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Research Article

A Note on Topological Properties of Non-Hausdorff Manifolds

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The notion of compatible apparition points is introduced for non-Hausdorff manifolds, and properties of these points are studied. It is well known that the Hausdorff property is independent of the other conditions given in the standard definition of a topological manifold. In much of literature, a topological manifold of dimension n is a Hausdorff topological space which has a countable base of open sets and is locally Euclidean of dimension n. We begin with the definition of a non-Hausdorff topological manifold.

1. Topological Properties of Non-Hausdorff Manifolds

Definition 1.1. A non-Hausdorff manifold of dimension n is a topological space which has a countable base of open sets and is locally Euclidean of dimension n.

Since every point of a non-Hausdorff manifold has a Euclidean neighborhood, it is easy to show that every non-Hausdorff manifold is T_1.

We now briefly review some of the well-known properties of non-Hausdorff manifolds. Since $\mathbb{R}^n$ is locally compact, a non-Hausdorff manifold of dimension n is locally compact. In some of literature, compactness is only defined in Hausdorff spaces. In such cases, compact subsets must be closed. Compact subsets of T_1-spaces, however, need not to be closed. This remains true for non-Hausdorff manifolds (Example 1.2). A non-Hausdorff manifold of dimension n must be locally connected. Since a non-Hausdorff manifold M has a countable base of open sets, M is Lindelöf; that is, every open cover of M has a countable subcover. Further, since locally compact Lindelöf spaces are sigma-compact, it follows that a non-Hausdorff manifold M of dimension n is sigma-compact. Finally, we note that when M is not Hausdorff, it is not regular.

We now consider the property of paracompactness. A Hausdorff space X is paracompact if every open covering $\mathcal{U}$ of X has a locally finite refinement $\mathcal{V}$. That is, each

$V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$ and each $x \in X$ has a neighborhood N which meets only finitely many sets in $\mathcal{V}$. Paracompactness can be defined for T_1-spaces as follows. A T_1-space X is paracompact if and only if each open covering of X has an open barycentric refinement, where $\mathcal{V}$ is a barycentric refinement of $\mathcal{U}$ if the collection $\{St(x, \mathcal{V}) : x \in X\}$ refines $\mathcal{U}$, where $St(x, \mathcal{V}) = \cup\{V \in \mathcal{V} : x \in V\}$. A space is metacompact if every open cover has a point finite refinement. Since Hausdorff second countable manifolds are metrizable, they are paracompact and hence metacompact. In [1], an example of a non-Hausdorff manifold which is not metacompact is given. We present another one.

Example 1.2. A non-Hausdorff manifold M need not to be metacompact.

Let $M = \mathbb{R} \cup (\mathbb{Q} \times \{1\})$ and define a topology on M as follows.

(i) For each $x \in \mathbb{R}$, a basic open neighborhood of x is open in $\mathbb{R}$ with the usual topology.

(ii) For each $(q, 1) \in \mathbb{Q} \times \{1\}$, a basic open neighborhood of $(q, 1)$ is of the form $[\{(q, 1)\} \cup U] \setminus \{q\}$ where U is an open neighborhood of q in $\mathbb{R}$ with the usual topology.

Claim 1. The non-Hausdorff manifold M is not metacompact.

Proof. Let $\mathcal{U} = \{\{(q, 1)\} \cup \mathbb{R} : q \in \mathbb{Q}\}$. To see that $\mathcal{U}$ has no point finite refinement, let $\mathcal{V}$ be a refinement of $\mathcal{U}$. Let $q_0 \in \mathbb{Q}$ and $\varepsilon_0 > 0$ such that $(q_0 - \varepsilon_0, q_0 + \varepsilon_0)$ is a subset of some element of $\mathcal{V}$. For each $n \in \mathbb{N}$, choose $q_n \in \mathbb{Q}$, $\varepsilon_n > 0$, and $V_n \in \mathcal{V}$ such that $[q_n - \varepsilon_n, q_n + \varepsilon_n] \subseteq (q_{n-1} - \varepsilon_{n-1}, q_{n-1} + \varepsilon_{n-1}) \setminus \{q_{n-1}\}$, $\varepsilon_n < 1/n$, and $(\{(q_n, 1)\} \cup [q_n - \varepsilon_n, q_n + \varepsilon_n]) \setminus \{q_n\} \subseteq V_n$. By the way $\mathcal{U}$ is defined, no element of $\mathcal{U}$ contains more than one element of $\mathbb{Q} \times \{1\}$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, no element of $\mathcal{V}$ contains more than one element of $\mathbb{Q} \times \{1\}$. Hence, $V_j \neq V_k$ whenever $j \neq k$. By Cantor's Intersection theorem, there exists $x \in \mathbb{R}$ such that $\{x\} = \bigcap_{n=1}^{\infty} [q_n - \varepsilon_n, q_n + \varepsilon_n] \subseteq \bigcap_{n=1}^{\infty} V_n$. Therefore, $\mathcal{V}$ is not point finite. □

Remark 1.3. In the above example, $[0, 1]$ is compact and Hausdorff but not closed.

