first.

2. Regularity of $T(X,Y)$

To give a necessary and sufficient condition for the semigroup $T(X,Y)$ to be regular, we first note the following.

(1) If $|Y| = 1$, say $Y = \{a\}$, then $T(X,Y)$ contains exactly one element (namely, the constant map X_a with range $\{a\}$), so $T(X,Y)$ is regular.

(2) If $Y = X$, then $T(X,Y) = T(X)$ which is a regular semigroup.

(3) If $|X| \leq 2$, then $|Y| = 1$ or $Y = X$, and $T(X,Y)$ is regular by (1) and (2).

Now, we need some notation. We adopt the convention introduced in ([[3], page 241]), namely, if $\alpha \in T(X,Y)$, then we write

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}, \tag{2.1}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

Theorem 2.1. *$T(X,Y)$ is a regular semigroup if and only if $|Y| = 1$ or $Y = X$.*

Proof. Assume that $|Y| \neq 1$ and $Y \neq X$. Let $a, b \in Y$ be such that $a \neq b$ and choose $c \in X \setminus Y$. Let

$$\alpha = \begin{pmatrix} X_i \\ y_i \end{pmatrix} \tag{2.2}$$

be any element in $T(X,Y)$, where $X\alpha = \{y_i\} \subseteq Y$ and $X_i = y_i\alpha^{-1}$.

We define $\beta = \begin{pmatrix} c & X\setminus\{c\} \\ a & b \end{pmatrix}$, and it is clear that $\beta \in T(X,Y)$.

Since $c \notin Y$, so $y_i \neq c$ for all i, and

$$\alpha\beta = \begin{pmatrix} X_i \\ y_i \end{pmatrix} \begin{pmatrix} c & X \setminus \{c\} \\ a & b \end{pmatrix} = \begin{pmatrix} X \\ b \end{pmatrix} \neq \beta. \tag{2.3}$$

So, we conclude that $\alpha\beta \neq \beta$ for all $\alpha \in T(X,Y)$, this implies that β is a nonregular element in $T(X,Y)$. Therefore, $T(X,Y)$ is not a regular semigroup. The converse is clear by the previous note. □

Now, we consider the set

$$F = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}. \tag{2.4}$$

It is easy to see that $F = \{\alpha \in T(X,Y) : (X \setminus Y)\alpha \subseteq Y\alpha\} = \{\alpha \in T(X,Y) : X\alpha = Y\alpha\}$. Since $Y \neq \varnothing$, there exists $a \in Y$ and we see that the constant map X_a with range $\{a\}$ satisfies the condition in F, therefore, $X_a \in F$ and so $F \neq \varnothing$. And for each $\alpha \in F$ and $\beta \in T(X,Y)$, we have $X\alpha\beta = (X\alpha)\beta \subseteq (Y\alpha)\beta = Y\alpha\beta$, and thus $\alpha\beta \in F$. This proves the following.

Lemma 2.2. *F is a right ideal of $T(X,Y)$.*

In general, F is not a left ideal of $T(X,Y)$ as shown in the following example.

Example 2.3. Let $X = \mathbb{N}$ denote the set of positive integers, let Y denote the set of all positive even integers, and define

$$\alpha = \begin{pmatrix} n \\ 2n \end{pmatrix}, \qquad \beta = \begin{pmatrix} 2n & X \setminus Y \\ 2n & 2 \end{pmatrix}. \tag{2.5}$$

Then $\alpha \in T(X,Y) \setminus F$ and $\beta \in F$, but $\alpha\beta = \alpha \notin F$. Thus F is not a left ideal of $T(X,Y)$.

Theorem 2.4. *F is the largest regular sub-semigroup of $T(X,Y)$.*

Proof. From Lemma 2.2, we see that F is a sub-semigroup of $T(X,Y)$. Let $\alpha \in F$ and write

$$\alpha = \begin{pmatrix} x_i\alpha^{-1} \\ x_i \end{pmatrix}, \tag{2.6}$$

where $\bigcup_{i\in I} x_i\alpha^{-1} = X$ and $X\alpha = \{x_i : i \in I\} = Y\alpha$. For each $x \in Y\alpha$, choose $d_x \in x\alpha^{-1} \bigcap Y$, so $d_x\alpha = x$, and $d_y \neq d_z$, for all $y, z \in Y\alpha$ such that $y \neq z$. Choose $k \in I$ and let $J = I \setminus \{k\}$. Define

$$\beta = \begin{pmatrix} x_j & X \setminus \{x_j\} \\ d_{x_j} & d_{x_k} \end{pmatrix}, \tag{2.7}$$

where $\{x_j : j \in J\} = Y\alpha \setminus \{x_k\}$. Then $\beta \in T(X,Y)$ and $\alpha\beta\alpha = \alpha$. Since $X\beta = \{d_{x_i} : i \in I\} \subseteq (Y\alpha)\beta \subseteq Y\beta$, we have $\beta \in F$. Hence F is a regular sub-semigroup of $T(X,Y)$. Now, let α be any regular element in $T(X,Y)$. Then $\alpha\beta\alpha = \alpha$, for some $\beta \in T(X,Y)$, so $X\alpha = X\alpha\beta\alpha = (X\alpha\beta)\alpha \subseteq Y\alpha$, and thus $\alpha \in F$. Therefore, F is the largest regular sub-semigroup of $T(X,Y)$ as required. □

