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Research Article

First Hitting Place Probabilities for a Discrete Version of the Ornstein-Uhlenbeck Process

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A Markov chain with state space $\{0, \ldots, N\}$ and transition probabilities depending on the current state is studied. The chain can be considered as a discrete Ornstein-Uhlenbeck process. The probability that the process hits N before 0 is computed explicitly. Similarly, the probability that the process hits N before $-M$ is computed in the case when the state space is $\{-M, \ldots, 0, \ldots, N\}$ and the transition probabilities $p_{i,i+1}$ are not necessarily the same when i is positive and i is negative.

1. Introduction

The Ornstein-Uhlenbeck process $\{X(t), t \geq 0\}$ is defined by the stochastic differential equation

$$dX(t) = -cX(t)dt + dW(t), \tag{1.1}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion and c is a positive constant. Discrete versions of this very important diffusion process have been considered by various authors. In particular, Larralde [1, 2] studied the discrete-time process $\{X_n, n = 0, 1, \ldots\}$ for which

$$X_{n+1} = \gamma X_n + Y_{n+1}, \quad (X_0 = x_0), \tag{1.2}$$

where the random variables Y_{n+1} are i.i.d. with zero mean and a common probability distribution. Larralde computed the probability that $\{X_n, n = 0, 1, \ldots\}$ will hit the negative semiaxis for the first time at the nth step, starting from $X_0 = 0$. The problem was solved exactly in the case when the distribution of the random variables Y_n is continuous and such that

$$f_{Y_n}(y) = \frac{1}{2}e^{-|y|} \tag{1.3}$$

for $y \in \mathbb{R}$ and for all $n \in \{1, 2, \ldots\}$.

Versions of the discrete Ornstein-Uhlenbeck process have also been studied by, among others, Renshaw [3], Anishchenko et al. [4, page 53], Bourlioux et al. [5, page 236], Sprott [6, page 234], Kontoyiannis and Meyn [7], and Milstein et al. [8]. In many cases, the distribution of the Y_n's is taken to be $N(0, \sigma^2)$.

For discrete versions of diffusion processes, in general, see Kac [9] and the references therein. A random walk leading to the Ornstein-Uhlenbeck process is considered in Section 4 of Kac's paper.

Next, consider a Markov chain for which the displacements take place every Δt units of time. When the process is in state x, it moves to $x + \Delta x$ (resp., $x - \Delta x$) with probability $\theta(x)$ (resp., $\phi(x)$) and remains in x with probability $1 - \theta(x) - \phi(x)$.

Assume that $(\Delta x)^2 = A\Delta t$, and let

$$\theta(x) = \frac{1}{2A}\left[\alpha(x) + \beta(x)\Delta x\right], \qquad \phi(x) = \frac{1}{2A}\left[\alpha(x) - \beta(x)\Delta x\right], \tag{1.4}$$

where A is a positive constant such that $\alpha(x) < A$ for all x. Then, when Δx and Δt decrease to zero, the Markov chain converges to a diffusion process having infinitesimal mean $\beta(x)$ and infinitesimal variance $\alpha(x)$ (see [10, page 213]). In the case of the Ornstein-Uhlenbeck process, $\beta(x) = -cx$ (with $c > 0$) and $\alpha(x) \equiv 1$. Hence, with $A = 2$, we have that

$$\theta(x) = \frac{1}{4}(1 - cx\Delta x), \qquad \phi(x) = \frac{1}{4}(1 + cx\Delta x). \tag{1.5}$$

In the present paper, we first consider the Markov chain with state space $\{0, \ldots, N\}$ and

$$p_{i,i+1} = \frac{1}{4}(1 - ci), \qquad p_{i,i-1} = \frac{1}{4}(1 + ci), \qquad p_{i,i} = \frac{1}{2} \tag{1.6}$$

for $i = 1, \ldots, N-1$. Notice that $p_{i,i+1}$ (resp., $p_{i,i-1}$) could be denoted by θ_i (resp., ϕ_i). To respect the condition $p_{i,j} \in [0, 1]$ for all i, j, the positive constant c must be such that

$$c < \frac{1}{N-1}. \tag{1.7}$$

This Markov chain with state-dependent transition probabilities may also clearly be regarded as a discrete version of the Ornstein-Uhlenbeck process. It corresponds to the case when $\gamma = 1$ in (1.2) and

$$Y_{n+1} = \begin{cases} -1 & \text{with probability } p_{X_n, X_{n-1}}, \\ 0 & \text{with probability} \dfrac{1}{2}, \\ 1 & \text{with probability } p_{X_n, X_{n+1}}, \end{cases} \tag{1.8}$$

for $X_n \in \{1, 2, \ldots, N-1\}$.

