

Research Article

Asymptotic Behavior of Tail Density for Sum of Correlated Lognormal Variables

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Received 8 August 2008; Revised 3 February 2009; Accepted 5 March 2009

Recommended by Michael Evans

We consider the asymptotic behavior of a probability density function for the sum of any two lognormally distributed random variables that are nontrivially correlated. We show that both the left and right tails can be approximated by some simple functions. Furthermore, the same techniques are applied to determine the tail probability density function for a ratio statistic, and for a sum with more than two lognormally distributed random variables under some stricter conditions. The results yield new insights into the problem of characterization for a sum of lognormally distributed random variables and demonstrate that there is a need to revisit many existing approximation methods.

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1. Introduction

The lognormal distribution has applications in many fields such as survival analysis [1], genetic studies [2, 3], financial modelling [4, 5], telecommunication studies [6, 7] amongst others. It has been found that many types of data can be modeled by lognormal distributions, which include human blood pressure, microarray data, stock options, survival rate for different groups of human beings, and the received power's long-term fluctuation. In these occasions, we wish to make some inferences based on the collected data involving the addition of a few lognormally distributed random variables (RVs). Deriving the statistical properties of a sum of lognormally distributed RVs is therefore desirable [6, 8]. Also note that the number of summands is so small in practice that the central limit theorem is not applicable.

Many research works assume that all the summands are independent, either justified by practical considerations or for the sake of simplicity. However, there are some applications

(e.g., the Asian option pricing model [4]) in which correlations among the summands are inevitable. Our study will address the correlation problem.

In some cellular mobile systems (see [9]), the signal quality is largely dictated by signal to interference ratio (SIR). On a large scale, both useful signals and interfering signals experience lognormal shadow fading. That is to say, SIR can be modeled by $(X_1 + \dots + X_n)/(Y_1 + \dots + Y_m)$ where all the RVs $X_1, \dots, X_n, Y_1, \dots, Y_m$ are lognormally distributed. SIR only characterizes the instantaneous quality. For ordinary users and network operators, one important factor to consider is the outage probability target to a certain SIR threshold. For example, for the users of a data transfer service, it is required that the outage probability $P_0 = Pr(\text{SIR} < \gamma_0 \text{ such that } \text{BER}(\gamma_0) = 10^{-6})$ needs to be less than 0.01. Here BER stands for bit error rate that usually depends on SIR and other factors. In this paper, Theorem 2.8 provides an approximation to $Pr(\text{SIR} < \gamma_0)$ when $n = 2, m = 2$.

In addition, Theorems 2.6 and 2.7 try to characterize the left and right tails of a sum of lognormally distributed RVs. The theorems are useful to construct a Padé approximation to the probability density function (PDF) of a sum of lognormally distributed RVs. For example, if that approximation is available, and if useful signals and interfering signals are independent, the outage probability can be numerically estimated as follows: Let $Z_1 = \log(\sum_{i=1}^n X_i)$ and $Z_2 = \log(\sum_{j=1}^m Y_j)$, then

$$Pr(\text{SIR} < \gamma_0) = Pr(Z_1 - Z_2 < \log(\gamma_0)) \approx \int_{-\infty}^{\log(\gamma_0)} \int_{\mathbb{R}} f_{Z_1}^{\text{approx}}(x - y) f_{Z_2}^{\text{approx}}(-y) dy dx. \quad (1.1)$$

Since Fenton [10] addressed the problem, many methods have been developed, but none of them have been successful in finding a closed form representation for the PDF of a sum of multiple lognormally distributed RVs. These methods can be divided into three categories.

- (i) The first type of methods attempt to characterize the PDF by calculating the moment generating function [11, 12] or the characteristic function [13, 14]. The results obtained can be used in the numerical computations of a PDF or a cumulative density function (CDF). To our knowledge, no work has succeeded in using the results of this category to describe the shape of a PDF or CDF.
- (ii) The second type of methods [15–18] use the bound technique for the CDF of an underlying statistic.
- (iii) The third type of methods focuses on finding a good approximation to either the PDF or CDF of the underlying statistic. Most published works belong to this category. The way to find the approximation can often be described as follows: first, assume a specific distribution that the sum (or the ratio of sum) of the lognormally distributed RV follows; then use a variety of methods to identify the parameters for that specific distribution. The specific distributions in the literature include lognormal [15, 19], reciprocal Gamma [4], log shifted Gamma [20], and user-defined PDF [21, 22]. In some works [23], only the CDF approximation is defined. Moment matching [10, 24, 25], moment generating function matching [19], and the least squares fitting [26, 27] are a few popular methods used to determine the parameters associated with the distribution.

In this paper, we will rigorously characterize the right and left tails behavior of a PDF for a random variable $Z = \log(e^{X_1} + e^{X_2})$, where (X_1, X_2) are jointly distributed with $N(\mu, \Sigma)$

as distribution. This is our first step towards understanding the more general problem: the characterization of the PDF of $Z = \log(\sum_{k=1}^N e^{X_k})$ where (X_1, \dots, X_N) are jointly distributed as $N(\mu, \Sigma)$. Note here we do not assume that the X_k are independent (except for Theorem 2.7), nor do we assume that X_1, \dots, X_N have the same marginal distribution. We hope that our study can lead to a better solution to the works presented in [28] or [29].

Janos [15] is the first one to study the right tail probability of a sum of lognormals. More advanced and more general studies can be found in [30]. We have not found any theoretical results regarding the left tails. In addition, the right tails results we show cannot be deduced from the results in [5, 30–32].

Our results show that it is possible to find some elementary functions g_L, g_R such that

$$\lim_{z \rightarrow -\infty} \frac{f_Z(z)}{g_L(z)} = 1, \quad \lim_{z \rightarrow +\infty} \frac{f_Z(z)}{g_R(z)} = 1. \quad (1.2)$$

The explicit forms of g_L and g_R enable us to assess the performance of the existing approximation methods and to determine how to improve these methods. By Theorem 2.3 (see also the subsequent remark and Corollary 2.5), we can determine that at the left tail region, even under the independence assumption, given any function g_* within the families of PDFs such as lognormal, reciprocal Gamma or log shifted Gamma, either $\lim_{z \rightarrow -\infty} g_*(z)/f_Z(z)$ does not exist or $\lim_{z \rightarrow -\infty} g_*(z)/f_Z(z)$ can be only zero or ∞ . No previous works have led to this discovery. Szyszkowicz [18] has pointed out that some precedent models are wrong in the tail region, but this work was based on a hypothesis that was only justified by the numerical results, and it still focused on finding the best lognormal type approximation. In view of our results, such efforts are unlikely to succeed.

Our characterization of the behavior of the tail of the PDF of a sum of two lognormals is complete in the sense that our results cover all nondegenerate covariance matrices. Our work regarding the ratio RV is obtained under more stringent conditions. This new result shows that the ratio RV is neither lognormal nor log Gamma. This indicates that others should be cautious with the method in [9] despite the successful examples demonstrated therein.

When the number of summands exceeds two, the situation for a PDF approximation becomes much more complicated. We are able to show some left tail and right tail results by imposing some conditions on the covariance matrix that covers the independent case. The result of Theorem 2.7 could be well-known to experts working with functions from the subexponential class. Unfortunately, we did not find any references that explicitly state the result, so we provide a short proof in Appendix F. Further in this line, [5] has presented the complete CDF approximation for the right tail with an arbitrary covariance matrix. However our results cannot be deduced trivially from the CDF behavior and are interesting in themselves. For example, for any polynomial growth continuous function h , we can say that

$$\lim_{z \rightarrow +\infty} \frac{E[h(e^Z)1_{Z>z}]}{\int_z^\infty h(e^x)g_R(x)dx} = 1, \quad (1.3)$$

whereas such an approximation cannot be a direct consequence from the result in [5].

In the following sections, we will first present our results followed by a numerical validation. We will then discuss some future studies that this paper does not cover. Also we present the proofs in the appendix.

2. Main Results

Let (X_1, X_2) be a jointly normally distributed random vector. Let ρ be the correlation coefficient, and $\mu_i = E(X_i)$, $\sigma_i^2 = \text{Var}(X_i)$, for $i = 1, 2$. Then the joint PDF of (e^{X_1}, e^{X_2}) is given by

$$f(u, v) \equiv \frac{h(u, v)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}uv}, \quad (2.1)$$

where

$$h(u, v) = \exp\left\{\frac{1}{2(1-\rho^2)}\left[\left(\frac{\log(u)-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(\log(u)-\mu_1)(\log(v)-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{\log(v)-\mu_2}{\sigma_2}\right)^2\right]\right\}. \quad (2.2)$$

We wish to study the left and right tail probabilities of $Z = \log(e^{X_1} + e^{X_2})$, which has PDF as $f_Z(z) \equiv e^z g(e^z)$ with

$$g(z) \equiv \int_0^z f(y, z-y) dy. \quad (2.3)$$

We hope to understand the asymptotic behavior of $g(e^z)$ when $z \rightarrow \pm\infty$. Direct calculus yields

$$g(e^z) = \frac{e^{-z}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^1 \frac{e^{-A(z,t)}}{(1-t)t} dt, \quad (2.4)$$

where the exponent $A(z, t) =$

$$\frac{1}{2(1-\rho^2)} \left[\left(\frac{\log(1-t) + z - \mu_1}{\sigma_1}\right)^2 + \left(\frac{\log t + z - \mu_2}{\sigma_2}\right)^2 - 2\rho \frac{\log(1-t) + z - \mu_1}{\sigma_1} \frac{\log t + z - \mu_2}{\sigma_2} \right]. \quad (2.5)$$

