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Research Article

On the Tensor Products of Maximal Abelian JW-Algebras

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It is well known in the work of Kadison and Ringrose (1983) that if A and B are maximal abelian von Neumann subalgebras of von Neumann algebras M and N, respectively, then $A\overline{\otimes}B$ is a maximal abelian von Neumann subalgebra of $M\overline{\otimes}N$. It is then natural to ask whether a similar result holds in the context of JW-algebras and the JW-tensor product. Guided to some extent by the close relationship between a JW-algebra M and its universal enveloping von Neumann algebra $W^*(M)$, we seek in this article to investigate the answer to this question.

1. Introduction

A JC-*algebra* A is a norm (uniformly) closed Jordan subalgebra of the Jordan algebra $B(H)_{s.a}$ of all bounded self adjoint operators on a Hilbert space H. The Jordan product is given by $a \circ b = (ab + ba)/2$. A subspace I of a JC-algebra A is called *a Jordan ideal* if $a \circ b \in I$ for every $a \in A$ and every $b \in I$. A JC-algebra is said to be *simple* if it has no nontrivial norm closed Jordan ideals. A JW-*algebra* $M \subseteq B(H)_{s.a}$ is a weakly closed JC-algebra. If M is a JC-algebra (resp., JW-algebra), let $C^*(M)$ (resp., $W^*(M)$) be the universal enveloping C^*-algebra (resp., von Neumann algebra) of M, and let θ_M (resp., Φ_M) be the canonical involutive $*$-antiautomorphism of $C^*(M)$ (resp., $W^*(M)$). Usually we will regard M as a generating Jordan subalgebra of $C^*(M)$) and $W^*(M)$ so that θ_M and Φ_M fix each point of M. The real C^*-algebra $R^*(M) = \{x \in C^*(M) : \theta_M(x) = x^*\}$ satisfies

$$R^*(M) \cap iR^*(M) = 0, \qquad C^*(M) = R^*(M) \oplus iR^*(M), \tag{1.1}$$

and the real von Neumann algebra $RW^*(M) = \{x \in W^*(M) : \Phi_M(x) = x^*\}$ satisfies

$$RW^*(M) \cap iRW^*(M) = 0, \qquad W^*(M) = RW^*(M) \oplus iRW^*(M). \tag{1.2}$$

The reader is refered to [1–5] for a detailed account of the theory of JC-algebras and JW-algebras. The relevant background on the theory of C^*-algebras and von Neumann algebras can be found in [6–8].

A projection e of a JW-algebra M is said to be *abelian* if eMe is associative, and it is called *minimal* if it is nonzero and contains no other nonzero projections of M, or equivalently, e is minimal if and only if $eMe = \mathbb{R}e$. *A JW-factor* is a JW-algebra with trivial centre; a Type *I JW-factor* is a JW-factor which contains a minimal projection. A *JW-algebra* is said to be of *Type $I_{\rm n}$* if there is a family of abelian projections $(e_\alpha)_{\alpha\in J}$ such that the central support $c_{\rm M}(e_\alpha)$ of e_α in M equals the unit $1_{\rm M}$ of M, $\sum_{\alpha\in J} e_\alpha = 1_{\rm M}$ and card $J = n$ (see [1, Section 5.3]). *A spin factor* $V = H \oplus \mathbb{R}1_V$ is a real Jordan algebra with identity 1_V, where H is a real Hilbert space of dimension at least two. The Jordan product on V is defined by

$$(a + \lambda 1_V) \circ (b + \mu 1_V) = (\mu a + \lambda b) + (\langle a, b\rangle + \lambda\mu)1_V, \quad a, b \in V,\ \lambda, \mu \in \mathbb{R}, \tag{1.3}$$

and the norm on V is given by

$$\|a + \lambda 1_V\| = \langle a, a\rangle^{1/2} + |\lambda|. \tag{1.4}$$

A spin factor V is universally reversible when $\dim V = 3$ or 4, nonreversible when $\dim V \neq 3, 4$ or 6, and it can be either reversible or nonreversible when $\dim V = 6$. A spin factor is a simple reflexive JW-algebra and constitutes the Type I_2 JW-factor (see [2, Section 6.1]).