Remark 1.4. For each $n \in \mathbb{N}, \mathbb{R}^n$ is a complete metric space and $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$. Therefore, a construction similar to the one above can be used to create a non-Hausdorff manifold of dimension n that is not metacompact.

2. Compatible Apparition Points

If a manifold M of dimension n is non-Hausdorff, there exist at least two points x and y which cannot be separated by disjoint open sets. Also, the points x and y cannot be contained in the same Euclidean neighborhood since Euclidean neighborhoods are Hausdorff.

Definition 2.1. Let M be a non-Hausdorff manifold and let x and y be distinct points of M. Then x and y are compatible apparition points if there do not exist disjoint open sets U and V with $x \in U$ and $y \in V$. By a "set of compatible apparition points," we will mean that any pair of distinct points in the set are compatible apparition points.

Remark 2.2. Since a non-Hausdorff manifold is locally Hausdorff, then no more than one element of a set of compatible apparition points can be contained in a single Euclidean neighborhood. Hence, a set of compatible apparition points is a closed discrete set.

Remark 2.3. Since a non-Hausdorff manifold has a countable base and each point is contained in its own Euclidean neighborhood, any set of compatible apparition points must be countable.

A non-Hausdorff manifold can have an uncountable collection of sets of compatible apparition points.

Example 2.4. Let C denote the Cantor ternary set and define $X = \mathbb{R} \cup (C \times \{0\})$. Define a topology on X as follows.

(i) For each $x \in \mathbb{R}$, a basic open neighborhood of x is open in $\mathbb{R}$ with the usual topology.

(ii) For each $x \in C$, a basic open neighborhood of $(x,0)$ is of the form $[(x-\varepsilon, x+\varepsilon) \cap C] \times \{0\} \cup (x-\varepsilon, x+\varepsilon) \setminus C$.

Note that for each $x \in C$, $\{x, (x,0)\}$ is a set of compatible apparition points. Also, note that since each ε can be chosen to be rational, X is second countable.

Recall that a subset of a topological space is nowhere dense if the interior of its closure is empty.

Proposition 2.5. *Let S be a set of compatible apparition points in a non-Hausdorff manifold M. Then S is nowhere dense in M.*

Proof. Since S is closed and discrete and every element of M has a Euclidean neighborhood, S is the frontier of $M \setminus S$ which is open. Hence, S is nowhere dense by [2, 4G part 2 on page 37]. □

Proposition 2.6. *Let M be an n-dimensional non-Hausdorff manifold. Suppose that M contains a nonempty set S of compatible apparition points. Then every continuous function from M to a Hausdorff space X is constant on S.*

Proof. Suppose that $f : M \to X$ is continuous. Attempting a contradiction, suppose that $x_1, x_2 \in S$ such that $f(x_1) \neq f(x_2)$. Since X is Hausdorff, there are disjoint open sets $U_1, U_2 \subseteq X$ such that $f(x_1) \in U_1$ and $f(x_2) \in U_2$. Then $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are disjoint open subsets of M with $x_1 \in f^{-1}(U_1)$ and $x_2 \in f^{-1}(U_2)$, a contradiction. □

Theorem 2.7. *In a non-Hausdorff manifold, the set of points which are not apparition points is dense.*

Proof. Suppose that M is a non-Hausdorff manifold. Since M is locally Hausdorff, Lemma 4.2 of [3] implies that each $x \in M$ has a dense open Hausdorff neighborhood U_x. Since M is Lindelöf, the cover $\{U_x\}_{x \in M}$ has a countable subcover $\mathcal{C}$. Since M is Baire, $\cap\mathcal{C}$ is dense in M. Since the elements of $\mathcal{C}$ are Hausdorff, any point in $\cap\mathcal{C}$ can be separated from any other point in M. Therefore, $\cap\mathcal{C}$ is a dense set of nonapparition points. □

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