Rewrite A in three terms $A(t) = A_1(z) + 2zA_2(t) + A_3(t)$, where the A_i are defined as

$$\begin{aligned} A_1(z) &= \frac{1}{2(1-\rho^2)} \left\{ z^2 \left(\frac{1}{\sigma_1^2} - \frac{2\rho}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2} \right) + 2z \left[\left(\frac{\rho}{\sigma_1} - \frac{1}{\sigma_2} \right) \frac{\mu_2}{\sigma_2} + \left(\frac{\rho}{\sigma_2} - \frac{1}{\sigma_1} \right) \frac{\mu_1}{\sigma_1} \right] \right\}, \\ A_2(t) &= \frac{1}{2(1-\rho^2)} \left[\left(\frac{1}{\sigma_1} - \frac{\rho}{\sigma_2} \right) \frac{\log(1-t)}{\sigma_1} + \left(\frac{1}{\sigma_2} - \frac{\rho}{\sigma_1} \right) \frac{\log t}{\sigma_2} \right], \\ A_3(t) &= \frac{1}{2(1-\rho^2)} \left[\frac{(\log(1-t) - \mu_1)^2}{\sigma_1^2} - \frac{2\rho(\log(1-t) - \mu_1)(\log t - \mu_2)}{\sigma_1\sigma_2} + \frac{(\log t - \mu_2)^2}{\sigma_2^2} \right]. \end{aligned} \quad (2.6)$$

Hence, by changing the variable t to e^{-u} ,

$$f_Z(z) = \frac{e^{-A_1(z)}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_0^\infty \frac{e^{-2zA_2(e^{-u})-A_3(e^{-u})}}{1-e^{-u}} du. \quad (2.7)$$

We regroup the integrand in (2.7) in the form

$$\int_0^\infty \frac{e^{-2zA_2(e^{-u})-A_3(e^{-u})}}{1-e^{-u}} du = \int_0^\infty H(u)e^{-zG(u)} du, \quad (2.8)$$

with

$$G(u) = \frac{1}{1-\rho^2} \left[\frac{(\sigma_2 - \sigma_1\rho) \log(1-e^{-u})}{\sigma_1^2\sigma_2} - \frac{(\sigma_1 - \sigma_2\rho)u}{\sigma_2^2\sigma_1} \right], \quad H(u) = \frac{e^{-A_3(e^{-u})}}{1-e^{-u}}. \quad (2.9)$$

Remark 2.1. Without loss of generality, in this paper, we always assume

$$0 < \sigma_1 \leq \sigma_2, \quad |\rho| < 1. \quad (2.10)$$

We also use the following notation.

Definition 2.2. We say that two functions f and h are equivalent near some point $a \in \overline{\mathbb{R}}$, denoted by $f(z) \sim_a h(z)$, if we have $\lim_{z \rightarrow a} f(z)/h(z) = 1$.

For the left tail, we have the following result.

Theorem 2.3. Let $X_i \sim N(\mu_i, \sigma_i^2)$ be defined as above for $i = 1, 2$ and $\rho \in (-1, 1)$ be the correlation coefficient. Let f_Z be the PDF of $Z = \log(e^{X_1} + e^{X_2})$, then $f_Z(z) \sim_{-\infty} h_L(z)$ as follows.

(i) If $\rho < \sigma_1/\sigma_2$, one has

$$h_L(z) = \frac{H(u_0)e^{-zG(u_0)-A_1(z)}}{\sqrt{2\pi z(1-\rho^2)G''(u_0)\sigma_1\sigma_2}} \quad \text{with } u_0 = \log \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{\sigma_1^2 - \rho\sigma_1\sigma_2}. \quad (2.11)$$

Here the functions G, H, A_1 are defined by (2.9) and (2.6).

(ii) If $\rho = \sigma_1/\sigma_2$, one has

$$h_L(z) = \frac{N(\mu_1, \sigma_1^2)}{\sqrt{\log(-z)}} e^{1/(2(\sigma_2^2 - \sigma_1^2)) \{-\log z_\rho - \log(\log z_\rho) + \mu_2 - \mu_1\}^2 - 2\log z_\rho} \quad \text{with } z_\rho = -z(\rho^{-2} - 1). \quad (2.12)$$

(iii) If $\rho > \sigma_1/\sigma_2$, one has

$$h_L(z) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-(z-\mu_1)^2/(2\sigma_1^2)}. \quad (2.13)$$

In particular, when $\rho = 0$, we find that $f_Z \sim_{-\infty} C_1 e^{-C_2 z - C_3 z^2} / \sqrt{z}$, which means that any lognormal, reciprocal Gamma or log shifted Gamma cannot be used to fit the left tail, under the independence hypothesis. The situation for the right tail of f_Z is simpler. It is interesting to remark that the result does not depend on the correlation coefficient ρ (see also [5]). Here and later on, we employ the lexicographical order to the couple (σ, μ) .

Theorem 2.4. Let Z and f be defined as above, then $f_Z(z) \sim_{\infty} h_R(z)$, where $h_R(z)$ is defined as follows.

(i) If $(\sigma_1, \mu_1) \neq (\sigma_2, \mu_2)$, one has

$$h_R(z) = N(\mu, \sigma^2) \quad \text{with} \quad (\sigma, \mu) = \max\{(\sigma_1, \mu_1), (\sigma_2, \mu_2)\}. \quad (2.14)$$

(ii) If $(\sigma_1, \mu_1) = (\sigma_2, \mu_2) = (\sigma, \mu)$, one has $h_R(z) = 2/(\sigma\sqrt{2\pi})e^{-(z-\mu)^2/(2\sigma^2)}$.

The following corollary is an immediate consequence of Theorems 2.3 and 2.4.

Corollary 2.5. Let f_V be the PDF of $V = e^{X_1} + e^{X_2}$, where X_1, X_2 are i.i.d. RVs following $N(0, \sigma^2)$ distributions. Then, one has

$$\lim_{z \rightarrow 0} \frac{f_V(z)}{f_{VL}(z)} = 1, \quad \text{where} \quad f_{VL}(z) = \frac{\exp[-(\log z - \log 2)^2/\sigma^2]}{z\sigma\sqrt{-\pi \log z}}; \quad (2.15)$$

$$\lim_{z \rightarrow +\infty} \frac{f_V(z)}{f_{VR}(z)} = 1, \quad \text{where} \quad f_{VR}(z) = \frac{\sqrt{2} \exp[-\log^2 z/(2\sigma^2)]}{\sqrt{\pi}\sigma z}.$$

The results in Corollary 2.5 confirm those results reported in [18]. Furthermore, we can easily show that the models in [4, 21] will also fail in the tail regions.

Next we show that our left tail and right tail study can be extended for some special cases in higher dimension by using the Laplace methods.

Theorem 2.6. Let $(X_i)_{1 \leq i \leq n}$ ($n \geq 3$) be a joint normally distributed random variable with distribution $N(\mu, \Sigma)$. Let $\mathbf{M} = \Sigma^{-1}$. Let f_Z be the PDF of random variable $Z = \log(\sum_{1 \leq i \leq n} e^{X_i})$. If $\mathbf{M} = (m_{ij})$ satisfies $m_k = \sum_{1 \leq j \leq n} m_{kj} > 0$ for all $k = 1, \dots, n$, then the left tail of f_Z satisfies

$$f_Z(z) \sim_{-\infty} \frac{H_1(U_0)}{\sqrt{2\pi|z|^{n-1}|\det \Sigma||\det \mathbf{H}_F(U_0)|}} \exp \left[-z \sum_{k=1}^n m_k \log \frac{m_k}{m} - \frac{1}{2} \sum_{i,j=1}^n (z - \mu_i) m_{ij} (z - \mu_j) \right] \quad (2.16)$$

where $m = \sum_{1 \leq j \leq n} m_j$, $F(u_2, \dots, u_n) = \sum_{1 \leq i \leq n} m_i C_i$, $\mathbf{H}_F = (\partial_{ij}^2 F)$ denotes the Hessian matrix of F . Here,

$$H_1(u_2, \dots, u_n) \equiv \frac{e^{-(C, M(C-2\mu))/2}}{1 - \sum_{2 \leq i \leq n} e^{u_i}}, \quad U_0 = \left(\log \frac{m_k}{m} \right)_{2 \leq k \leq n} \in \mathbb{R}^{n-1}, \quad (2.17)$$

and $C = (C_i) \in \mathbb{R}^n$ is given by

$$C = \left(\log \left(1 - \sum_{2 \leq k \leq n} e^{u_k} \right), u_2, u_3, \dots, u_n \right). \quad (2.18)$$

Theorem 2.7. Let (X_1, \dots, X_n) be independent normally distributed RVs, that is, $X_i \sim N(\mu_i, \sigma_i^2)$. Let $Z = \log(\sum_{i=1}^n e^{X_i})$. Define $(\sigma, \mu) = \max\{(\sigma_i, \mu_i), 1 \leq i \leq n\}$ for the lexicographical order and m_n the number of maximum points, that is $m_n = \#\{1 \leq i \leq n, \text{ s.t. } (\sigma_i, \mu_i) = (\sigma, \mu)\}$. Then the PDF of Z , $f_Z(x)$ satisfies

$$\lim_{x \rightarrow \infty} \frac{f_Z(x)}{N(\mu, \sigma^2)(x)} = m_n \quad \text{where } N(\mu, \sigma^2)(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}. \quad (2.19)$$

Finally, we show a result for the quotient of sums of i.i.d lognormal variables.

Theorem 2.8. Let X_1, X_2, Y_1, Y_2 be i.i.d random variables. Each of them follows $N(0, \sigma^2)$ distribution. Let $W = \log((e^{X_1} + e^{X_2}) / (e^{Y_1} + e^{Y_2}))$ and $f_W(w)$ be the PDF of W , then

$$f_W(w) \sim_{\pm\infty} \frac{2e^{-(|w|+\log 2)^2/(3\sigma^2)}}{\sigma\sqrt{\pi}|w|}. \quad (2.20)$$

This result can be generalized to the case where X_1, X_2 follow $N(0, \sigma_x^2)$, Y_1, Y_2 follow $N(0, \sigma_y^2)$, with any positive constants σ_x and σ_y . Indeed, we can prove that

$$f_W(w) \sim_{-\infty} \frac{2}{\sigma_x \sqrt{-\pi w}} e^{-(w-\log 2)^2/(\sigma_x^2+2\sigma_y^2)}, \quad f_W(w) \sim_{+\infty} \frac{2}{\sigma_y \sqrt{\pi w}} e^{-(w+\log 2)^2/(\sigma_y^2+2\sigma_x^2)}. \quad (2.21)$$

For the sake of brevity, we only present the proof for the special case where both variances are equal.

3. Numerical Validation

We have validated the two-dimensional theoretical results by performing Monte-Carlo simulations. The curve generated by Monte Carlo method is obtained through bin-based density estimation. In all of the presented cases, we can see that our approximations match the numerical results closely.