Note that if $Y = \{a\}$, then for each $\alpha \in T(X,Y)$, we have $X\alpha = \{a\} = Y\alpha$ which implies that $F = T(X,Y)$ consists of only one element and so is a regular semigroup, and if $X = Y$, then $F = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\} = \{\alpha \in T(X) : X\alpha \subseteq X\alpha\} = T(X)$ which is also a regular semigroup.

3. Green's relations on $T(X,Y)$

Let S be a semigroup. Then we define S^1 to be a semigroup of adding an identity to S if S does not already have an identity element in it and $S^1 = S$ if S contains an identity. The following definitions are due to J. A. Green. For any $a, b \in S$, we define

$$a\mathcal{L}b \quad \text{iff } S^1 a = S^1 b, \tag{3.1}$$

or equivalently; $a\mathcal{L}b$ if and only if $a = xb,\ b = ya$ for some $x, y \in S^1$.

Dually, we define

$$a\mathcal{R}b \text{ to means } aS^1 = bS^1, \tag{3.2}$$

or equivalently; $a\mathcal{R}b$ if and only if $a = bx,\ b = ay$ for some $x, y \in S^1$.

And we define

$$a\mathcal{J}b \text{ to means } S^1 aS^1 = S^1 bS^1, \tag{3.3}$$

or equivalently; $a\mathcal{J}b$ if and only if $a = xby,\ b = uav$ for some $x, y, u, v \in S^1$.

Finally, we define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathfrak{D} = \mathcal{L} \circ \mathcal{R}$.

In [2, 3], Clifford and Preston characterized Green's relations on the full transformation semigroup $T(X)$, where X is an arbitrary set. They proved that

$$\begin{aligned} \alpha\mathcal{L}\beta &\quad \text{iff } X\alpha = X\beta, \\ \alpha\mathcal{R}\beta &\quad \text{iff } \pi_\alpha = \pi_\beta. \end{aligned} \tag{3.4}$$

Here, we do the same for the semigroup $T(X,Y)$ and we obtain results which generalize the same results on $T(X)$.

Lemma 3.1. *Let $\alpha, \beta \in T(X,Y)$. If $\beta \in F$, then $X\alpha \subseteq X\beta$ if and only if $\alpha = \gamma\beta$ for some $\gamma \in T(X,Y)$.*

Proof. Let β be an element of F. It is clear that if $\alpha = \gamma\beta$ for some $\gamma \in T(X,Y)$, then $X\alpha \subseteq X\beta$. Now, we assume that $X\alpha \subseteq X\beta$ and write

$$\alpha = \begin{pmatrix} a_i\alpha^{-1} \\ a_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} b_j\beta^{-1} \\ b_j \end{pmatrix}, \tag{3.5}$$

where $\{a_i\} \subseteq \{b_j\}$. For each $a \in X\alpha \subseteq X\beta \subseteq Y\beta$ (since $\beta \in F$), we get $a = y\beta$ for some $y \in Y$ which implies $y \in a\beta^{-1}$ and thus $y \in Y \cap a\beta^{-1} \neq \varnothing$. Choose $d_a \in Y \cap a\beta^{-1}$, so $d_a \in Y$ and $d_a\beta = a$. Since $X = \bigcup_{a_i \in X\alpha} a_i\alpha^{-1}$ is the disjoint union of the $a_i\alpha^{-1}$, we can define

$$\gamma = \begin{pmatrix} a_i\alpha^{-1} \\ d_{a_i} \end{pmatrix}. \tag{3.6}$$

Then $\gamma \in T(X,Y)$ and $\gamma\beta = \alpha$. □

From now on, the notations L_α $(R_\alpha, H_\alpha, D_\alpha)$ denote the set of all elements of $T(X,Y)$ which are $\mathcal{L}$-related ($\mathcal{R}$-related, $\mathcal{H}$-related, $\mathcal{D}$-related) to α, where $\alpha \in T(X,Y)$.

Theorem 3.2. *For $\alpha \in T(X,Y)$, the following statements hold.*

(1) *If $\alpha \in F$, then $L_\alpha = \{\beta \in F : X\alpha = X\beta\}$.*

(2) *If $\alpha \in T(X,Y) \setminus F$, then $L_\alpha = \{\alpha\}$.*

Proof. Let α be any element in $T(X,Y)$ and let $\beta \in L_\alpha$. Then $\alpha\mathcal{L}\beta$ which implies that $\alpha = \alpha'\beta$ and $\beta = \beta'\alpha$, for some $\alpha', \beta' \in T(X,Y)^1$.