In Section 2, the probability

$$p_i := P[X_\tau = N \mid X_0 = i], \tag{1.9}$$

where

$$\tau := \inf\{n > 0 : X_n = 0 \text{ or } N\} \tag{1.10}$$

and $i \in \{1, 2, \ldots, N-1\}$, will be computed explicitly. In Section 3, the problem will be extended by assuming that the state space of the Markov chain $\{X_n, n = 0, 1, \ldots\}$ is $\{-M, \ldots, 0, \ldots, N\}$. Furthermore, the transition probabilities $p_{i,j}$ will be assumed to be (possibly) sign-dependent (see [11]). Finally, some concluding remarks will be made in Section 4.

2. First Hitting Place Probabilities

To obtain the first hitting place probability defined in (1.9), we may try to solve the following difference equation:

$$\begin{gathered} p_i = \frac{(1-ci)}{4} p_{i+1} + \frac{1}{2} p_i + \frac{(1+ci)}{4} p_{i-1} \\ \Longleftrightarrow p_i = \frac{(1-ci)}{2} p_{i+1} + \frac{(1+ci)}{2} p_{i-1} \end{gathered} \tag{2.1}$$

for $i = 1, \ldots, N-1$, subject to the boundary conditions

$$p_0 = 0, \qquad p_N = 1. \tag{2.2}$$

For N small, it is a relatively simple task to calculate explicitly p_i for all i by solving a system of linear equations. However, we want to obtain an exact expression for any positive N.

Next, setting $x = i - 1$ and letting $y(x) = p_{i-1}$, (2.1) can be rewritten as

$$\left[\frac{1 - c(x+1)}{2}\right] y(x+2) - y(x+1) + \left[\frac{1 + c(x+1)}{2}\right] y(x) = 0 \tag{2.3}$$

for $x = 0, \ldots, N-2$ (with $y(0) = 0$ and $y(N) = 1$). This second-order homogeneous equation with linear coefficients is called the *hypergeometric difference equation*, due to the fact that its solutions can be expressed in terms of the hypergeometric function (see [12, page 68]).

Equation (2.3) can be transformed into its *normal form*, namely,

$$(x + \beta_1 + \beta_2 + 2)y(x+2) - \left[(\rho_1 + \rho_2)(x+1) + \beta_1\rho_2 + \beta_2\rho_1\right]y(x+1) + \rho_1\rho_2 x y(x) = 0. \tag{2.4}$$

In our case, we have (see [12, pages 68-69])

$$\rho_1 = 1, \qquad \rho_2 = -1, \qquad \beta_1 = -1, \qquad \beta_2 = -1 - \frac{2}{c}, \tag{2.5}$$

so that we must solve

$$(x-a)y(x+2) + ay(x+1) - xy(x) = 0, \tag{2.6}$$

where

$$a := \frac{2}{c}. \tag{2.7}$$

Furthermore, the variable x now belongs to the set $\{1 + 1/c, \ldots, N - 1 + 1/c\}$ (because the new argument of the function y is $x' = x + \beta_3$, where $\beta_3 = 1 + 1/c$ in our problem).

Using the results in Batchelder [12, Chapter III], we can state that a fundamental system of solutions of (2.6) is

$$\begin{gathered} y_1(x) \equiv \gamma \in \mathbb{R}, \\ y_2(x) = (-1)^x \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right), \end{gathered} \tag{2.8}$$

where $F(\cdot,\cdot,\cdot,\cdot)$ is the hypergeometric function defined by (see [13, page 556])

$$F(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{2.9}$$

with

$$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1), \quad (\text{and } (\alpha)_0 = 1). \tag{2.10}$$

Remarks. (i) The function F is sometimes denoted by ${}_2F_1$. It can also be expressed as (see, again, [13, page 556])

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!}. \tag{2.11}$$

(ii) The ratio $F(\alpha, \beta, \gamma, z_0)/\Gamma(\gamma)$ is an entire function of α, β, and γ if z_0 is fixed and such that $|z_0| < 1$ (see [14, page 68]).