For the simulation parameters of the Z statistic, in order to test our results in the extreme cases, we have chosen $\rho = 0.7, 0.8, 0.9$. The mean values were arbitrarily set. The values of σ were chosen to be 9.6 and 12 so that their ratio σ_1/σ_2 is 0.8.

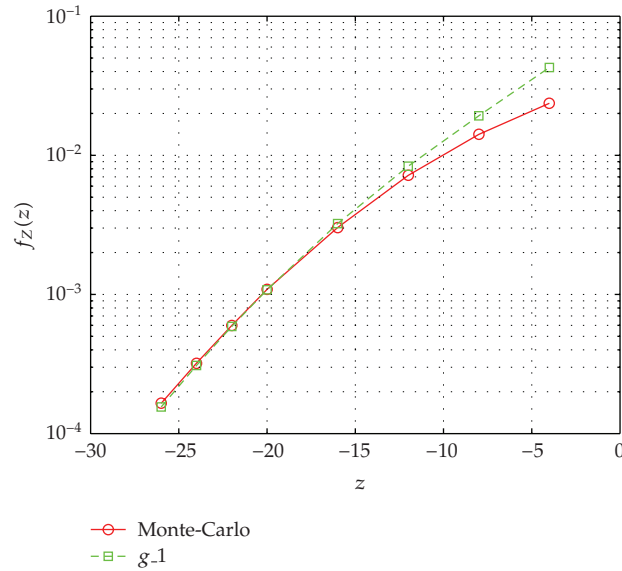


Figure 1: Left tail $f_Z, \rho < \sigma_1/\sigma_2$. g-1 is the approximation.

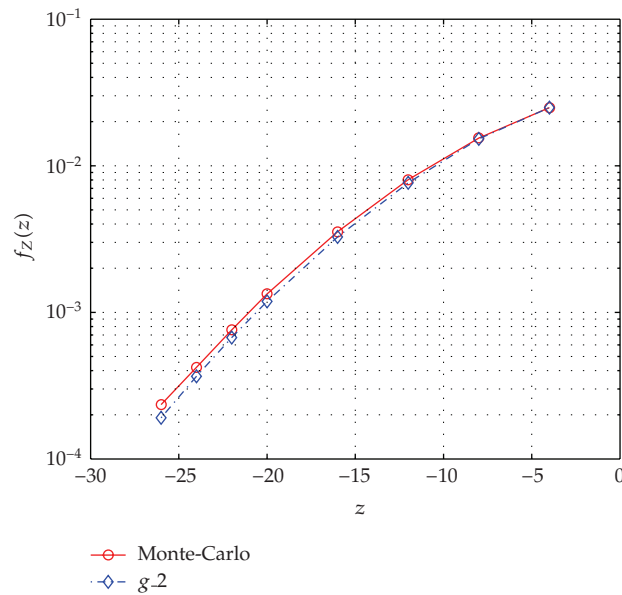


Figure 2: Left tail $f_Z, \rho = \sigma_1/\sigma_2$. g-2 is the approximation.

For the parameters of the ratio statistic (denoted by W), we used $(\sigma_x, \sigma_y) = (12, 9.6)$ for two groups of normally distributed RVs. The mean for these RVs were set to 0. Due to the symmetry properties that W has, it is sufficient to show the verification results for the left tail of f_W .

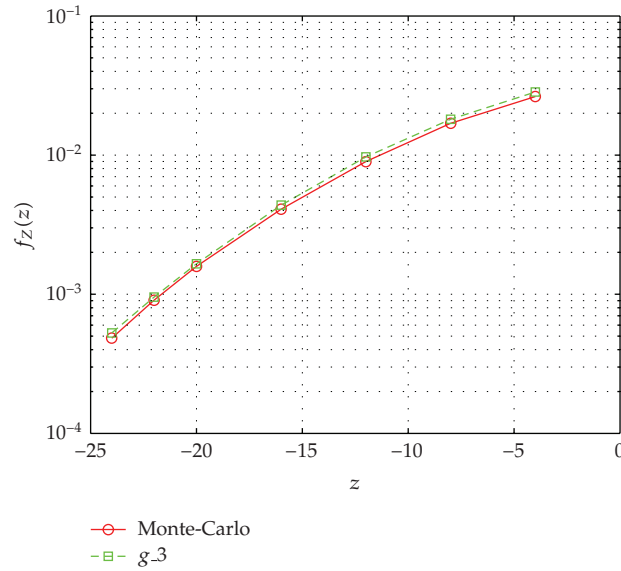


Figure 3: Left tail $f_Z, \rho > \sigma_1/\sigma_2$. g.3 is the approximation.

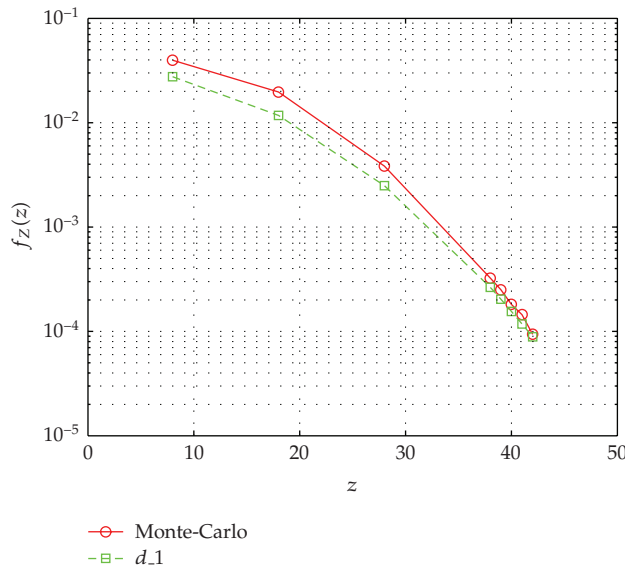


Figure 4: Right tail $f_Z, \rho < \sigma_1/\sigma_2$. d.1 is the approximation.

4. Further Remarks

We have seen that our tail density approximations (for a sum of lognormal RVs) do not deal with an arbitrary covariance matrix. In the 2D proofs, we used the classical approximation technique for integrals, called the Laplace method (see Lemmas A.1, E.2). Then we divided the study into a few subcases and then proceeded in different ways. Comparing Theorems 2.3, 2.6 to Theorems 2.4, 2.7, it seems that in general the left tail behavior is more involved than the right tail case. We hope to adapt our approach to the higher-dimensional space,

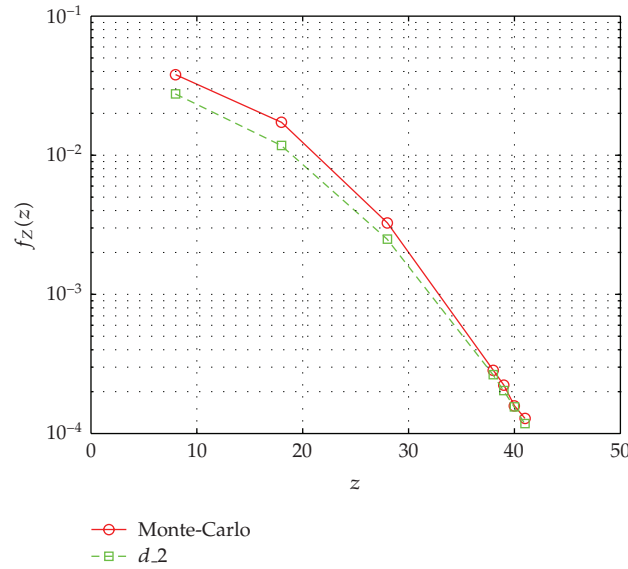


Figure 5: Right tail $f_Z, \rho > \sigma_1/\sigma_2$. d.2 is the approximation.

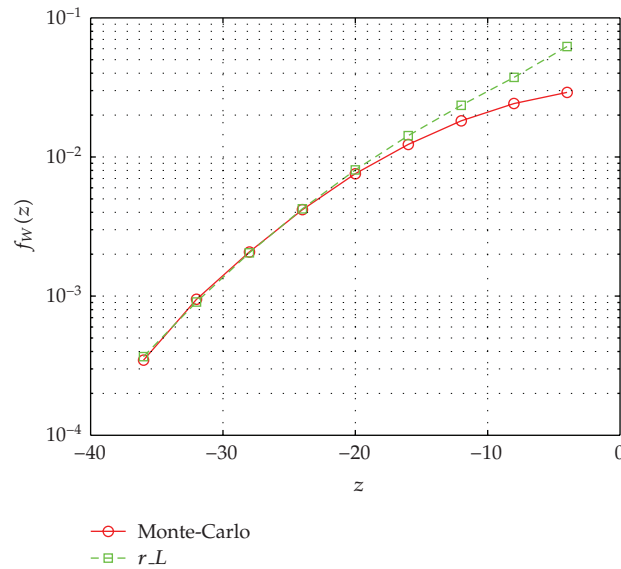


Figure 6: Left tail $f_W, (\sigma_x, \sigma_y) = (12, 9.6)$. r.L is the approximation.

especially for the right tail behavior, which will lead to the result in [5]. It is also useful to perform a higher order approximation for both tails so that an efficient Padé approximation can be developed accordingly.

In view of the work in [32], it should be worthwhile to extend our lognormal work (at least for the right tail) to a more general family such as the subexponential class. The importance of this distribution family can be found in [5, 32, 33]. Perhaps future work for the sum of lognormals mentioned above may shed some insight on the subexponential class problem.

Appendices

A. Preliminaries

Here we list some basic lemmas useful in the subsequent discussion. Their proofs use standard techniques and hence are omitted.

Lemma A.1. *Let H be a positive, integrable function on an interval $(a, b) \subset \overline{\mathbb{R}}$. Let $G \in C^2(a, b)$ be concave such that $x_0 \in (a, b)$ verifies $G'(x_0) = 0, G''(x_0) < 0$, then*

$$\lim_{z \rightarrow \infty} \frac{\sqrt{-zG''(x_0)}e^{-zG(x_0)}}{\sqrt{2\pi}H(x_0)} \times \int_a^b H(x)e^{zG(x)} dx = 1. \tag{A.1}$$

This result is the so-called Laplace method in modern analysis (see [34]), which is more often cited as saddle point approximation in other fields such as statistics or physics. Later on, we also give a higher dimensional version (see Lemma E.2).