A linear map $\varphi : A \to B$ between JC-algebras A and B is called *a (Jordan) homomorphism* if it preserves the Jordan product. A Jordan homomorphism which is one to one is called *a Jordan isomorphism*. *A factor representation* of a JC-algebra A is a (Jordan) homomorphism of A onto a weakly dense subalgebra of a JW-factor M. *Type I factor representations* are defined accordingly.

A JC-algebra A is said to be *reversible* if $a_1a_2\cdots a_n + a_na_{n-1}\cdots a_1 \in A$ whenever $a_1, a_2, \ldots, a_n \in A$ and is said to be *universally reversible* if $\pi(A)$ is reversible for every representation π of A [2, page 5]. The only universally reversible spin factors are $V_2 = M_2(\mathbb{R})_{s.a}$ and $V_3 = M_2(\mathbb{C})_{s.a}$ [2, Theorem 2.1]. A JC-algebra A is universally reversible if and only if it has no spin factor representations other than onto V_2 and V_3 [2, Theorems 2.2]. Every JW-algebra without a direct summand of Type I_2 is universally reversible [1, 5.1.5, 5.3.5, 6.2.3].

Two elements a and b of a JC-algebra A are said to *operator commute* if $T_aT_b = T_bT_a$, where $T_a : A \to A$ is the multiplication operator defined by $T_a(x) = a \circ x$, for all $x \in A$. A JW-algebra M is called *associative* if all its elements operators commute. A JW-subalgebra A of a JW-algebra M is called *maximal associative* if it is not contained in any larger associative JW-subalgebra of M. If A is a JW-subalgebra of a JW-algebra $M \subseteq B(H)_{s.a}$ and A' is the set of all elements of $B(H)_{s.a}$ which operator commutes with all elements of A, then A is a maximal associative JW-subalgebra of M if and only if $A = A' \cap M$. Indeed, since A is associative, $A \subseteq A' \cap M$ and A together with any element of $A' \cap M$ generates an associative JW-subalgebra of M which implies that $A' \cap M \subseteq A$ since A is maximal abelian. In particular, if $A \subseteq B(H)_{s.a}$ is an associative JW-algebra, then A is maximal associative if and only if $A = A'$.

This article aims to study the relationship between the maximality of an associative JW-subalgebra B of a JW-algebra M and that of $W^*(B)$ in $W^*(M)$. We give a counterexample which rules out the establishing of a result in the theory of JW-tensor products analog to that given in [6, Theorem 11.2.18] for von Neumann tensor products (cf. Example 2.2). Then we prove that a Jordan analog of Theorem 11.2.18 in [6] can be established in some particular cases.

Theorem 1.1 (see [9, Proposition 1]). *Let $M \subseteq B(H)_{s.a}$ be a JW-algebra, and let $a, b \in M$. Then the following are equivalent:*

(i) $ab = ba$;

(ii) $T_a T_b = T_b T_a$;

(iii) $a^2 \circ b = aba$.

That is, a and b operators commute if and only if they commute under ordinary operator multiplication.

Definition 1.2. Let M and N be a pair of JW-algebras canonically embedded in their respective universal enveloping von Neumann algebras $W^*(M)$ and $W^*(N)$.Then the JW-tensor product $JW(M \overline{\otimes} N)$ of M and N is the JW-algebra generated by $M \otimes N$ in $W^*(M) \overline{\otimes} W^*(N)$. The reader is referred to [10] for the properties of the JW-tensor product of JW-algebras.

Theorem 1.3 (see [10, Theorem 2.9]). *Let M and N be JW- algebras. If $JW(M \overline{\otimes} N)$ is universally reversible, then*

$$W^*\big(JW(M \overline{\otimes} N)\big) = W^*(M)\overline{\otimes}W^*(N). \tag{1.5}$$

2. Maximal Abelian JW-algebras

Let $\mathfrak{A}$ and $\mathfrak{B}$ be maximal abelian von Neumann subalgebras of von Neumann algebras $\mathfrak{M}$ and $\mathfrak{N}$, respectively, then $\mathfrak{A} \overline{\otimes} \mathfrak{B}$ is a maximal abelian von Neumann algebra of $\mathfrak{M} \overline{\otimes} \mathfrak{N}$ (see [6, 11.2.18]). In Example 2.2, we show that the Jordan analog of this result, in the context of JW-algebras and the JW-tensor product, is not true in general. However, it is shown in Theorem 2.11 that the result does hold in special circumstances.