(1) Assume that $\alpha \in F$. If $\beta = \alpha$, then $\beta \in F$ and $X\alpha = X\beta$. If $\beta \neq \alpha$, then α' and β' both belong to $T(X,Y)$. Thus $X\beta = (X\beta'\alpha')\beta \subseteq Y\beta$, and hence $\beta \in F$. From $\alpha \in F$ and $\beta = \beta'\alpha$, we get $X\beta \subseteq X\alpha$ by Lemma 3.1. Similarly, from $\beta \in F$ and $\alpha = \alpha'\beta$, we get $X\alpha \subseteq X\beta$. Therefore, $X\alpha = X\beta$. Now, if $\gamma \in F$ and $X\alpha = X\gamma$, then it is clear by Lemma 3.1 that $\gamma \in L_\alpha$.

(2) Assume that $\alpha \in T(X,Y) \setminus F$. If $\alpha', \beta' \in T(X,Y)$, then $X\alpha = X(\alpha'\beta) = X(\alpha'(\beta'\alpha)) = (X\alpha'\beta')\alpha \subseteq Y\alpha$. Thus $\alpha \in F$ which is a contradiction, so $\alpha' = 1$ or $\beta' = 1$ and $\beta = \alpha$. □

We note that for any $\alpha \in T(X,Y)$, $\pi_\alpha = \{(a,b) \in X \times X : a\alpha = b\alpha\}$ is an equivalence on X and $|X/\pi_\alpha| = |X\alpha|$. The relation π_α is usually called the *kernel* of α.

Theorem 3.3. *Let $\alpha, \beta \in T(X,Y)$. Then $\pi_\beta \subseteq \pi_\alpha$ if and only if $\alpha = \beta\gamma$ for some $\gamma \in T(X,Y)$. Hence $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$.*

Proof. It is clear that if $\alpha = \beta\gamma$ for some $\gamma \in T(X,Y)$, then $\pi_\beta \subseteq \pi_\alpha$. Now, suppose that $\pi_\beta \subseteq \pi_\alpha$. If $x \in X\beta$, then $x = z\beta$ for some $z \in X$, so we define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} z\alpha, & \text{if } x \in X\beta, \\ x\beta, & \text{if } x \in X \setminus X\beta. \end{cases} \tag{3.7}$$

Then γ is well defined (since $\pi_\beta \subseteq \pi_\alpha$) and $\gamma \in T(X,Y)$. For each $x \in X$, let $y = x\beta \in X\beta$, so $x\beta\gamma = (x\beta)\gamma = y\gamma = x\alpha$ by the definition of γ. Thus $\alpha = \beta\gamma$ as required, and the remaining assertion is clear. □

Lemma 3.4. *Let $\alpha, \beta \in T(X,Y)$. If $\pi_\alpha = \pi_\beta$ then either both α and β are in F, or neither is in F.*

Proof. Assume that $\pi_\alpha = \pi_\beta$ and suppose that $\alpha, \beta \in F$ is false. So one of α or β is not in F, we suppose that $\alpha \notin F$. Thus $(X \setminus Y)\alpha \nsubseteq Y\alpha$, so there is $x_0 \in X \setminus Y$ such that $x_0\alpha \neq y\alpha$ for all $y \in Y$. Thus $(x_0, y) \notin \pi_\alpha$ for all $y \in Y$. If $\beta \in F$, then $X\beta = Y\beta$, so $(x_0, y) \in \pi_\beta$ for some $y \in Y$ which contradicts $\pi_\alpha = \pi_\beta$. Therefore, $\beta \notin F$. □

Using Theorem 3.3 and Lemma 3.4, we have the following corollary.

Corollary 3.5. *For $\alpha \in T(X,Y)$, the following statements hold.*

(1) *If $\alpha \in F$, then $R_\alpha = \{\beta \in F : \pi_\alpha = \pi_\beta\}$.*

(2) *If $\alpha \in T(X,Y) \setminus F$, then $R_\alpha = \{\beta \in T(X,Y) \setminus F : \pi_\alpha = \pi_\beta\}$.*

As a direct consequence of Theorems 3.2 and 3.3, we have the following.

Theorem 3.6. *For $\alpha \in T(X,Y)$, the following statements hold.*

(1) *If $\alpha \in F$, then $H_\alpha = \{\beta \in F : X\alpha = X\beta$ and $\pi_\alpha = \pi_\beta\}$.*

(2) *If $\alpha \in T(X,Y) \setminus F$, then $H_\alpha = \{\alpha\}$.*

In [2, 3], volume 1, Clifford and Preston proved that two elements of $T(X)$ are $\mathfrak{D}$-related if and only if they have the same rank (i.e., the ranges of the two elements have the same cardinality). But for $T(X,Y)$, we have the following.

Theorem 3.7. *For $\alpha \in T(X,Y)$, the following statements hold.*

(1) *If $\alpha \in F$, then $D_\alpha = \{\beta \in F : |X\alpha| = |X\beta|\}$.*

(2) *If $\alpha \in T(X,Y) \setminus F$, then $D_\alpha = \{\beta \in T(X,Y) \setminus F : \pi_\alpha = \pi_\beta\}$.*

Proof. Let α be any element in $T(X,Y)$ and let $\beta \in D_\alpha$. Then $\alpha\mathcal{L}\gamma$ and $\gamma\mathcal{R}\beta$ for some $\gamma \in T(X,Y)$.