Now, because of the term $(-1)^x$, the function $y_2(x)$ defined previously is generally complex-valued. Since the function $y(x)$ in our application is obviously real, we can take the real part of $y_2(x)$. That is, we simply have to replace $(-1)^x$ by $\cos(\pi x)$. Alternatively, because $[x] = [i + 1 + 1/c] = i + 1 + [1/c]$, where [ ] denotes the integer part, we can write that

$$(-1)^x = (-1)^{[x]+1/c-[1/c]}. \tag{2.12}$$

With the difference equation (2.6) being homogeneous, we can state that

$$y_2(x) = (-1)^{[x]} \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right) \tag{2.13}$$

is a real-valued function that is also a solution of this equation. Hence, the general solution of (2.6) can be expressed as

$$y(x) = \gamma_1 + \gamma_2 (-1)^{[x]} \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right), \tag{2.14}$$

where γ_1 and γ_2 are arbitrary (real) constants.

(iii) We must be careful when the constant c is of the form

$$c = \frac{1}{N+j}, \tag{2.15}$$

where $j \in \{0, 1, \ldots\}$. Indeed, because $\beta_1 = -1$, (2.6) is *reducible*. Moreover, it is *completely reducible* if β_2 is also a negative integer (see [12, pages 123-124]), that is, if

$$\beta_2 = -1 - \frac{2}{c} = -k \tag{2.16}$$

with $k \in \{1, 2, \ldots\}$. Since c must be smaller than $1/(N-1)$ (see (1.7)), this condition translates into

$$c = \frac{1}{N+j} \quad \text{or} \quad c = \frac{2}{2(N+j)-1}, \tag{2.17}$$

where $j \in \{0, 1, \ldots\}$. We find that the case when $c = 2/[2(N+j)-1]$ does not really cause any problem. However, when $c = 1/(N+j)$, we can show that although $y_1(x) \equiv \gamma$ and $y_2(x)$ defined in (2.13) are obviously linearly independent when we consider all possible values of the argument x, it turns out that in our problem $y_2(x)$ always takes on the same value. More precisely, we can show that

$$y_2(x) = (-1)^{[x]} P_a(x), \tag{2.18}$$

where $P_a(x)$ is a polynomial of degree a, with

$$a = \frac{2}{c} = 2(N + j), \tag{2.19}$$

given by

$$P_a(x) = \frac{a!}{x}\left(-\frac{1}{2}\right)^a \sum_{n=0}^{a} (-2)^n \frac{x^{(n+1)}}{n!}, \tag{2.20}$$

with

$$x^{(i)} := x(x-1)\cdots(x-i+1) \tag{2.21}$$

for any natural number i.

Remark. The formula for $P_a(x)$ is valid if $a = 2(N + j) - 1$ as well, so that we can set a equal to $2N + j$, with $j \in \{-1, 0, 1, \ldots\}$, above.

Now, we find that

$$y_2(x) = (-1)^{a+1}\frac{a!}{2^a} \quad \text{if } x = 1, 2, \ldots, a+1. \tag{2.22}$$

For example, suppose that $N = 3$, so that the state space of the Markov chain is $\{0, 1, 2, 3\}$, and that $c = 1/3$. Because $x = i + 4$, the possible values of x are $4, 5, 6, 7$. Furthermore, $a = 2/c = 6$. The solution $y_2(x)$ can be written as

$$y_2(x) = (-1)^{[x]}\left\{x^6 - 24x^5 + \frac{455}{2}x^4 - 1080x^3 + 2674x^2 - 3216x + \frac{5715}{4}\right\}. \tag{2.23}$$

It is a simple matter to show that this function satisfies (2.6) with $a = 6$. However, we calculate

$$y_2(x) = -\frac{45}{4} \quad \text{for } x \in \{1, 2, \ldots, 7\}. \tag{2.24}$$

Thus, $y_1(x)$ and $y_2(x)$ are both constant for the values of interest of x in our problem.