Remark 2.1. (i) Note that any JW-subalgebra of a spin factor which is not a spin factor is of dimension at most 2. Indeed, let A be a JW-subalgebra of a spin factor $V \subseteq B(H)_{s.a}$. If $1_A \neq 1_V$,then 1_A is the only projection in A, since every projection in V is minimal, and hence $\dim A = 1$. If $1_A = 1_V$, then any family of orthogonal central projections of A contains at most two projections. Indeed if $e_1 + e_2 + e_3 = 1_A, e_i \in Z(A), i = 1,2,3$, then $e_2 + e_3 \leq 1_A - e_1$. Since $1_A - e_1$ is a minimal projection, we see that one of $e_i, i = 1,2,3$ must be zero. It is clear that if A is a factor, then it is of Type I_2, and hence it is a spin factor. (ii)Recall that $W^*(V) = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, where V is the 4-dimensional spin factor $M_2(\mathbb{C})_{s.a}$ [1, 6.2.1]:

$$M_2(\mathbb{C}) = \operatorname{span}\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\right\} = \mathbb{C} \underset{\mathbb{R}}{\otimes} M_2(\mathbb{R}), \tag{2.1}$$

which is an 8-dimensional real C^*-algebra.

Example 2.2. Let A be a maximal abelian JW-subalgebra of $V = M_2(C)_{s.a}$. Then $JW(A \overline{\otimes} A)$ is not a maximal abelian subalgebra of $JW(V \overline{\otimes} V)$.

Proof. By the above remark, $\dim A = 2$, and hence $A = \mathbb{R}e + \mathbb{R}f$ for some minimal projections e, f. Therefore,

$$JW\left(A \overline{\otimes} A\right) = A \underset{\mathbb{R}}{\otimes} A = \mathbb{R}(e \otimes e) \oplus \mathbb{R}(e \otimes f) \oplus \mathbb{R}(f \otimes e) \oplus \mathbb{R}(f \otimes f), \tag{2.2}$$

and hence $\dim JW(A \overline{\otimes} A) = 4$, since $\dim A \underset{\mathbb{R}}{\otimes} A = \dim A \cdot \dim A$ (see [11, Corollary 7.5]). On the other hand, $JW(V \overline{\otimes} V)$ is universally reversible, by [10, Proposition 2.7] which implies that

$$\begin{aligned} JW(V \overline{\otimes} V) &= RW^*(JW(V \overline{\otimes} V))_{s.a} \\ &= \left(RW^*(V) \underset{\mathbb{R}}{\overline{\otimes}} RW^*(V)\right)_{s.a} \\ &= \left(M_2(\mathbb{C}) \underset{\mathbb{R}}{\otimes} M_2(\mathbb{C})\right)_{s.a} \\ &= M_{2^2}(\mathbb{C})_{s.a} \oplus M_{2^2}(\mathbb{C})_{s.a}, \end{aligned} \tag{2.3}$$

since $RW^*(M_2(\mathbb{C})_{s.a}) = M_2(\mathbb{C})$ [3, page 385]. It can be seen that a maximal abelian JW-subalgebra of $JW(V \overline{\otimes} V)$ is of dimension 8, which implies that $JW(A \overline{\otimes} A)$ is not maximal abelian in $JW(V \overline{\otimes} V)$. □

Remark 2.3. Note that if B is an associative JW-subalgebra of a JW-algebra M such that $W^*(B)$ is a maximal abelian subalgebra of $W^*(M)$, then B is a maximal associative JW-subalgebra of M, since $B = W^*(B) \cap M$.

Lemma 2.4. *Let B be an associative JW-subalgebra of a JW-algebra M. Then,*

$$W^*(B) = \overline{[B]} = B \oplus iB, \tag{2.4}$$

is an abelian von Neumann algebra, where $\overline{[B]}$ is the weak*-closure of the C^*-subalgebra $[B]$ of $W^*(M)$ generated by B.