(1) If $\alpha \in F$, then since $\alpha\mathcal{L}\gamma$, we must have $\gamma \in F$ and $X\alpha = X\gamma$. From $\gamma\mathcal{R}\beta$, we get $\pi_\gamma = \pi_\beta$ and $\beta = \gamma\lambda$ for some $\lambda \in T(X,Y)^1$. Since F is a right ideal of $T(X,Y)$, so $\beta \in F$. And $|X\alpha| = |X\gamma| = |X/\pi_\gamma| = |X/\pi_\beta| = |X\beta|$. Conversely, assume that $\lambda \in F$ and $|X\alpha| = |X\lambda|$. Then there is a bijection $\theta : X\lambda \to X\alpha$. We let $\mu = \lambda\theta$, then $\mu \in T(X,Y)$ and $X\mu = X\lambda\theta = (X\lambda)\theta = X\alpha$. Since $\lambda \in F$ implies $X\lambda \subseteq Y\lambda$, so $X\mu = X\lambda\theta \subseteq Y\lambda\theta = Y\mu$, hence $\mu \in F$. Since $\alpha, \mu \in F$ and $X\alpha = X\mu$, so $\alpha\mathcal{L}\mu$ by Theorem 3.2. Now, since $\mu = \lambda\theta$ and θ is injective on $X\lambda$, we get $\pi_\mu = \pi_\lambda$, so $\mu\mathcal{R}\lambda$. Therefore, α and λ are $\mathfrak{D}$-related and $\lambda \in D_\alpha$.

(2) If $\alpha \in T(X,Y) \setminus F$, then $\gamma = \alpha$ (since $\alpha\mathcal{L}\gamma$) and thus $\alpha\mathcal{R}\beta$ which implies that $\pi_\alpha = \pi_\beta$. So by Lemma 3.4 we must have $\beta \in T(X,Y) \setminus F$. The other containment is clear since $\mathcal{R} \subseteq \mathfrak{D}$. □

In order to characterize the $\mathcal{J}$-relation on $T(X,Y)$, the following lemma is needed.

Lemma 3.8. *Let $\alpha, \beta \in T(X,Y)$. If $\alpha = \lambda\beta\mu$ for some $\lambda \in T(X,Y)$ and $\mu \in T(X,Y)^1$, then $|X\alpha| \le |Y\beta|$.*

Proof. If $\alpha = \lambda\beta\mu$ for some $\lambda \in T(X,Y)$ and $\mu \in T(X,Y)^1$. Then $X\lambda \subseteq Y$, which implies that $(X\lambda)\beta \subseteq Y\beta$ and so $|X\lambda\beta| \le |Y\beta|$. If $\mu = 1$, then $\alpha = \lambda\beta$ and so $|X\alpha| = |X\lambda\beta| \le |Y\beta|$. If $\mu \in T(X,Y)$, then $|X\alpha| = |X(\lambda\beta\mu)| = |(X\lambda\beta)\mu| \le |X\lambda\beta| \le |Y\beta|$. Thus $|X\alpha| \le |Y\beta|$ as required. □

Theorem 3.9. *Let $\alpha, \beta \in T(X,Y)$. Then*

$$\alpha\mathcal{J}\beta \quad \text{iff } \pi_\alpha = \pi_\beta \text{ or } |X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|. \tag{3.8}$$

Proof. First, assume that $\alpha\mathcal{J}\beta$. Then $\alpha = \gamma\beta\lambda$ and $\beta = \gamma'\alpha\lambda'$ for some $\gamma, \lambda, \gamma', \lambda' \in T(X,Y)^1$. If $\gamma = 1 = \gamma'$, then $\alpha = \beta\lambda$ and $\beta = \alpha\lambda'$ which imply $\alpha\mathcal{R}\beta$ and thus $\pi_\alpha = \pi_\beta$. If $\gamma \in T(X,Y)$ or $\gamma' \in T(X,Y)$, then we conclude that $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$ for some $\sigma, \sigma' \in T(X,Y)$ and $\delta, \delta' \in T(X,Y)^1$. For example, if $\gamma = 1$ and $\gamma' \in T(X,Y)$, then $\alpha = \beta\lambda$ and $\beta = \gamma'\alpha\lambda'$ imply $\alpha = \beta\lambda = (\gamma'\alpha\lambda')\lambda = \gamma'\alpha(\lambda'\lambda) = \gamma'(\beta\lambda)\lambda'\lambda = \gamma'\beta(\lambda\lambda'\lambda)$. By using Lemma 3.8, we get that $|Y\beta| \ge |X\alpha| \ge |Y\alpha| \ge |X\beta| \ge |Y\beta|$, so it follows that $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$.