Lemma A.2. *Let $x_0(z)$ be a nonnegative function defined on \mathbb{R} , and $\lim_{z \rightarrow z_0} x_0(z) = \infty$. Assume that $f(z, t)$ is a nontrivial, nonnegative function such that for any fixed z near $z_0 \in \overline{\mathbb{R}}$, one has $f(z, t) \in L^1(\mathbb{R})$ and for any $\epsilon > 0$,*

$$\int_{x_0(z)-\epsilon}^{x_0(z)+\epsilon} f(z, t) dt \sim_{z_0} \|f(z, \cdot)\|_{L^1}. \tag{A.2}$$

If moreover G is a bounded uniformly continuous function over \mathbb{R} , such that $\liminf_{x \rightarrow \infty} G(x) > 0$, then

$$\int_{\mathbb{R}} f(z, t)G(t) dt \sim_{z_0} G(x_0(z)) \|f(z, \cdot)\|_{L^1}. \tag{A.3}$$

In fact, we need a special case of lemma 16. When $\lim_{z \rightarrow \infty} G(x) = G_0 \neq 0$ exists, we can simply require that G is a continuous function and replace the term $G(x_0(z))$ in (A.3) by G_0 .

Lemma A.3. *Let G be a bounded measurable function defined on (a, ∞) with $a \in \mathbb{R} \cup \{-\infty\}$, such that $\lim_{\xi \rightarrow \infty} G(\xi) = G_0$ exists. Let λ be a positive constant. Then*

$$\lim_{z \rightarrow \infty} e^{-z^2/(4\lambda)} \int_a^\infty G(\xi) e^{z\xi - \lambda\xi^2} d\xi = G_0 \sqrt{\frac{\pi}{\lambda}}. \tag{A.4}$$

Using usual developments, we also have the following asymptotic expansion.

Lemma A.4. *Fix $\lambda \in \mathbb{R}$, if $u(z)$ satisfies $z = (e^{u(z)} - 1)(u(z) + \lambda)$, then as $z \rightarrow \infty$, $u(z)$ is uniquely determined and*

$$u(z) = \log z - \log \log z + \frac{\log \log z - \lambda}{\log z} + o\left(\frac{\log \log z}{\log^2 z}\right). \tag{A.5}$$

B. The Left Tail Behavior

In this section, we will prove Theorem 2.3. We discuss the cases $\rho < \sigma_1/\sigma_2$, $\rho = \sigma_1/\sigma_2$ and $\rho > \sigma_1/\sigma_2$, respectively. Recall that $0 < \sigma_1 \leq \sigma_2$ and $|\rho| < 1$.

B.1.

Case 1. $\rho < \sigma_1/\sigma_2$. Using the formulas (2.7) and (2.8), we need only to understand the behavior of

$$T(z) = \int_0^\infty \frac{e^{-2zA_2(e^{-u})-A_3(e^{-u})}}{1-e^{-u}} du = \int_0^\infty H(u)e^{-zG(u)} du \quad (\text{B.1})$$

as $z \rightarrow -\infty$. Here G and H are defined by (2.9). Since

$$G'(u) = \frac{1}{1-\rho^2} \left(\frac{\sigma_2\rho - \sigma_1}{\sigma_1\sigma_2^2} + \frac{\sigma_1\rho - \sigma_2}{\sigma_2\sigma_1^2} \frac{e^{-u}}{e^{-u}-1} \right), \quad G''(u) = \frac{(\sigma_1\rho - \sigma_2)e^{-u}}{\sigma_2\sigma_1^2(1-\rho^2)(e^{-u}-1)^2}. \quad (\text{B.2})$$

Thus, $G'(u) = 0$ has a unique solution

$$u_0 = \log \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{\sigma_1^2 - \rho\sigma_1\sigma_2}. \quad (\text{B.3})$$

We also have $G''(u) < 0$ in \mathbb{R} and $H > 0$, integrable over \mathbb{R}_+ . Hence, Lemma A.1 allows us to conclude

$$f_Z(z) \sim_{-\infty} \frac{H(u_0)e^{-zG(u_0)-A_1(z)}}{\sqrt{2\pi z(1-\rho^2)G''(u_0)\sigma_1\sigma_2}}. \quad (\text{B.4})$$

B.2.

Case 2. $\rho = \sigma_1/\sigma_2$. Here we can simplify A_1 and A_2 as

$$A_1(z) = \frac{z^2 - 2\mu_1 z}{2\sigma_1^2}, \quad A_2(t) = \frac{\log(1-t)}{2\sigma_1^2}. \quad (\text{B.5})$$

Rewrite

$$T(z) = \int_0^\infty e^{L(z,u)} G_2(u) du \quad (\text{B.6})$$

with

$$L(z, u) = -\frac{z \log(1 - e^{-u})}{\sigma_1^2} - \frac{(u + \mu_2)^2 - 2\mu_1 u}{2(1 - \rho^2)\sigma_2^2} + \frac{\mu_1 \mu_2}{(1 - \rho^2)\sigma_2^2}, \tag{B.7}$$

$$G_2(u) = \frac{1}{1 - e^{-u}} \exp \left[-\frac{1}{2(1 - \rho^2)} \left(\frac{[\log(1 - e^{-u}) - \mu_1]^2}{\sigma_1^2} + \frac{2\rho(u + \mu_2) \log(1 - e^{-u})}{\sigma_1 \sigma_2} \right) \right]. \tag{B.8}$$

Clearly

$$\lim_{u \rightarrow 0^+} G_2(u) = 0, \quad \lim_{u \rightarrow \infty} G_2(u) = e^{-\mu_1^2 / (2(1 - \rho^2)\sigma_1^2)} = G_0 > 0. \tag{B.9}$$

Thus, G_2 is uniformly bounded and uniformly continuous over \mathbb{R}_+ . Furthermore, we have

$$\begin{aligned} \partial_u L(z, u) &= -\frac{z}{\sigma_1^2} \frac{1}{e^u - 1} - \frac{u + \mu_2 - \mu_1}{(1 - \rho^2)\sigma_2^2}, \\ \partial_u^2 L(z, u) &= \frac{z}{\sigma_1^2} \frac{e^u}{(e^u - 1)^2} - \frac{1}{\sigma_2^2(1 - \rho^2)}, \\ \partial_u^3 L(z, u) &= -\frac{z}{\sigma_1^2} \frac{e^u(e^u + 1)}{(e^u - 1)^3}. \end{aligned} \tag{B.10}$$

If $z < 0$, $\partial_u^2 L(z, u) < 0$, $\partial_u^3 L(z, u) > 0$ for any $u \in \mathbb{R}_+$. Let u_z be the unique solution of $\partial_u L(z, u) = 0$. Obviously, u_z satisfies the equation $(e^{u_z} - 1)(u_z + \mu_2 - \mu_1) = -(\rho^{-2} - 1)z$. According to Lemma A.4, we obtain $\lim_{z \rightarrow -\infty} u_z = \lim_{z \rightarrow -\infty} -ze^{-u_z} = \infty$ and

$$\begin{aligned} \partial_u^2 L(z, u_z) &\sim_{-\infty} -\frac{\log(-z)}{\sigma_2^2 - \sigma_1^2}, \\ e^{L(z, u_z)} &\sim_{-\infty} e^{1/(2(\sigma_2^2 - \sigma_1^2))(-[\log z \rho - \log(\log z \rho) + \mu_2 - \mu_1]^2 - 2 \log z \rho + \mu_1^2)} \end{aligned} \tag{B.11}$$

where $z_\rho = -(\rho^{-2} - 1)z$.

The situation here is more delicate than in Lemma A.1, but we can follow the same idea. For any $\epsilon \in (0, 1)$ fixed, there exists $R > 0$ such that $|G_2(x)/G_0 - 1| \leq \epsilon$, $\partial_u^2 L(z, x) < 0$ for any $x > R$. Choosing now z near $-\infty$ such that $u_z > R + 1$ and

$$1 \leq \frac{\partial_u^2 L(z, u_z - \epsilon)}{\partial_u^2 L(z, u_z)} \leq 1 + C\epsilon. \tag{B.12}$$

We will decompose the integral into three parts: First we consider the integral of $e^{L(z,u)}G_2(u)$ over $[u_z - \epsilon, u_z + \epsilon]$. Using Taylor expansion and the monotonicity of $\partial_u^2 L$, we get

$$\int_{u_z - \epsilon}^{u_z + \epsilon} e^{L(z,u)} G_2(u) du \geq (1 - \epsilon) G_0 \int_{u_z - \epsilon}^{u_z + \epsilon} e^{L(z,u_z) + \partial_u^2 L(z,u_z - \epsilon)(u - u_z)^2 / 2} du. \quad (\text{B.13})$$

By (B.12), for $\epsilon > 0$ small enough,

$$\liminf_{z \rightarrow -\infty} \frac{e^{-L(z,u_z)} \sqrt{-\partial_u^2 L(z,u_z)}}{G_0 \sqrt{2\pi}} \int_{u_z - \epsilon}^{u_z + \epsilon} e^{L(z,u)} G_2(u) du \geq \frac{1 - \epsilon}{\sqrt{1 + C\epsilon}}. \quad (\text{B.14})$$

Using $L(z, u_z + \epsilon)$, we also have (for small $\epsilon > 0$)

$$\limsup_{z \rightarrow -\infty} \frac{e^{-L(z,u_z)} \sqrt{-\partial_u^2 L(z,u_z)}}{G_0 \sqrt{2\pi}} \int_{u_z - \epsilon}^{u_z + \epsilon} e^{L(z,u)} G_2(u) du \leq \frac{1 + \epsilon}{\sqrt{1 - C\epsilon}}. \quad (\text{B.15})$$

Consider now the integral of $e^{L(z,u)}G_2(u)$ on $[u_z + \epsilon, \infty)$. Since $L(z, u)$ is strictly concave in u ,

$$\frac{1}{\|G_2\|_\infty} \int_{u_z + \epsilon}^\infty e^{L(z,u)} G_2(u) du \leq \int_{u_z + \epsilon}^\infty e^{L(z,u_z + \epsilon) + \partial_u L(z,u_z + \epsilon)(u - u_z - \epsilon)} du = \frac{e^{L(z,u_z + \epsilon)}}{\partial_u L(z, u_z + \epsilon)}. \quad (\text{B.16})$$