Proof. Being associative, B has no representation into a spin factor of the form V_{4n+1} and is, therefore, universally reversible. It follows from [3, page 383] that

$$B = RW^*(B)_{s.a}. \tag{2.5}$$

Therefore, by [3, Corollary 3.2], $RW^*(B)$ is isomorphic to the weak*-closure $\overline{R(B)}$ of the real C^*-subalgebra $R(B)$ of $W^*(M)$ generated by B, and the result follows. □

Recall that if M is a JW-algebra isomorphic to the self adjoint part $\mathcal{N}_{s.a}$ of a von Neumann algebra $\mathcal{N}$ and has no one-dimensional representations, then $W^*(M)$ is $*$-isomorphic to $\mathcal{N} \oplus \mathcal{N}^\circ$, where $\mathcal{N}^\circ$ is the opposite algebra of $\mathcal{N}$ [2, 7.4.15]. A real C^*-algebra $\mathfrak{A}$

can be realized as a complex C^*-algebra if there is a C^*-algebra isomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ of a complex C^*-algebra $\mathfrak{B}$ onto $\mathfrak{A}$. In this case, the real linear isometry j on $\mathfrak{A}$ defined, for each a in $\mathfrak{B}$, by

$$j\phi(a) = \phi(ia) \tag{2.6}$$

is such that j^2 and $-id_{\mathfrak{A}}$ coincide.

Lemma 2.5. *Let B be a maximal associative JW-subalgebra of a JW-algebra M. Suppose that M is isomorphic to the self adjoint part $\mathcal{N}_{s.a}$ of a von Neumann algebra $\mathcal{N}$ and has no one-dimensional representations. Then $W^*(B)$ is not a maximal abelian on Neumann subalgebra of $W^*(M)$.*

Proof. Identifying M with $\mathcal{N}_{s.a}$, $\overline{[B]}$ is a von Neumann subalgebra of both $\mathcal{N}$ and $\mathcal{N}^\circ$, and hence, the von Neumann subalgebra $\overline{[B]} \oplus \overline{[B]}$ of $\mathcal{N} \oplus \mathcal{N}^\circ \cong W^*(M)$ is abelian and contains $W^*(B) = \overline{[B]} \cong \overline{[B]} \oplus \{0\}$, which implies that $W^*(B)$ is not maximal abelian in $W^*(M)$. □

Lemma 2.6. *Let B be a maximal associative JW-subalgebra of a JW-algebra M. If $RW^*(M)$ is $*$-isomorphic to a complex C^*-algebra, then $W^*(B)$ is not a maximal abelian von Neumann subalgebra of $W^*(M)$.*

Proof. Since $C^*(M)$ is the complex C^*-algebra $[M]$ generated by M in $W^*(M)$ [12, Theorem 2.7], $RW^*(M)$ is the weak*-closure of $R^*(M)$ in $W^*(M)$. Therefore, $R^*(M)$ is a complex C^*-algebra, which implies that $C^*(M) = \mathcal{I} \oplus \Phi_M(\mathcal{I})$ for some norm closed ideal $\mathcal{I}$ of $C^*(M)$ isomorphic to $R^*(M)$ [13, Lemma 1], so that $W^*(M) = \mathcal{J} \oplus \Phi_M(\mathcal{J})$, where $\mathcal{J}$ is the weak*-closure $\overline{\mathcal{I}}$ of $\mathcal{I}$ in $W^*(M)$. Hence, $\mathcal{J}$ is isomorphic to $RW^*(M)$. Let ϕ be the isomorphism of $\mathcal{J}$ onto $RW^*(M)$, and let j be the corresponding real linear operator on $RW^*(M)$, defind above. Then, using Lemma 2.4, there exists an isomorphism π from the W*-algebra $B \oplus iB$ into $RW^*(M)$ such that, for elements b_1 and b_2 in B,

$$\pi(b_1 + ib_2) = b_1 + jb_2. \tag{2.7}$$

It follows that $\phi^{-1} \circ \pi$ and $\Phi_M \circ \phi^{-1} \circ \pi$ are *-isomorphisms of $\overline{[B]} = B \oplus iB$ into $\mathcal{J}$ and $\Phi_M(\mathcal{J})$, respectively. Since a *-isomorphism between C^*-algebras is an isometry [7, Corollary 1.5.4], we may identify $\overline{[B]}$ with $\phi^{-1} \circ \pi(\overline{[B]})$ and $\Phi_M \circ \phi^{-1} \circ \pi(\overline{[B]})$. It follows that $\overline{[B]} \oplus \overline{[B]}$ is an abelian von Neumann subalgebra of $W^*(M)$, proving that $W^*(B) = \overline{[B]} \cong \overline{[B]} \oplus \{0\}$ is not maximal abelian in $W^*(M)$. □