Conversely, if $\pi_\alpha = \pi_\beta$, then $\alpha\mathcal{R}\beta$ which implies that $\alpha\mathcal{J}\beta$ since $\mathcal{R} \subseteq \mathcal{J}$. If $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$, then by applying Lemma 2.7 in [2, 3], to $|X/\pi_\alpha| = |X\alpha| = |Y\beta|$ and $|X/\pi_\beta| = |X\beta| = |Y\alpha|$, we get there are $\gamma, \lambda \in T(X,Y)$ such that $\pi_\gamma = \pi_\alpha$, $X\gamma = Y\beta$; and $\pi_\lambda = \pi_\beta$, $X\lambda = Y\alpha$. From

$\pi_\gamma = \pi_\alpha$ and $\pi_\lambda = \pi_\beta$, we get $\gamma\mathcal{R}\alpha$ and $\lambda\mathcal{R}\beta$, so $\alpha = \gamma\gamma'$ and $\beta = \lambda\lambda'$ for some $\gamma', \lambda' \in T(X,Y)^1$. And from $X\gamma = Y\beta$, we write $Y\beta = \{y_i : i \in I\}$, so

$$\gamma = \begin{pmatrix} y_i\gamma^{-1} \\ y_i \end{pmatrix}. \tag{3.9}$$

For each $i \in I$, choose $a_i \in y_i\beta^{-1} \cap Y$ and define $\beta' : X \to X$ by

$$\beta' = \begin{pmatrix} y_i\gamma^{-1} \\ a_i \end{pmatrix}. \tag{3.10}$$

Then $\beta' \in T(X,Y)$ and $\gamma = \beta'\beta$. Similarly, from $X\lambda = Y\alpha$, we can prove that $\lambda = \alpha'\alpha$ for some $\alpha' \in T(X,Y)$.

Therefore, $\alpha = \gamma\gamma' = \beta'\beta\gamma'$ and $\beta = \lambda\lambda' = \alpha'\alpha\lambda'$ which implies that $\alpha\mathcal{J}\beta$ as required. □

Recall that $\mathfrak{D} \subseteq \mathcal{J}$ on any semigroup and $\mathfrak{D} = \mathcal{J}$ on $T(X)$; but in $T(X,Y)$, this is not always true, as shown in the following example.

Example 3.10. Let $X = \mathbb{N}$ denote the set of positive integers and let Y denote the set of all positive even integers. Then we define

$$\alpha = \begin{pmatrix} n \\ 2n \end{pmatrix}, \qquad \beta = \begin{pmatrix} 2n & X \setminus Y \\ 4n & 2 \end{pmatrix}. \tag{3.11}$$

Hence $\alpha, \beta \in T(X,Y) \setminus F$ and $|X\alpha| = |Y\alpha| = \aleph_0 = |Y\beta| = |X\beta|$, so $\alpha\mathcal{J}\beta$. Since $\pi_\alpha \neq \pi_\beta$, we have α and β are not $\mathfrak{D}$-related on $T(X,Y)$.

As a consequence of Theorems 3.7 and 3.9, we see that $\mathfrak{D} = \mathcal{J}$ on the sub-semigroup F of $T(X,Y)$.

Corollary 3.11. *If $\alpha, \beta \in F$, then $\alpha\mathcal{J}\beta$ on $T(X,Y)$ if and only if $\alpha\mathfrak{D}\beta$ on $T(X,Y)$.*

Proof. In general, we have $\mathfrak{D} \subseteq \mathcal{J}$. Let $\alpha, \beta \in F$ and $\alpha\mathcal{J}\beta$ on $T(X,Y)$. Then $\pi_\alpha = \pi_\beta$ or $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$. If $\pi_\alpha = \pi_\beta$, then $|X\alpha| = |X/\pi_\alpha| = |X/\pi_\beta| = |X\beta|$. Thus, both cases imply $|X\alpha| = |X\beta|$ and $\alpha\mathfrak{D}\beta$ on $T(X,Y)$ by Theorem 3.7. □

If we replace Y with X in the above corollary, we then get $\mathfrak{D} = \mathcal{J}$ on $T(X,Y) = T(X)$. Next, we will consider the case when Y is a finite subset of X.

Theorem 3.12. *If Y is a finite subset of X, then $\mathfrak{D} = \mathcal{J}$ on $T(X,Y)$.*

Proof. Let Y be a finite subset of X and let $\alpha, \beta \in T(X,Y)$ be such that α and β are $\mathcal{J}$-related. Then $\pi_\alpha = \pi_\beta$ or $|X\alpha| = |Y\alpha| = |Y\beta| = |X\beta|$. We note that if $\alpha \notin F$, then $|X\alpha| > |Y\alpha|$. For if $|X\alpha| = |Y\alpha|$, then $X\alpha$ is a finite set (since Y is finite) which implies that $X\alpha = Y\alpha$ and thus $\alpha \in F$. Now, if $\alpha \in F$ and $\beta \notin F$, then $|X\alpha| = |Y\alpha|$ but $|Y\beta| < |X\beta|$, so $\pi_\alpha = \pi_\beta$ which contradicts Lemma 3.4. Therefore, either both α and β are in F, or neither is in F. If $\alpha, \beta \in F$, then $\alpha\mathfrak{D}\beta$ by Corollary 3.11. If $\alpha, \beta \notin F$, then $|X\alpha| > |Y\alpha|$ which implies that $\pi_\alpha = \pi_\beta$ and thus $\alpha\mathfrak{D}\beta$ by Theorem 3.7. Therefore, $\mathcal{J} \subseteq \mathfrak{D}$ and the other containment is clear. □