Actually we easily find that $p_1 = 1/13$ and $p_2 = 3/13$ in this example. Therefore, we cannot make use of $y_1(x)$ and $y_2(x)$ to obtain p_i. Nevertheless, because $y_2(x)$ is a continuous function of the parameter c, we simply have to take the limit as c tends to $1/(N + j)$ to get the solution we are looking for.

Next, we have obtained the general solution of (2.6) in (2.14). We must find the constants γ_1 and γ_2 for which the boundary conditions

$$y\left(1 + \frac{a}{2}\right) = 0, \qquad y\left(N + 1 + \frac{a}{2}\right) = 1 \tag{2.25}$$

are satisfied. We can state the following proposition.

Proposition 2.1. *When $c \neq 2/[2(N+j)-1]$ for $j \in \{0,1,\ldots\}$, the probability p_i defined in* (1.9) *is given by*

$$p_i = y\left(i+1+\frac{1}{c}\right), \tag{2.26}$$

where the function $y(\cdot)$ is defined in (2.14), *with*

$$\begin{aligned}\gamma_2 &= (-1)^{[a/2]}\left\{\frac{a}{2}\sqrt{\pi}\frac{\Gamma(a/2)}{\Gamma((1-a)/2)}\right.\\ &\qquad\left. -(-1)^N\frac{\Gamma(N+1+a/2)}{\Gamma(N+1-a/2)}F\left(-a,1,N+1-\frac{a}{2},\frac{1}{2}\right)\right\}^{-1},\\ \gamma_1 &= \gamma_2(-1)^{[a/2]}\frac{a}{2}\sqrt{\pi}\frac{\Gamma(a/2)}{\Gamma((1-a)/2)}.\end{aligned} \tag{2.27}$$

In the case when $c = 2/[2(N+j)-1]$, the constants γ_1 and γ_2 become

$$\gamma_1 = 0, \qquad \gamma_2 = (-1)^N\frac{\Gamma(N+1-a/2)}{\Gamma(N+1+a/2)F(-a,1,N+1-a/2,1/2)}. \tag{2.28}$$

Proof. We find (see [13, page 557]) that $y_2(x)$ evaluated at $x = 1+a/2$ (i.e., $i = 0$) can be expressed as

$$y_2\left(1+\frac{a}{2}\right) = (-1)^{1+[a/2]}\frac{a}{2}\sqrt{\pi}\frac{\Gamma(a/2)}{\Gamma((1-a)/2)}. \tag{2.29}$$

This is actually obtained as a limit when $c = 1/(N+j)$ with $j \in \{0,1,\ldots\}$. Moreover, it follows that as c tends to $2/[2(N+j)-1]$, we have

$$y_2\left(1+\frac{a}{2}\right) \longrightarrow 0. \tag{2.30}$$

Hence, for any $c \neq 2/[2(N+j)-1]$, the constants γ_1 and γ_2 are uniquely determined from the boundary conditions (2.25), while $c = 2/[2(N+j)-1]$ immediately yields that $\gamma_1 = 0$ and that γ_2 is as in (2.28). □

Remarks. (i) We see that the case when the difference equation is completely reducible is rather special. When $c = 2/[2(N+j)-1]$, the constant γ_1 vanishes, while when $c = 1/(N+j)$, the probability p_i is obtained by taking the limit of the previous solution when c tends to this particular value.

(ii) We can obtain an approximate formula for the probability p_i, valid for N large, by proceeding as follows. First, because (by assumption) $c < 1/(N-1)$, we can write that

$$c = \frac{1}{N-1+\kappa} \Longleftrightarrow \frac{1}{c} = N-1+\kappa, \tag{2.31}$$

where $\kappa > 0$. Hence,

$$\frac{1}{c} \approx N - 1 + [\kappa] + \frac{1}{2} = N + [\kappa] - \frac{1}{2}. \tag{2.32}$$

Notice that the relative error ϵ_r committed by replacing $1/c$ by its approximate value is such that

$$\epsilon_r \leq \frac{1/2}{N - 1 + \kappa}, \tag{2.33}$$

so that it is negligible when N is large. Moreover, for this approximate value of the constant c, we can express the solution in terms of the polynomial in (2.20), with $a = 2(N + [\kappa]) - 1$. Making use of the boundary conditions, we deduce that

$$y(x) \approx \frac{(-1)^{[x]} P_a(x)}{(-1)^N P_a(N + 1 + a/2)}. \tag{2.34}$$

We can simply write that

$$y(x) \approx \left| \frac{P_a(x)}{P_a(N + 1 + a/2)} \right|. \tag{2.35}$$

Since $P_a(x)$ is a polynomial of degree a, we find that we have approximated the function $y(x)$ by a polynomial of degree $2[1/c] + 1$.