Moreover, $\partial_u L(z, u_z + \epsilon) \geq \epsilon \partial_u^2 L(z, u_z)$, $L(z, u_z + \epsilon) < L(z, u_z)$, so we get

$$\lim_{z \rightarrow -\infty} e^{-L(z,u_z)} \sqrt{-\partial_u^2 L(z,u_z)} \int_{u_z + \epsilon}^\infty e^{L(z,u)} G_2(u) du = 0. \quad (\text{B.17})$$

Similarly,

$$\lim_{z \rightarrow -\infty} e^{-L(z,u_z)} \sqrt{-\partial_u^2 L(z,u_z)} \int_0^{u_z - \epsilon} e^{L(z,u)} G_2(u) du = 0. \quad (\text{B.18})$$

Combining all these estimates, we deduce

$$\begin{aligned} \frac{1 + \epsilon}{\sqrt{1 - C\epsilon}} &\geq \limsup_{z \rightarrow -\infty} \frac{e^{-L(z,u_z)} \sqrt{-\partial_u^2 L(z,u_z)}}{G_0 \sqrt{2\pi}} \int_0^\infty e^{L(z,u)} G_2(u) du \\ &\geq \liminf_{z \rightarrow -\infty} \frac{e^{-L(z,u_z)} \sqrt{-\partial_u^2 L(z,u_z)}}{G_0 \sqrt{2\pi}} \int_0^\infty e^{L(z,u)} G_2(u) du \geq \frac{1 - \epsilon}{\sqrt{1 + C\epsilon}}. \end{aligned} \quad (\text{B.19})$$

As $\epsilon > 0$ can be arbitrarily small, by (2.7),

$$f_Z(z) \sim_{-\infty} \frac{G_0 e^{L(z, u_z) - A_1(z)}}{\sqrt{-2\pi(1 - \rho^2) \partial_u^2 L(z, u_z) \sigma_1 \sigma_2}}. \tag{B.20}$$

Applying (B.5), (B.9) and (B.11), we complete the proof.

B.3.

Case 3. $\rho > \sigma_1/\sigma_2$. Rewrite

$$A_2(e^{-u}) = \xi(u) = B_1 \log(1 - e^{-u}) + B_2 u \tag{B.21}$$

with two positive constants

$$B_1 = \frac{\sigma_2 - \sigma_1 \rho}{2(1 - \rho^2) \sigma_1^2 \sigma_2}, \quad B_2 = \frac{\sigma_2 \rho - \sigma_1}{2(1 - \rho^2) \sigma_2^2 \sigma_1}. \tag{B.22}$$

Thus,

$$\int_0^\infty \frac{e^{-2zA_2(e^{-u}) - A_3(e^{-u})}}{1 - e^{-u}} du = \int_0^\infty e^{L_2(z, u)} G_2(u) du \tag{B.23}$$

with G_2 given by (B.8) and

$$\begin{aligned} L_2(z, u) &= -2z[B_1 \log(1 - e^{-u}) + B_2 u] - \frac{(u + \mu_2)^2}{2(1 - \rho^2) \sigma_2^2} + \frac{\rho \mu_1 (u + \mu_2)}{(1 - \rho^2) \sigma_1 \sigma_2} \\ &= -2z[B_1 \log(1 - e^{-u}) + B_2 u] - \frac{1}{2(1 - \rho^2)} \left[\left(\frac{u + \mu_2}{\sigma_2} - \frac{\rho \mu_1}{\sigma_1} \right)^2 - \frac{\mu_1^2 \rho^2}{\sigma_1^2} \right]. \end{aligned} \tag{B.24}$$

Notice that ξ is C^∞ diffeomorphism from $(0, \infty)$ into \mathbb{R} , with $\xi'(u) = B_1(e^u - 1)^{-1} + B_2 > 0$ in $(0, \infty)$. Let η be the inverse function of ξ , namely $u = \eta(\xi)$. We have the following properties:

$$\lim_{\xi \rightarrow +\infty} \eta(\xi) = \infty, \quad \lim_{\xi \rightarrow +\infty} \eta'(\xi) = \frac{1}{B_2}, \quad \lim_{\xi \rightarrow +\infty} \frac{\xi - B_2 \eta(\xi)}{-B_1 e^{-\eta(\xi)}} = 1. \tag{B.25}$$

The change of variable $u = \eta(\xi)$ yields

$$\int_0^\infty \frac{e^{-2zA_2(e^{-u}) - A_3(e^{-u})}}{1 - e^{-u}} du = \int_{\mathbb{R}} G_2(\eta(\xi)) \eta'(\xi) e^{L_2(z, \eta(\xi))} d\xi. \tag{B.26}$$

For $z < 0$, as $\eta'(\xi)\|G_2\|_\infty \in L^1(\mathbb{R}_-)$, $L_2(z, \eta(\xi)) \leq C$ in \mathbb{R}_- and $\lim_{z \rightarrow -\infty} L_2(z, \eta(\xi)) = -\infty$ for $\xi < 0$, we obtain

$$\lim_{z \rightarrow -\infty} \int_{\mathbb{R}_-} G_2(\eta(\xi)) \eta'(\xi) e^{L_2(z, \eta(\xi))} d\xi = 0. \quad (\text{B.27})$$

Furthermore,

$$\int_{\mathbb{R}_+} G_2(\eta(\xi)) \eta'(\xi) e^{L_2(z, \eta(\xi))} d\xi = \int_{\mathbb{R}_+} H_2(\xi, \eta(\xi)) e^{-2z\xi - 1/(2(1-\rho^2))(\xi/(B_2\sigma_2) + \mu_2/\sigma_2 - \rho\mu_1/\sigma_1)^2} d\xi, \quad (\text{B.28})$$

where

$$H_2(\xi, \eta(\xi)) = G_2(\eta(\xi)) \eta'(\xi) e^{\mu_1^2 \rho^2 / (2(1-\rho^2)\sigma_1^2)} \times e^{[(\xi/(B_2\sigma_2) + \mu_2/\sigma_2 - \rho\mu_1/\sigma_1)^2 - (\eta(\xi)/\sigma_2 + \mu_2/\sigma_2 - \rho\mu_1/\sigma_1)^2] / (2(1-\rho^2))} \quad (\text{B.29})$$

is a bounded function in \mathbb{R}_+ by properties of η and G_2 . Otherwise, using (B.9) and (B.25)

$$\lim_{\xi \rightarrow \infty} H_2(\xi, \eta(\xi)) = \frac{G_0}{B_2} e^{\mu_1^2 \rho^2 / (2(1-\rho^2)\sigma_1^2)} = \frac{e^{-\mu_1^2 / (2\sigma_1^2)}}{B_2} > 0. \quad (\text{B.30})$$

Applying Lemma A.3, we get

$$\int_{\mathbb{R}_+} G_2(\eta(\xi)) \eta'(\xi) e^{L_2(z, \eta(\xi))} d\xi \sim_{-\infty} \sigma_2 \sqrt{2\pi(1-\rho^2)} e^{-\mu_1^2 / (2\sigma_1^2)} e^{2B_2^2 \sigma_2^2 (1-\rho^2) z^2 + 2B_2(\mu_2 - \rho\mu_1 \sigma_2 / \sigma_1) z}. \quad (\text{B.31})$$

Finally, combining (B.27), (B.31), (B.22) and (2.7)

$$f_Z(z) \sim_{-\infty} \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\mu_1^2 / (2\sigma_1^2)} e^{-A_1(z) + 2B_2^2 \sigma_2^2 (1-\rho^2) z^2 + 2B_2(\mu_2 - \rho\mu_1 \sigma_2 / \sigma_1) z} = \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-(z-\mu_1)^2 / (2\sigma_1^2)}, \quad (\text{B.32})$$

which is just the claimed result.

C. The Right Tail Behavior

Here we prove Theorem 2.4. We begin with the formulae (2.7), (2.6) and divide the study into two cases: $\rho < \sigma_1/\sigma_2$ and $\rho \geq \sigma_1/\sigma_2$. Since the arguments are often similar to the previous consideration and the situation is simpler, we will proceed with less details.

C.1.

Case 1. $\rho < \sigma_1/\sigma_2$. We have $A_2(e^{-u}) = \xi(u) = B_1 \log(1 - e^{-u}) + B_2 u$ with B_i given by (B.22). Since $B_1 > 0$ and $B_2 < 0$, it's clear that $-\xi''(u) = B_1 e^u (e^u - 1)^{-2} > 0$ in $(0, \infty)$ and $u_0 = \log(1 - B_1/B_2)$

is the unique solution of $\xi'(u) = 0$. Hence ξ is decreasing on $(0, u_0)$, increasing on (u_0, ∞) and $\lim_{u \rightarrow 0^+} \xi(u) = \lim_{u \rightarrow \infty} \xi(u) = -\infty$. Thus for any $t \in (-\infty, \xi(u_0))$, there exist exactly two solutions $\eta_1(\xi) < \eta_2(\xi)$ for $\xi(u) = t$. Before proceeding, we list some properties of η_i

$$\lim_{\xi \rightarrow -\infty} \eta_1(\xi) = \lim_{\xi \rightarrow -\infty} \xi \eta_1(\xi) = 0, \quad \lim_{\xi \rightarrow -\infty} \frac{\eta_1'(\xi)}{1 - e^{-\eta_1(\xi)}} = \frac{1}{B_1}, \tag{C.1}$$

$$\lim_{\xi \rightarrow -\infty} \eta_2(\xi) = \infty, \quad \lim_{\xi \rightarrow -\infty} \eta_2'(\xi) = \frac{1}{B_2}, \quad \lim_{\xi \rightarrow -\infty} \frac{\xi - B_2 \eta_2(\xi)}{-B_1 e^{-\eta_2(\xi)}} = 1. \tag{C.2}$$