4. Maximal inverse sub-semigroups on $T(X,Y)$

We first recall that a semigroup is said to be an *inverse semigroup* if it is regular and any two idempotents commute. In this section, we give one class of maximal inverse sub-semigroups on $T(X,Y)$. If $|Y| = 1$, then there is only one element in $T(X,Y)$, the constant map. Hence, in this case, there is no maximal inverse sub-semigroup on $T(X,Y)$. Therefore, from now on, we assume that $|Y| \geq 2$.

In 1976, Nichols [4] gave a class of maximal inverse sub-semigroups of $T(X)$. Later in 1978, Reilly [5] generalized Nichols' result. Here, with some mild modifications of the proof given in [4], we get one class of maximal inverse sub-semigroups of $T(X,Y)$ which generalizes Nichols' result.

Let X be a set and Y a nonempty subset of X. For each $a \in Y$, define

$$F_a = \{\alpha \in F : a\alpha = a \text{ and } \alpha \text{ is injective on } X \setminus a\alpha^{-1}\}. \tag{4.1}$$

We see that $F_a \neq \varnothing$, since the constant map $X_a \in F_a$. To describe maximal inverse sub-semigroups on $T(X,Y)$, we first prove the following.

Theorem 4.1. *Let $\alpha \in T(X,Y)$ and $a \in Y$. Then $\alpha \in F_a$ if and only if $\{a\} \cup (X \setminus Y) \subseteq a\alpha^{-1}$ and α is injective on $Y \setminus a\alpha^{-1}$.*

Proof. If $Y = X$, then $X \setminus Y = \varnothing$ and $F = T(X)$, thus $\alpha \in F_a$ if and only if $\alpha \in T(X)$, $\{a\} \subseteq a\alpha^{-1}$ and α is injective on $X \setminus a\alpha^{-1}$.

Now, we prove for the case $Y \neq X$. Assume that $\alpha \in F_a$. So $\alpha \in F$, $a\alpha = a$ and α is injective on $X \setminus a\alpha^{-1}$. We show that $(X \setminus Y)\alpha = \{a\}$. Let $b \in (X \setminus Y)\alpha$, so there exists $x \in X \setminus Y$ such that $x\alpha = b$. Thus $b \in (X \setminus Y)\alpha \subseteq Y\alpha$ since $\alpha \in F$. Hence $b = y\alpha$ for some $y \in Y$. So, $x, y \in b\alpha^{-1}$ and $x \neq y$. By the definition of F_a, we must have $b = a$. Therefore, $\{a\} \cup (X \setminus Y) \subseteq a\alpha^{-1}$ and α is injective on $Y \setminus a\alpha^{-1}$. Conversely, assume that the conditions hold. Since $a \in Y$ and $\{a\} \subseteq a\alpha^{-1}$, we get $a = a\alpha \in Y\alpha$, and thus $\{a\} \subseteq Y\alpha$. So $(X \setminus Y)\alpha = \{a\} \subseteq Y\alpha$ and therefore $\alpha \in F$. Since α is injective on $Y \setminus a\alpha^{-1}$ and $Y \setminus a\alpha^{-1} = X \setminus a\alpha^{-1}$, it follows that α is injective on $X \setminus a\alpha^{-1}$, and so $\alpha \in F_a$. □

Recall that for each $\alpha \in T(X)$, α is an idempotent in $T(X)$ if and only if $x\alpha = x$ for all $x \in X\alpha$. Since $T(X,Y)$ is a sub-semigroup of $T(X)$, we conclude that α is an idempotent in $T(X,Y)$ if and only if $x\alpha = x$ for all $x \in X\alpha$. And by Theorem 4.1, if α is an idempotent in F_a, then $x\alpha = x$ for all $x \in Y \setminus a\alpha^{-1}$.