In the next section, the state space of the Markov chain will be extended to $\{-M, \ldots, 0, \ldots, N\}$ and the (possibly) *asymmetric* case will be treated.

3. The Asymmetric Case

We extend the problem considered in the previous section by assuming that the state space of the Markov chain $\{X_n, n = 0, 1, \ldots\}$ is the set

$$S := \{-M, \ldots - 1, 0, 1 \ldots, N\}, \tag{3.1}$$

where $M \in \{1, 2, \ldots\}$. Furthermore, we set

$$p_{0,1} = p_0, \qquad p_{0,-1} = q_0, \qquad p_{0,0} = \frac{1}{2}, \tag{3.2}$$

where $p_0, q_0 \in (0, 1)$ and $p_0 + q_0 = 1/2$.

When i is a negative state, we define

$$p_{i,i+1} = \frac{1 - di}{4}, \qquad p_{i,i-1} = \frac{1 + di}{4}, \qquad p_{i,i} = \frac{1}{2} \tag{3.3}$$

for $i \in \{-M+1, \dots, -1\}$. In order to respect the condition $p_{i,j} \in [0,1]$ for all i, j, we find that the positive constant d must be such that

$$d < \frac{1}{M-1}. \tag{3.4}$$

Let

$$T = \inf\{n > 0 : X_n = N \text{ or } -M\}. \tag{3.5}$$

We want to compute the first hitting place probability

$$\pi_i := P[X_T = N \mid X_0 = i] \tag{3.6}$$

for $i \in \{-M+1, \dots, 0, \dots, N-1\}$. We have

$$\pi_N = 1, \qquad \pi_{-M} = 0. \tag{3.7}$$

Let us denote the probability p_i defined in (1.9) by $p_N(i)$ and define

$$p_M(i) = P[X_\sigma = -M \mid X_0 = i], \tag{3.8}$$

where $i \in \{-M+1, \dots, -1\}$, and

$$\sigma := \inf\{n > 0 : X_n = -M \text{ or } 0\}. \tag{3.9}$$

Proceeding as in Section 2, we can show that

$$p_M(i) = l_1 + l_2(-1)^{i+1+[1/d]}\frac{\Gamma(i+1+1/d)}{\Gamma(i+1-1/d)}F\left(-\frac{2}{d}, 1, i+1-\frac{1}{d}, \frac{1}{2}\right), \tag{3.10}$$

where the constants l_1 and l_2 are uniquely determined from the boundary conditions

$$p_M(-M) = 1, \qquad p_M(0) = 0. \tag{3.11}$$

Again, we must be careful in the case when the difference equation is completely reducible.

Next, we define the events

$$\begin{aligned} E_i &= \text{the process hits } N \text{ before } -M \text{ from } i \in S; \\ F_i &= \text{the process hits } N \text{ before } 0 \text{ from } i > 0; \\ G_i &= \text{the process hits } -M \text{ before } 0 \text{ from } i < 0. \end{aligned} \tag{3.12}$$

Assume first that i is positive. Then, we can write that

$$\begin{aligned}\pi_i \equiv P[E_i] = P[E_i \cap F_i] &+ P\left[E_i \cap F_i^c\right] \\ &= p_N(i) + 2\left[1 - p_N(i)\right]\left\{\pi_1 p_0 + \pi_{-1} q_0\right\}.\end{aligned} \tag{3.13}$$

When i is negative, we have

$$\begin{aligned}\pi_i &= P[E_i \cap G_i] + P\left[E_i \cap G_i^c\right] = P\left[E_i \cap G_i^c\right] \\ &= 2p_M(i)\left\{\pi_1 p_0 + \pi_{-1} q_0\right\}.\end{aligned} \tag{3.14}$$