Note that

$$\begin{aligned} I(z) &= \int_{u_0}^{\infty} \frac{e^{-2zA_2(e^{-u})-A_3(e^{-u})}}{1 - e^{-u}} du = - \int_{-\infty}^{\xi(u_0)} G_2(\eta_2(\xi)) \eta_2'(\xi) e^{L_2(z, \eta_2(\xi))} d\xi \\ &= - \int_{-\infty}^{\xi(u_0)} H_2(\xi, \eta_2(\xi)) e^{-2z\xi - 1/(2(1-\rho^2))(\xi/(B_2\sigma_2) + \mu_2/\sigma_2 - \rho\mu_1/\sigma_1)^2} d\xi, \end{aligned} \tag{C.3}$$

where G_2, L_2 and H_2 are defined by (B.8), (B.24), (B.29) respectively. Then we can repeat the above proof for the left tail (the third case), substituting the function η by η_2 , using the properties in (C.2), we conclude that

$$I(z) \sim_{\infty} \sigma_2 \sqrt{2\pi(1-\rho^2)} e^{A_1(z) - (z-\mu_1)^2/(2\sigma_1^2)}. \tag{C.4}$$

For the integral over $(0, u_0)$, we have

$$\begin{aligned} &2(1-\rho^2)A_3(e^{-\eta_1(\xi)}) \\ &= \frac{[\log(1 - e^{-\eta_1(\xi)}) - \mu_1]^2}{\sigma_1^2} - \frac{2\rho[\log(1 - e^{-\eta_1(\xi)}) - \mu_1](-\eta_1(\xi) - \mu_2)}{\sigma_1\sigma_2} + \frac{(-\eta_1(\xi) - \mu_2)^2}{\sigma_2^2} \\ &= \left[\frac{\xi - B_2\eta_1(\xi) - B_1\mu_1}{B_1\sigma_1} \right]^2 + \frac{2\rho(\eta_1(\xi) + \mu_2)}{B_1\sigma_1\sigma_2} [\xi - B_2\eta_1(\xi) - B_1\mu_1] + \frac{(\eta_1(\xi) + \mu_2)^2}{\sigma_2^2} \\ &= \left(\frac{\xi}{B_1\sigma_1} - \frac{\mu_1}{\sigma_1} + \frac{\rho\mu_2}{\sigma_2} \right)^2 + G_1(\xi, \eta_1(\xi)) + \frac{\mu_2^2(1-\rho^2)}{\sigma_2^2}, \end{aligned} \tag{C.5}$$

with

$$\begin{aligned} G_1(\xi, \eta_1(\xi)) &= -2 \frac{(\xi - B_1\mu_1)B_2\eta_1(\xi)}{B_1^2\sigma_1^2} + \frac{B_2^2\eta_1^2(\xi)}{B_1^2\sigma_1^2} \\ &+ \frac{2\rho\eta_1(\xi)}{B_1\sigma_1\sigma_2} [\xi - B_2\eta_1(\xi) - B_1\mu_1 - B_2\mu_2] + \frac{\eta_1^2(\xi) + 2\eta_1(\xi)\mu_2}{\sigma_2^2}. \end{aligned} \tag{C.6}$$

On the other hand,

$$J(z) = \int_0^{u_0} \frac{e^{-2zA_2(e^{-u}) - A_3(e^{-u})}}{1 - e^{-u}} du = \int_{-\infty}^{\xi(u_0)} H_1(\xi, \eta_1(\xi)) e^{-2z\xi - 1/(2(1-\rho^2))(\xi/(B_1\sigma_1) - \mu_1/\sigma_1 + \rho\mu_2/\sigma_2)^2} d\xi, \quad (\text{C.7})$$

where

$$H_1(\xi, \eta_1(\xi)) = \frac{\eta_1'(\xi)}{1 - e^{-\eta_1(\xi)}} \exp\left[\frac{-G_1(\xi, \eta_1(\xi))}{2(1-\rho^2)} - \frac{\mu_2^2}{2\sigma_2^2}\right]. \quad (\text{C.8})$$

Using the properties (C.1),

$$\lim_{\xi \rightarrow -\infty} H_1(\xi, \eta_1(\xi)) = \frac{e^{-\mu_2^2/(2\sigma_2^2)}}{B_1}. \quad (\text{C.9})$$

Using Lemma A.3, we get the behavior of J as $z \rightarrow \infty$, and a simplification leads to

$$J(z) \sim_{\infty} \sigma_1 \sqrt{2\pi(1-\rho^2)} e^{A_1(z) - (z-\mu_2)^2/(2\sigma_2^2)}. \quad (\text{C.10})$$

Since $2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}f_Z(z) = e^{-A_1(z)}[I(z) + J(z)]$, the dominant term of $f_Z(z)$ is clearly given by $J(z)$ if $\sigma_1 < \sigma_2$, However when $\sigma_1 = \sigma_2$, we need to compare μ_1 and μ_2 . Finally,

$$f_Z(z) \sim_{\infty} \begin{cases} N(\mu, \sigma^2) \text{ with } (\sigma, \mu) = \max\{(\sigma_i, \mu_i)\}, & \text{if } (\sigma_1, \mu_1) \neq (\sigma_2, \mu_2); \\ 2N(\mu, \sigma^2), & \text{if } (\sigma_1, \mu_1) = (\sigma_2, \mu_2) = (\sigma, \mu). \end{cases} \quad (\text{C.11})$$

C.2.

Case 2. $\rho \geq \sigma_1/\sigma_2$. We always have $B_1 > 0$, but now $B_2 \geq 0$. Thus, $\xi'(u) > 0$ and $\xi''(u) < 0$ for $u > 0$, so ξ is a diffeomorphism from $(0, \infty)$ to \mathbb{R} . Denote $\eta(\xi)$ as the inverse function of ξ . $\eta(\xi)$ satisfies (C.1) and

$$\lim_{\xi \rightarrow \infty} \eta(\xi) = \infty, \quad \lim_{\xi \rightarrow \infty} \eta'(\xi) = \frac{1}{B_2} \text{ if } B_2 > 0, \quad \lim_{\xi \rightarrow \infty} \frac{\eta'(\xi)}{e^{\eta(\xi)} - 1} = \frac{1}{B_1} \text{ if } B_2 = 0. \quad (\text{C.12})$$

Let H_1 and G_1 be defined in (C.8), (C.6) respectively. Replace η_1 by η . It is not difficult to prove that we always have $\lim_{\xi \rightarrow \infty} G_1(\xi, \eta(\xi)) = \infty$ and

$$\int_0^{\infty} \frac{e^{-2zA_2(e^{-u}) - A_3(e^{-u})}}{1 - e^{-u}} du = \int_{\mathbb{R}} H_1(\xi, \eta(\xi)) e^{-2z\xi - 1/(2(1-\rho^2))(\xi/(B_1\sigma_1) - \mu_1/\sigma_1 + \rho\mu_2/\sigma_2)^2} d\xi. \quad (\text{C.13})$$

Moreover, when z tends to ∞ , the behavior of $f_Z(z)$ is dominated by the integral over \mathbb{R} . By (C.1) and (C.12), $H_1(\xi, \eta(\xi))$ is uniformly bounded in \mathbb{R} . Using Lemma A.3 again, as for $J(z)$ in Case 1, we come to the conclusion $f_Z(z) \sim_{\infty} N(\mu_2, \sigma_2^2)$.

D. Quotient of Sum of Lognormal

Here is the proof for Theorem 2.8. Let $Z_1 = \log(e^{X_1} + e^{X_2})$, $Z_2 = \log(e^{Y_1} + e^{Y_2})$, as before,

$$f_{Z_1}(t) = f_{Z_2}(t) = \frac{1}{2\pi\sigma^2} \int_0^1 \frac{1}{(1-u)u} \exp\left[-\frac{(t + \log u)^2 + (t + \log(1-u))^2}{2\sigma^2}\right] du. \tag{D.1}$$

Denoting $\Omega_1 = [0, 1] \times [0, 1] \times \mathbb{R}$, we get

$$\begin{aligned} f_Z(z) &= \int_{\mathbb{R}} f_{Z_1}(t) f_{Z_2}(t-z) dt. \\ &= \frac{1}{4\pi^2\sigma^4} \iiint_{\Omega_1} \frac{e^{-1/(2\sigma^2)[(\log u+t)^2+(\log(1-u)+t)^2+(\log v+t-z)^2+(\log(1-v)+t-z)^2]}}{u(1-u)v(1-v)} du dv dt. \end{aligned} \tag{D.2}$$

Direct calculus yields

$$\begin{aligned} &(\log u + t)^2 + (\log(1-u) + t)^2 + (\log v + t - z)^2 + (\log(1-v) + t - z)^2 \\ &= \left(2t - z + \frac{\log[uv(1-u)(1-v)]}{2}\right)^2 + z^2 + z \log\left[\frac{u(1-u)}{v(1-v)}\right] + R(u, v), \end{aligned} \tag{D.3}$$

with

$$R(u, v) = \log^2 u + \log^2(1-u) + \log^2 v + \log^2(1-v) - \frac{\log^2[uv(1-u)(1-v)]}{4}. \tag{D.4}$$

Thus,

$$\begin{aligned} 4\pi^2\sigma^4 f_Z(z) &= \int_0^1 \int_0^1 \int_{\mathbb{R}} \frac{e^{-1/(2\sigma^2)\{(2t-z+\log[uv(1-u)(1-v)]/2\}^2+z^2+z\log[u(1-u)/(v(1-v))]+R(u,v)\}}}{u(1-u)v(1-v)} du dv dt \\ &= \int_0^1 \int_0^1 \int_{\mathbb{R}} \frac{e^{-1/(2\sigma^2)\{4t^2+z^2+z\log[u(1-u)/(v(1-v))]+R(u,v)\}}}{u(1-u)v(1-v)} du dv dt \\ &= \frac{\sigma\sqrt{\pi}e^{-z^2/(2\sigma^2)}}{\sqrt{2}} \int_0^1 \int_0^1 \frac{e^{-1/(2\sigma^2)\{z\log[u(1-u)/(v(1-v))]+R(u,v)\}}}{u(1-u)v(1-v)} du dv \\ &= 2\sqrt{2\pi}\sigma e^{-z^2/(2\sigma^2)} \int_0^{1/2} \int_0^{1/2} \frac{e^{-1/(2\sigma^2)\{z\log[u(1-u)/(v(1-v))]+R(u,v)\}}}{u(1-u)v(1-v)} du dv. \end{aligned} \tag{D.5}$$