Lemma 4.2. *Let L be a regular sub-semigroup of $T(X,Y)$ such that $F_a \subsetneq L$ and suppose that $\alpha \in L \setminus F_a$. Then*

(i) *$\alpha \mathcal{L} \beta$ on L for some idempotent $\beta \in F_a$.*

(ii) *If $a\alpha = a$, then α is not $\mathcal{R}$-related on L to any element in F_a.*

Proof. (i) We write

$$\alpha = \begin{pmatrix} a\alpha^{-1} & A_i \\ a & a_i \end{pmatrix}, \tag{4.2}$$

where $\bigcup A_i = X \setminus a\alpha^{-1}$, and define $\beta \in T(X,Y)$ by

$$\beta = \begin{pmatrix} a_i & X \setminus B \\ a_i & a \end{pmatrix}, \tag{4.3}$$

where $\{a_i\} = B$. Then $\beta : X \to Y$, is an identity map on B and $X \setminus B = a\beta^{-1}$. So $B = X \setminus a\beta^{-1}$ and β is injective on $X \setminus a\beta^{-1} = B$. Since $a \in X \setminus B$, so $a = a\beta \in Y\beta$ and $\{a\} \subseteq Y\beta$. From $B = \{a_i\} \subseteq Y$, we get $(X \setminus Y)\beta \subseteq (X \setminus B)\beta = \{a\} \subseteq Y\beta$, so $\beta \in F$. Thus $\beta \in F_a$ and it is also an idempotent. From the fact that $L \subseteq F$ is a regular sub-semigroup of $T = T(X,Y)$, it follows from Hall's theorem that $\mathcal{L}^L = \mathcal{L}^T \cap (L \times L) \subseteq \mathcal{L}^T \cap (F \times F)$, where $(a,b) \in \mathcal{L}^S$ means there exist s,t in S^1 such that $a = sb,\ b = ta$. Since $X\alpha = X\beta$, we must have by Theorem 3.2 that $\alpha\mathcal{L}\beta$ on L.

(ii) Assume that $a\alpha = a$ and suppose that $\alpha\mathcal{R}\beta$ on L for some $\beta \in F_a$. Thus $\pi_\alpha = \pi_\beta$ by Theorem 3.3, and hence $a\alpha^{-1} = a\beta^{-1}$. Now, let $x_1, x_2 \in X \setminus a\alpha^{-1}$ be such that $x_1\alpha = x_2\alpha$. Then $(x_1,x_2) \in \pi_\alpha$ implies $(x_1,x_2) \in \pi_\beta$ (since $\pi_\alpha = \pi_\beta$), so $x_1\beta = x_2\beta$ and since β is injective on $X \setminus a\beta^{-1} = X \setminus a\alpha^{-1}$, we get $x_1 = x_2$ which implies that α is injective on $X \setminus a\alpha^{-1}$, which contradicts $\alpha \notin F_a$. Therefore, α is not $\mathcal{R}$-related to any element in F_a. □

Theorem 4.3. *F_a is a maximal inverse sub-semigroup of $T(X,Y)$.*

Proof. First, we prove that F_a is a sub-semigroup of $T(X,Y)$.

Let $\alpha,\ \beta$ be elements in F_a. Then $\alpha,\beta \in F,\ a\alpha = a = a\beta$, and $\alpha,\ \beta$ are injective on $X \setminus a\alpha^{-1}$ and $X \setminus a\beta^{-1}$, respectively. Since F is a right ideal of $T(X,Y)$, it follows that $\alpha\beta \in F$. Clearly $a(\alpha\beta) = a$, and $\alpha\beta$ is injective on $X \setminus a(\alpha\beta)^{-1}$. Therefore, $\alpha\beta \in F_a$.

Next, we show that F_a is a regular sub-semigroup of $T(X,Y)$. For each $\alpha \in F_a$, $a\alpha = a$ and $|x\alpha^{-1}| = 1$ for all $x \in Y\alpha \setminus \{a\}$ (see Theorem 4.1). Let $\{x_i\} = Y\alpha \setminus \{a\}$ and write $x_i\alpha^{-1} = y_i$ for all i, thus

$$\alpha = \begin{pmatrix} y_i & a\alpha^{-1} \\ x_i & a \end{pmatrix}, \tag{4.4}$$

where $\bigcup\{y_i\} = Y \setminus a\alpha^{-1}$, $X = Y \bigcup a\alpha^{-1}$ and define $\beta \in T(X,Y)$ by

$$\beta = \begin{pmatrix} x_i & A \\ y_i & a \end{pmatrix}, \tag{4.5}$$

where $\{x_i\} = Y\alpha \setminus \{a\}$ and $A = X \setminus \{x_i\}$. Since $a \in A$, we get $a = a\beta \in Y\beta$, and so $\{a\} \subseteq Y\beta$. And for each $x \in X \setminus Y$, $x \neq x_i$ for all i since $x_i \in Y$. Thus $(X \setminus Y)\beta = \{a\} \subseteq Y\beta$, and so $\beta \in F$. Since $a\beta = a$ and β is injective on $\{x_i\} = X \setminus a\beta^{-1}$, it follows that $\beta \in F_a$. And, it is clear that $\alpha = \alpha\beta\alpha$.

Now, we prove that any two idempotents in F_a commute, which is enough to show that F_a is an inverse semigroup. Assume that $\alpha,\ \beta$ are idempotents in F_a. Then $x\alpha = x$ for all $x \in X \setminus a\alpha^{-1}$ and $x\beta = x$ for all $x \in X \setminus a\beta^{-1}$. Let x be any element in X.