Proposition 2.7. *Let M be a universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra and without direct summands of type I_1. If B is a maximal associative subalgebra of M, then $W^*(B)$ is a maximal abelian von Neumann subalgebra of $W^*(M)$.*

Proof. By Lemma 2.4, $W^*(B) = \overline{[B]} = B \oplus iB \hookrightarrow W^*(M)$. If $W^*(B)$ is not maximal abelian in $W^*(M)$, there exists an element $z \in W^*(M) = RW^*(M) \oplus iRW^*(M)$, $z \notin W^*(B)$ such that z together with $W^*(B)$ generate an abelian von Neumann subalgebra $Y \underset{\neq}{\supset} W^*(B) \supseteq B$ of $W^*(M) \supseteq M$. Let $z = x + iy$, $x, y \in W^*(M)_{s.a}$. Since $z \notin W^*(B)$, then either x or y (or both) does not belong to $W^*(B)$. Suppose that $x \notin W^*(B)$, since $W^*(M) = RW^*(M) \oplus iRW^*(M)$, then $x = a + ib$, for some $a, b \in RW^*(M)$. Then either a or b (or both) does not belong to

$W^*(B)$. Since $x \in W^*(M)_{s.a}$, we have $a = a^*$, and $b = -b^*$, and so $a \in M = RW^*(M)_{s.a}$, since M is a universally reversible [3, page 383]. Therefore, a must be the zero element, since it obviously commutes with all elements in B. On the other hand, $b^2 = -bb^* \in RW^*(M)_{s.a} = M$. Since $bu = ub$ for all $u \in W^*(B)$, $b^2u = bub = ub^2$ for all $u \in B$, and so b^2 and u operators commute relative to the Jordan product in B [9, Proposition 1]. Hence $b^2 \in B \subsetneq W^*(B)$, since B is a maximal associative subalgebra of M, which implies that $b \in W^*(B)$. Therefore, $x = ib \in W^*(B)$, a contradiction. This proves the result. □

Lemma 2.8. *Let M be a universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra. If B is a maximal associative subalgebra of M, then $W^*(B)$ is a maximal abelian von Neumann subalgebra of $W^*(M)$.*

Proof. Splitting $M = M_{I_1} \oplus M_{n.a}$ as the direct sum of a JW-algebra M_{I_1} of type I_1 (the abelian part) and a JW-algebra $M_{n.a}$ without direct summands of type I_1(the nonabelian part). It is clear that $B \supseteq M_{I_1}, B_{n.a} = B \cap M_{n.a}$ is a maximal associative subalgebra of $M_{n.a}$ and $B = M_{I_1} \oplus B_{n.a}$. By Proposition 2.7, $W^*(B_{n.a})$ is a maximal abelian von Neumann subalgebra of $W^*(M_{n.a})$, and hence $W^*(B) = W^*(M_{I_1}) \oplus W^*(B_{n.a})$ is a maximal abelian von Neumann subalgebra of $W^*(M)$, since $W^*(M) = W^*(M_{I_1}) \oplus W^*(M_{n.a})$ [12, Lemma 2.6]. □

Proposition 2.9. *Let B_i be a maximal associative subalgebra of a JW-algebra M_i, $i = 1, 2$, and suppose that M_i is universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra and without direct summands of type I_1. Then $JW(B_1 \overline{\otimes} B_2)$ is a maximal associative JW-subalgebra of $JW(M_1 \overline{\otimes} M_2)$.*