The last equality is obtained by the symmetry of our integrand between u, v and $1-u, 1-v$. Since $\zeta(u) = \log[u(1-u)]$ is increasing in $(0, 1/2]$, clearly for any $z < 0$,

$$\begin{aligned} I_1(z) &= \int_0^{1/2} \int_u^{1/2} \frac{e^{-1/(2\sigma^2)\{z \log[u(1-u)/(v(1-v))] + R(u,v)\}}}{u(1-u)(v(1-v))} du dv \\ &\leq \int_0^{1/2} \int_u^{1/2} \frac{e^{-R(u,v)/(2\sigma^2)}}{u(1-u)(v(1-v))} du dv = a_0 < \infty \end{aligned} \quad (\text{D.6})$$

Denote by $\gamma : (-\infty, -\log 4] \rightarrow (0, 1/2]$ the inverse function of ζ , then

$$\gamma(s) = \frac{1 - \sqrt{1 - 4e^s}}{2}, \quad \gamma'(s) = \frac{e^s}{\sqrt{1 - 4e^s}}. \quad (\text{D.7})$$

Moreover, by change of variables $u = \gamma(s), v = \gamma(t)$,

$$\begin{aligned} J_1(z) &= \int_0^{1/2} \int_0^u \frac{e^{-1/(2\sigma^2)\{z \log[u(1-u)/(v(1-v))] + R(u,v)\}}}{u(1-u)v(1-v)} du dv \\ &= \int_{-\infty}^{s_0} \int_{-\infty}^s \frac{e^{-1/(2\sigma^2)[z(s-t) + (3s^2 + 3t^2 - 2st)/4 - 2 \log(\gamma(s)) \log(1-\gamma(s)) - 2 \log(\gamma(t)) \log(1-\gamma(t))]}{\sqrt{1 - 4e^s} \sqrt{1 - 4e^t}} ds dt, \end{aligned} \quad (\text{D.8})$$

where $s_0 = -\log 4$. Making a new change of variables $x = -s + s_0, y = s - t$, we then get

$$J_1(z) = \int_0^\infty e^{-1/(2\sigma^2)(zy + y^2/2)} \int_0^\infty e^{-(x-s_0+y/2)^2/(2\sigma^2)} H(x)H(x+y) dx dy, \quad (\text{D.9})$$

with

$$H(x) = \frac{1}{\sqrt{1 - e^{-x}}} \exp \left[\frac{1}{\sigma^2} \log \left(\frac{1 - \sqrt{1 - e^{-x}}}{2} \right) \log \left(\frac{1 + \sqrt{1 - e^{-x}}}{2} \right) \right]. \quad (\text{D.10})$$

Obviously,

$$\lim_{x \rightarrow 0^+} H(x) \sqrt{x} e^{-(\log 2)^2/\sigma^2} = \lim_{x \rightarrow \infty} H(x) = 1. \quad (\text{D.11})$$

Consider

$$P(y) = \int_0^\infty e^{-(x+y/2-s_0)^2/(2\sigma^2)} H(x)H(x+y) dx, \quad (\text{D.12})$$

we claim that

$$\lim_{y \rightarrow \infty} \frac{\sqrt{y}}{\sqrt{2\pi\sigma}} e^{((y-2s_0)^2 - 8(\log 2)^2)/(8\sigma^2)} P(y) = 1. \quad (\text{D.13})$$

Indeed, for any $y > 0$

$$\begin{aligned} & \sqrt{y}e^{((y-2s_0)^2-8(\log 2)^2)/(8\sigma^2)}P(y) \\ &= \sqrt{y}e^{-(\log 2)^2/\sigma^2} \int_0^\infty e^{-(x^2+(y-2s_0)x)/(2\sigma^2)}H(x)H(x+y)dx \\ &= \frac{\sqrt{y}e^{-(\log 2)^2/\sigma^2}}{\sqrt{y-2s_0}} \int_0^\infty \frac{e^{-1/(2\sigma^2)[u+u^2/(y-2s_0)^2]}}{\sqrt{u}}H\left(\frac{u}{y-2s_0}+y\right) \times \sqrt{\frac{u}{y-2s_0}}H\left(\frac{u}{y-2s_0}\right)du. \end{aligned} \tag{D.14}$$

We use $x = u/(y - 2s_0)$ for the last equality. Using dominated convergence and (D.11), the claim (D.13) is obtained immediately. Defining $Q(y) = \sqrt{y}e^{(y-2s_0)^2/(8\sigma^2)}P(y)$, we finally get

$$J_1(z) = \int_0^\infty \frac{e^{-1/(2\sigma^2)[zy+y^2/2+(y/2-s_0)^2]}}{\sqrt{y}}Q(y)dy = e^{-s_0^2/(2\sigma^2)} \int_0^\infty \frac{e^{-1/(2\sigma^2)(zy-s_0y+3y^2/4)}}{\sqrt{y}}Q(y)dy. \tag{D.15}$$

As $z \rightarrow -\infty$, we need to understand the behavior of $W(b)$ when b tends to ∞ . More precisely, for $\lambda > 0$,

$$W(b) = \int_0^\infty \frac{e^{by-\lambda y^2}}{\sqrt{y}}Q(y)dy = 2\sqrt{b} \int_0^\infty e^{b^2(x^2-\lambda x^4)}Q(bx^2)dx. \tag{D.16}$$

Here, $G(x) = x^2 - \lambda x^4$ is not concave in $[0, \infty)$. But notice that G has a unique global maximum point at $x_0 = 1/\sqrt{2\lambda}$ in $[0, \infty)$ and $G''(x_0) = -4$. So the spirit of the Laplace method works. This implies

$$W(b) \sim_\infty \sqrt{\frac{2\pi}{b}}e^{b^2/(4\lambda)} \times \lim_{y \rightarrow \infty} Q(y). \tag{D.17}$$

Hence by (D.13),

$$J_1(z) \sim_{-\infty} \frac{2\sqrt{2}\pi\sigma^2}{\sqrt{-z}}e^{(z-s_0)^2/(6\sigma^2)-(\log 2)^2/\sigma^2}. \tag{D.18}$$

Finally, using

$$f_Z(z) = \frac{e^{-z^2/(2\sigma^2)}}{\sqrt{2\pi^3}\sigma^3} [I_1(z) + J_1(z)], \tag{D.19}$$

we conclude that $f_Z(z) \sim_{-\infty} r(z)$ where

$$r(z) = \frac{2e^{-z^2/(2\sigma^2) + (|z+s_0|^2)/(6\sigma^2) - (\log 2)^2/\sigma^2}}{\sigma\sqrt{\pi}|z|} = \frac{2e^{-(|z+\log 2|^2)/(3\sigma^2)}}{\sigma\sqrt{\pi}|z|}. \tag{D.20}$$

The proof is complete, since f_Z is an even function.

E. A Higher Dimension Left Tail Result

We will prove Theorem 2.6 in this section. Let $(X_1, X_2, \dots, X_n)_{n \geq 3}$ be a joint normal variable with distribution $N(\mu, \Sigma)$, where μ is the mean vector (μ_1, \dots, μ_n) , and Σ is a non-singular covariance matrix. Let $Z = \log(\sum_{1 \leq i \leq n} e^{X_i})$. We wish to study the left tail of $f_Z(z)$, and the PDF of z . Let f be the PDF of (X_1, X_2, \dots, X_n) , that is,

$$f(x_1, \dots, x_n) = \frac{1}{\sqrt{(2\pi)^n |\det \Sigma|}} \exp\left[-\frac{(x - \mu)\Sigma^{-1}(x - \mu)^t}{2}\right]. \tag{E.1}$$

Let $\mathbf{M} = (m_{ij}) = \Sigma^{-1}$. By change of variables, it is clear that

$$f_Z(z) = \frac{1}{\sqrt{(2\pi)^n |\det \Sigma|}} \int_{\Omega_0} \frac{1}{1 - \sum_{2 \leq i \leq n} e^{u_i}} \exp\left(-\frac{\sum_{1 \leq i, j \leq n} m_{ij} A_i A_j}{2}\right) du_2 \dots du_n, \tag{E.2}$$

where $\Omega_0 = \{(u_i)_{2 \leq i \leq n} \in \mathbb{R}^{n-1}, \sum_{2 \leq i \leq n} e^{u_i} < 1\}$, and $A \in \mathbb{R}^n$ is a vector defined by $A_i = z + C_i - \mu_i$ with C the vector given by (2.18). Consequently,

$$\sum_{1 \leq i, j \leq n} m_{ij} A_i A_j = \sum_{1 \leq i, j \leq n} m_{ij} C_i (C_j - 2\mu_j) + 2zF(u) + \sum_{1 \leq i, j \leq n} m_{ij} (z - \mu_i)(z - \mu_j), \tag{E.3}$$

with F in Theorem 2.6. Therefore, if $F(u)$ has only one saddle point, we can expect to apply the Laplace method to the higher dimensional case. One necessary and sufficient condition to ensure that is summarized in the following lemma.

Lemma E.1. *The function $F(u_2, \dots, u_n) = \sum_{1 \leq i \leq n} m_i C_i$ has a unique critical point within the set Ω_0 , if and only if for any $1 \leq k \leq n$, $m_k > 0$. Moreover, the Hessian of F , $\mathbf{H}_F = (\partial_{ij}^2 F)_{2 \leq i, j \leq n}$ is negative definite over Ω_0 when $m_1 > 0$.*

Indeed, it is easy to see that

$$\frac{\partial F}{\partial u_k} = m_k - \frac{m_1 e^{u_k}}{1 - S_1}, \quad \forall 2 \leq k \leq n, \tag{E.4}$$

where $S_1 = \sum_{2 \leq i \leq n} e^{u_i}$. Since $m_i \neq 0$, the solution to the system $\nabla F = 0$ would satisfy $e^{u_k} = m_k m^{-1}$ for all $2 \leq k \leq n$. Thus a solution exists in Ω_0 if and only if all m_i 's are positive. As

$$\frac{\partial^2 F}{\partial u_k \partial u_j} = -\frac{m_1}{1 - S_1} \left[e^{u_k} \delta_{kj} + \frac{e^{u_k} e^{u_j}}{1 - S_1} \right], \tag{E.5}$$

with δ_{kj} the Kronecker symbol, for any $\mathbf{v} = (v_i) \in \mathbb{R}^{n-1}$,

$$\langle \mathbf{v}, \mathbf{H}_F \mathbf{v} \rangle = -\frac{m_1}{1 - S_1} \left(\sum_{2 \leq k \leq n} e^{u_k} v_k^2 + \sum_{2 \leq k, j \leq n} \frac{e^{u_k} e^{u_j} v_k v_j}{1 - S_1} \right) = -\frac{m_1}{1 - S_1} \left[\sum_{2 \leq k \leq n} e^{u_k} v_k^2 + \frac{(\sum_{2 \leq k \leq n} e^{u_k} v_k)^2}{1 - S_1} \right]. \tag{E.6}$$

It is easy to conclude as long as $m_1 > 0$ and $S_1 < 1$. The following lemma is our key argument.