Case 1. $x \in X \setminus a\alpha^{-1}$. Then $x\alpha = x$. So, if $x \in X \setminus a\beta^{-1}$, we get $x\beta = x$ and $x(\alpha\beta) = (x\alpha)\beta = x\beta = x = x\alpha = (x\beta)\alpha = x(\beta\alpha)$. But if $x \in a\beta^{-1}$, then $x\beta = a$, and $x(\alpha\beta) = (x\alpha)\beta = x\beta = a = a\alpha = (x\beta)\alpha = x(\beta\alpha)$. Thus in this case $\alpha\beta = \beta\alpha$.

Case 2. $x \in a\alpha^{-1}$. Then $x\alpha = a$. So by the same proof as given in Case 1, we get $\alpha\beta = \beta\alpha$.
Therefore, F_a is an inverse sub-semigroup of $T(X, Y)$.

To prove the maximality, we suppose that F_a is properly contained in an inverse sub-semigroup L of $T(X, Y)$, where $L \subseteq F \subseteq T(X, Y)$ and let $\alpha \in L \backslash F_a$. Let β be the constant map with range $\{a\}$, so β is an idempotent in F_a, and thus $\beta\alpha$ is an idempotent in L. Since $\beta\alpha\beta = \beta$ and every two idempotents in L commute, it follows that $\beta = \beta\alpha\beta = \beta\beta\alpha = \beta\alpha$ and $a\alpha = (a\beta)\alpha = a(\beta\alpha) = a\beta = a$. Since L is regular, $\alpha = \alpha\alpha'\alpha$ for some $\alpha' \in L$ and $\alpha\mathcal{R}\alpha\alpha'$ on L such that $\alpha\alpha'$ is an idempotent in L. Let $\gamma = \alpha\alpha'$, then by Lemma 4.2 we must have $\gamma \in L \setminus F_a$ and $\gamma\mathcal{L}\sigma$ for some idempotent $\sigma \in F_a$. Since every idempotent e in a semigroup is a right identity for L_e, we have $\gamma = \gamma\sigma = \sigma\gamma = \sigma$ which is a contradiction since $\gamma \notin F_a$ but $\sigma \in F_a$. Therefore, $L = F_a$ as required. □

As an application of Theorem 4.3, we get the following corollary which first appeared in [4].

Corollary 4.4. *$F_a = \{\alpha \in T(X) : a\alpha = a$ and α is injective on $X \setminus a\alpha^{-1}\}$ is a maximal inverse sub-semigroup of $T(X)$.*

Proof. By taking $Y = X$ in Theorem 4.3, we get $T(X, Y) = T(X) = F$ and $F_a = \{\alpha \in T(X) : a\alpha = a$ and α is injective on $X \setminus a\alpha^{-1}\}$ which is a maximal inverse sub-semigroup of $T(X)$. □

Recall that the number of combinations of n distinct things taken r at a time written $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}. \tag{4.6}$$

That is, $\binom{n}{r}$ is the number of ways that r objects can be chosen from n distinct objects.

In the next result, we use the above information to find the number of elements in F_a when Y is a finite subset of X.

Theorem 4.5. *Suppose that X is an arbitrary set and Y is a nonempty subset of X such that $|Y| = n$. Then for each $a \in Y$, $|F_a| = \sum_{r=0}^{n-1} r!\binom{n-1}{r}^2$.*

Proof. Let $a \in Y$ and $\alpha \in F_a$. Then by Theorem 4.1 we see that

$$\alpha = \begin{pmatrix} (X \setminus Y)\bigcup Y_1 & Y_2 \\ a & Y_3 \end{pmatrix}, \tag{4.7}$$

where $Y = Y_1 \dot{\bigcup} Y_2$, $a \in Y_1$, $Y_3 \subseteq Y \setminus \{a\}$, and $|Y_2| = |Y_3|$. If $Y_2 = \varnothing$, then α can have only one form, the constant map X_a. If Y_2 has t elements, where $1 \le t \le n-1$, then Y_2 can have $\binom{n-1}{t}$ choices and for each choice of Y_2, Y_3 can have $\binom{n-1}{t}$ choices, thus there are $\binom{n-1}{t}^2$ ways to choose Y_2 and Y_3. Since the restriction of α to Y_2 is a permutation, for each choice of Y_2 and Y_3, the map α has $t!$ possible forms. Hence in this case α can have $t!\binom{n-1}{t}^2$ forms.

Therefore, $|F_a| = 1 + \sum_{r=1}^{n-1} r!\binom{n-1}{r}^2 = \sum_{r=0}^{n-1} r!\binom{n-1}{r}^2$ as required. □

We observe that the number of elements in F_a depends only on the elements of Y, and when we replace Y with X in Theorem 4.5, we have the following corollary which first appeared in [4].

Corollary 4.6. *If X is a finite set with $|X| = n$, then the number of elements in $F_a = \{\alpha \in T(X) : a\alpha = a$ and α is injective on $X \setminus a\alpha^{-1}\}$ equals $\sum_{r=0}^{n-1} r!\binom{n-1}{r}^2$.*

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