Setting $i = 1$ (resp., -1) in (3.13) (resp., (3.14)), we obtain a system of two linear equations for π_1 and π_{-1}:

$$\begin{aligned}\pi_1 &= p_N(1) + 2\left[1 - p_N(1)\right]\left\{\pi_1 p_0 + \pi_{-1} q_0\right\}, \\ \pi_{-1} &= 2p_M(-1)\left\{\pi_1 p_0 + \pi_{-1} q_0\right\}.\end{aligned} \tag{3.15}$$

Proposition 3.1. *The probability π_i defined in* (3.6) *is given for $i > 0$ (resp., $i < 0$) by* (3.13) *(resp.,* (3.14)*), in which*

$$\begin{aligned}\pi_1 &= \frac{p_N(1)\left[1 - 2q_0 p_M(-1)\right]}{1 - 2q_0 p_M(-1) - 2p_0\left[1 - p_N(1)\right]}, \\ \pi_{-1} &= \frac{2p_0 p_M(-1) p_N(1)}{1 - 2q_0 p_M(-1) - 2p_0\left[1 - p_N(1)\right]}.\end{aligned} \tag{3.16}$$

Remarks. (i) If $p_0 = q_0 = 1/4$, the formulas for π_1 and π_{-1} reduce to

$$\pi_1 = \frac{p_N(1)\left[2 - p_M(-1)\right]}{1 - p_M(-1) + p_N(1)}, \qquad \pi_{-1} = \frac{p_M(-1) p_N(1)}{1 - p_M(-1) + p_N(1)}. \tag{3.17}$$

Moreover, if $M = N$ and $d = c$, then (by symmetry) $p_M(-1) = p_N(1)$ and

$$\pi_1 = p_N(1)\left[2 - p_N(1)\right], \qquad \pi_{-1} = p_N^2(1). \tag{3.18}$$

(ii) The probability

$$\nu_i := P[X_T = -M \mid X_0 = i] \tag{3.19}$$

is of course given by $1 - \pi_i$, for $i = -M + 1, \ldots, 0, \ldots, N - 1$.

4. Concluding Remarks

In Section 2, we computed the probability p_i that a Markov chain with transition probabilities given by (1.6) and state space $\{0, 1, \ldots, N\}$ will hit N before 0, starting from $i \in \{1, \ldots, N-1\}$. If we let c decrease to 0 in (1.6), we obtain that

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{4}, \quad p_{i,i} = \frac{1}{2} \quad \text{for } i \in \{1, 2, \ldots, N-1\}. \tag{4.1}$$

That is, the Markov chain $\{X_n, n = 0, 1, \ldots\}$ is a (generalized) symmetric random walk having a probability $p_{i,i} = 1/2$ of remaining in its current state on each *transition*. The fact that $p_{i,i} > 0$ should not influence the probability p_i. Taking the limit as c decreases to 0 (i.e., $a \to \infty$) in Proposition 2.1, we indeed retrieve the well-known formula

$$p_i = \frac{i}{N} \quad \text{for } i = 0, 1, \ldots, N. \tag{4.2}$$

In Section 3, we were able to compute explicitly the probability π_i defined in (3.6) for a (possibly) asymmetric Markov chain with state space $\{-M, \ldots, 0, \ldots, N\}$. This type of Markov chain could have applications in mathematical finance, in particular. Indeed, if one is looking for the probability that the value of a certain stock reaches a given level before a lower one, it can be more realistic to assume that the stock price does not vary in the same way when the price is high or low. Hence, the assumption that the transition probabilities may be different when $X_n > 0$ and $X_n < 0$ seems plausible in some applications. In the application we have just mentioned, 0 could be the centered current value of the stock.

Next, another problem of interest is the determination of the average time D_i the process, starting from i, takes to hit either 0 or N (in Section 2), or $-M$ or N (in Section 3). To obtain an explicit expression for D_i, we must solve a nonhomogeneous linear difference equation. Finding a particular solution to this equation (in order to obtain the general solution by using the solution to the homogeneous equation obtained in the present work) is a surprisingly difficult problem.

Finally, we could try to take the limit of the Markov chain $\{X_n, n = 0, 1, \ldots\}$ in such a way as to obtain the Ornstein-Uhlenbeck process as a limiting process. We should retrieve the known formula for the probability p_i in the case of this process considered in the interval $[0, N]$ and generalize this formula to the asymmetric case, based on Section 3.

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