Proof. Note first that $W^*(B_1) \overline{\otimes} W^*(B_2)$ is a von Neumann $*$-subalgebra of $W^*(M_1) \overline{\otimes} W^*(M_2)$ [8, Theorem 11.2.10], and $JW(B_1 \overline{\otimes} B_2)$ is a JW-subalgebra of $JW(M_1 \overline{\otimes} M_2)$, since $B_1 \otimes B_2 \subseteq M_1 \otimes M_2$. By Proposition 5.2, $W^*(B_i)$ is maximal abelian in $W^*(M_i)$, and hence, $W^*(B_1) \overline{\otimes} W^*(B_2)$ is maximal abelian in $W^*(M_1) \overline{\otimes} W^*(M_2)$ [8, Corollary 11.2.18] and [10, Theorem 2.9]. The result is now obvious, since $W^*(JW(B_1 \overline{\otimes} B_2)) = W^*(B_1) \overline{\otimes} W^*(B_2)$, and $W^*(JW(M_1 \overline{\otimes} M_2)) = W^*(M_1) \overline{\otimes} W^*(M_2)$ [10, Theorem 2.9]. □

Proposition 2.10. *Let N be an associative JW-algebra, and let M be a universally reversible JW-algebra not isomorphic to the self adjoint part of a von Neumann algebra and without direct summands of type I_1. If B is a maximal associative subalgebra of M, then $JW(N \overline{\otimes} B)$ is a maximal associative JW-subalgebra of $JW(N \overline{\otimes} M)$.*

Proof. Let $M = M_{I_1} \oplus M_{n.a}$ be the decomposition of M into abelian part M_{I_1} and nonabelian part $M_{n.a}$. Then $B = M_{I_1} \oplus B_{n.a}$, where $B_{n.a} = B \cap M_{n.a}$ is obviously a maximal associative subalgebra of $M_{n.a}$. By [10, Remark 2.14],

$$\begin{aligned} JW(N \overline{\otimes} M) &= JW(N \overline{\otimes}(M_{I_1} \oplus M_{n.a})) = JW(A \overline{\otimes} M_{I_1}) \oplus JW(A \overline{\otimes} M_{n.a}), \\ JW(N \overline{\otimes} B) &= JW(N \overline{\otimes}(M_{I_1} \oplus B_{n.a})) = JW(N \overline{\otimes} M_{I_1}) \oplus JW(N \overline{\otimes} B_{n.a}). \end{aligned} \tag{2.8}$$

It is clear now that $JW(N \overline{\otimes} B)$ is a maximal associative JW-subalgebra of $JW(N \overline{\otimes} M)$, since $JW(N \overline{\otimes} M_{I_1})$ is obviously associative, and $JW(N \overline{\otimes} B_{n.a})$ is maximal in $JW(N \overline{\otimes} M_{n.a})$, by Proposition 2.9. □

Theorem 2.11. *Let M and N be universally reversible JW-algebras not isomorphic to the self adjoint parts of von Neumann algebras. If A and B are maximal associative subalgebra of M and N, respectively, then $JW(A \overline{\otimes} B)$ is a maximal associative JW-subalgebra of $JW(M \overline{\otimes} N)$.*

Proof. Let $M = M_{I_1} \oplus M_{n.a}, N = N_{I_1} \oplus N_{n.a}$ be the decomposition of M, N into abelian parts (M_{I_1}, N_{I_1}), and nonabelian parts $(M_{n.a}, N_{n.a})$. Then $A = M_{I_1} \oplus A_{n.a}$ and $B = N_{I_1} \oplus B_{n.a}$, where $A_{n.a} = A \cap M_{n.a}$ and $B_{n.a} = B \cap N_{n.a}$. Therefore,

$$\begin{aligned} JW(M \overline{\otimes} N) &= JW(M_{I_1} \overline{\otimes} N_{I_1}) \oplus JW(M_{I_1} \overline{\otimes} N_{n.a}) \\ &\quad \oplus JW(M_{n.a} \overline{\otimes} N_{I_1}) \oplus JW(M_{n.a} \overline{\otimes} N_{n.a}), \\ JW(A \overline{\otimes} B) &= JW(M_{I_1} \overline{\otimes} N_{I_1}) \oplus JW(M_{I_1} \overline{\otimes} B_{n.a}) \\ &\quad \oplus JW(A_{n.a} \overline{\otimes} N_{I_1}) \oplus JW(A_{n.a} \overline{\otimes} B_{n.a}), \end{aligned} \tag{2.9}$$

by [13, Remark 2.14]. The proof is complete, by Propositions 2.9 and 2.10. □

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