Lemma E.2. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let H, G be two continuous functions defined on Ω . We further assume H is a positive integrable function over Ω ; G is C^2 , strictly concave and G has a unique critical point x_0 in Ω . Then*

$$\lim_{z \rightarrow \infty} \int_{\Omega} H(x) e^{zG(x)} dx \times \frac{\sqrt{z^n |\det \mathbf{H}_G(x_0)|}}{H(x_0) e^{zG(x_0)} (2\pi)^{n/2}} = 1 \tag{E.7}$$

where \mathbf{H}_G denotes the Hessian matrix of G .

This lemma is an extension to higher dimension of Lemma A.1 and the proof is very similar, so we leave the details to interested readers. Returning to our proof, since all m_i 's are positive, we can verify that

- (i) Ω_0 is convex, since $\sum_{2 \leq i \leq n} e^{u_i}$ is a convex function.
- (ii) The function F has only one critical point $U_0 = \log(m_k/m) \in \Omega_0$ and $F(U_0) = \sum_{1 \leq k \leq n} m_k \log(m_k/m)$.
- (iii) $\mathbf{H}_F(u_2, \dots, u_n)$ is negative definite for any point $(u_2, \dots, u_n) \in \Omega_0$.
- (iv) The function H is clearly positive in Ω_0 . The integrability of H over Ω_0 is ensured by the fact that the matrix \mathbf{M} is positive definite. Because $|C|^2 = \log^2(1 - S_1) + \sum_{2 \leq i \leq n} u_i^2$, H is bounded over Ω_0 . Finally, as

$$f_Z(z) = \frac{1}{\sqrt{(2\pi)^n |\det \Sigma|}} e^{-\left(\sum_{1 \leq i, j \leq n} (z - \mu_i) m_{ij} (z - \mu_j)\right)/2} \times \int_{\Omega_0} H(u) e^{-zF(u)} du_2 \dots du_n, \tag{E.8}$$

a straightforward application of Lemma E.2 results in Theorem 2.6.

F. A Higher Dimension Right Tail Result

We will prove Theorem 2.7 by induction on the number n . It is trivially true for $n = 1$, since $Z = X_1$. Suppose that the result holds for $Z = \log(\sum_{i=1}^n e^{X_i})$. Consider now

$$Z_1 = \log\left(\sum_{i=1}^{n+1} e^{X_i}\right) = \log\left(e^Z + e^{X_{n+1}}\right), \quad (\text{F.1})$$

where (X_1, \dots, X_{n+1}) is a normally distributed random vector whose covariance matrix is diagonal. Without loss of generality, we can assume that the sequence $(\sigma_i, \mu_i)_{1 \leq i \leq n+1}$ is nonincreasing. There exist three possible cases:

- (i) $\sigma_{n+1} < \sigma = \sigma_1$;
- (ii) $\sigma_{n+1} = \sigma$ but $\mu_{n+1} < \mu = \mu_1$;
- (iii) $(\sigma_{n+1}, \mu_{n+1}) = (\sigma, \mu)$.

By translation, we can assume that $\mu = 0$.

Let us denote the PDF of $(e^Z + e^{X_{n+1}})$ by g , then $f_{Z_1}(z) = e^z g(e^z)$ is the PDF of Z_1 . Consider the asymptotic behavior of $g(z)$ as $z \rightarrow \infty$. First,

$$g(z) = \int_0^z \frac{f_Z(\log u)}{u} \times \frac{1}{\sqrt{2\pi}\sigma_{n+1}(z-u)} e^{-(\log(z-u)-\mu_{n+1})^2/(2\sigma_{n+1}^2)} du. \quad (\text{F.2})$$

For any $A > 0$ fixed (to be chosen later), as the function

$$h(t) = \frac{1}{\sqrt{2\pi}\sigma_{n+1}t} e^{-(\log t - \mu_{n+1})^2/(2\sigma_{n+1}^2)} \quad (\text{F.3})$$

is decreasing near $+\infty$, there exists $z_1 > A$ such that $h(z-u) \leq h(z-A)$ on $[0, A]$ for all $z \geq z_1$. Hence for such z ,

$$\begin{aligned} I_A(z) &= \int_0^A \frac{f_Z(\log u)}{u} h(z-u) du \leq h(z-A) \times \int_0^A \frac{f_Z(\log u)}{u} du \\ &= h(z-A) \times \int_{-\infty}^{\log A} f_Z(v) dv \\ &\leq h(z-A). \end{aligned} \quad (\text{F.4})$$

Noting that

$$\frac{[\log(z - A) - \mu_{n+1}]^2}{2\sigma_{n+1}^2} = \frac{[\log z + \log(1 - A/z) - \mu_{n+1}]^2}{2\sigma_{n+1}^2} \sim_{\infty} \frac{(\log z)^2}{2\sigma_{n+1}^2}, \tag{F.5}$$

for the case (i), we immediately see

$$I_A(z) \leq h(z - A) = o\left(\frac{1}{z} e^{-(\log z)^2/(2\sigma^2)}\right) \text{ as } z \rightarrow \infty. \tag{F.6}$$

For the case (ii), since $\mu_{n+1} < \mu = 0$, there exists $z_2 > z_1$ such that for all $z \geq z_2$,

$$\log z + \log\left(1 - \frac{A}{z}\right) - \mu_{n+1} \geq \log z - \frac{\mu_{n+1}}{2}. \tag{F.7}$$

With such z , we have

$$h(z - A) \times z e^{(\log z)^2/(2\sigma^2)} \leq \frac{1}{\sqrt{2\pi}\sigma_{n+1}} \times \frac{z}{z - A} e^{(4\mu_{n+1} \log z - \mu_{n+1}^2)/(8\sigma^2)}. \tag{F.8}$$

Recall that $\mu_{n+1} < 0$. So the estimate (F.6) still holds. Consider now $J_A = g(z) - I_A$. For any $\epsilon > 0$, by hypothesis of induction, we can fix $A > 0$ large enough such that

$$\left| \frac{f_Z(z)}{m_n N(0, \sigma^2)(z)} - 1 \right| \leq \epsilon \text{ in } [\log A, \infty). \tag{F.9}$$

Consequently $|J_A/K_A - 1| \leq \epsilon$ for $z \geq A$ where

$$K_A(z) = \int_A^z \frac{m_n N(0, \sigma^2)(\log u)}{u} \times h(z - u) du. \tag{F.10}$$

Using exactly the same proof as for Theorem 2.4 (the case $\rho < \sigma_1/\sigma_2$), we get that

$$K_A(z) \sim_{\infty} \frac{m_n N(0, \sigma^2)(\log z)}{z}. \tag{F.11}$$

Finally,

$$\begin{aligned} m_n(1 + \epsilon) &\geq \limsup_{z \rightarrow \infty} J_A(z) \times \sqrt{2\pi}\sigma z e^{(\log z)^2/(2\sigma^2)} \\ &\geq \liminf_{z \rightarrow \infty} J_A(z) \times \sqrt{2\pi}\sigma z e^{(\log z)^2/(2\sigma^2)} \geq m_n(1 - \epsilon) \end{aligned} \quad (\text{F.12})$$

(F.6) implies that $I_A = o(J_A)$ when z tends to ∞ . As $g(z) = I_A + J_A$, we conclude

$$\begin{aligned} m_n(1 + \epsilon) &\geq \limsup_{z \rightarrow \infty} g(z) \times \sqrt{2\pi}\sigma z e^{(\log z)^2/(2\sigma^2)} \\ &\geq \liminf_{z \rightarrow \infty} g(z) \times \sqrt{2\pi}\sigma z e^{(\log z)^2/(2\sigma^2)} \geq m_n(1 - \epsilon) \end{aligned} \quad (\text{F.13})$$

As ϵ is arbitrary, the proof is completed for the cases (i) and (ii) because $m_{n+1} = m_n$.

We now consider the case (iii). First we observe that (iii) simply means that $(X_i)_{1 \leq i \leq n+1}$ are i.i.d. following $N(0, \sigma^2)$ so that $m_n = n$ and $m_{n+1} = n + 1$. The estimate (F.12) is always true for any $A > 0$ satisfying (F.9). However, we need to estimate I_A differently. For any $\epsilon > 0$, we fix A such that (F.9) holds and

$$\int_0^A \frac{f_Z(\log u)}{u} = \int_{-\infty}^{\log A} f_Z(v) dv \in (1 - \epsilon, 1). \quad (\text{F.14})$$

Because in case (iii),

$$I_A \times \sqrt{2\pi}\sigma z e^{(\log z)^2/(2\sigma^2)} - \int_0^A \frac{f_Z(\log u)}{u} du = \int_0^A \frac{f_Z(\log u)}{u} R(u) du, \quad (\text{F.15})$$

with

$$R(u) = \frac{z}{z - u} e^{-1/(2\sigma^2)[2 \log z \log(1-u/z) + \log^2(1-u/z)]} - 1, \quad (\text{F.16})$$

which converges uniformly to 0 in $[0, A]$, we obtain

$$1 \geq \limsup_{z \rightarrow \infty} I_A(z) \times \sqrt{2\pi}\sigma z e^{(\log z)^2/(2\sigma^2)} \geq \liminf_{z \rightarrow \infty} I_A(z) \times \sqrt{2\pi}\sigma z e^{(\log z)^2/(2\sigma^2)} \geq 1 - \epsilon. \quad (\text{F.17})$$

Combining with (F.12), it is easy to conclude.

Acknowledgment

The authors would like to thank the Editor, the anonymous referee for their valuable comments and suggestions.

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