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Research Article

Convergence of Path and Approximation of Common Element of Null Spaces of Countably Infinite Family of m-Accretive Mappings in Uniformly Convex Banach Spaces

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We prove path convergence theorems and introduce a new iterative sequence for a countably infinite family of m-accretive mappings and prove strong convergence of the sequence to a common zero of these operators in uniformly convex real Banach space. Consequently, we obtain strong convergence theorems for a countably infinite family of pseudocontractive mappings. Our theorems extend and improve some important results which are announced recently by various authors.

1. Introduction

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \left\| f^* \right\|^2 \right\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between members of E and E^*. It is well known that if E^* is strictly convex, then J is single-valued (see, e.g., [1, 2]). In the sequel, we will denote the single-valued normalized duality mapping by j.

A mapping $A : D(A) \subseteq E \to E$ is called *accretive* if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0. \tag{1.2}$$

By the result of Kato [3], (1.2) is equivalent to

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\|, \quad \forall s > 0. \tag{1.3}$$

If E is a Hilbert space, accretive operators are also called *monotone*. An operator A is called *m-accretive* if it is accretive and $R(I + rA)$, range of $(I + rA)$, is E for all $r > 0$; and A is said to satisfy the range condition if $\text{cl}(D(A)) \subseteq R(I + rA)$, for all $r > 0$, where $\text{cl}(D(A))$ denotes the closure of the domain of A. It is easy to see that every m-accretive operator satisfies the range condition. An operator A is said to be *maximal accretive* if it is accretive and the inclusion $G(A) \subseteq G(B)$ implies $G(A) = G(B)$, where $G(A)$ denotes the graph of A and B is an accretive operator.

A mapping $T : D(T) \subseteq E \to R(T) \subseteq E$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T). \tag{1.4}$$

It is not difficult to deduce from (1.3) that a mapping A is accretive if and only if its resolvent $J_r := (I + rA)^{-1}$, for all $r > 0$, is nonexpansive and single valued on the range of $(I + rA)$. Thus, in particular, $J_A = J_1 := (I + A)^{-1}$ is nonexpansive and single valued on the range of $(I + A)$. Furthermore, $F(J_A) := A^{-1}(0) := \{x \in D(A) : Ax = 0\}$. For more details see, for example, [4, 5].

Closely related to the class of accretive operators is the class of pseudocontractive maps. An operator T with domain $D(T)$ in E and range $R(T)$ in E is called *pseudocontractive* if $A := I - T$ is accretive. The importance of these operators in application is well known (see, e.g., [6–9] and the references contained therein).

It is well known that the class of pseudocontractive mappings properly contains the class of nonexpansive mappings (see, e.g., [4]). Construction of fixed points of nonexpansive mappings is an important subject in nonlinear operator theory and its applications, in particular, in image recovery and signal processing (see, e.g., [10]).

Iterative approximation of fixed points and zeros of nonlinear operators have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequality problems (see, e.g., [11–15]). The iterative scheme

$$x_0 \in E, \quad x_{n+1} = J_{r_n} x_n, \quad n \geq 0, \tag{1.5}$$

(where J_{r_n} is the resolvent of an m-accretive operator A) for example, has been extensively studied over the past forty years or so for construction of zeros of accretive operators (see, e.g., [16–20]).

Kim and Xu [21] introduced a modification of Mann iterative scheme in a reflexive Banach space having weakly continuous duality mapping for finding a zero of an m-accretive operator A as follows:

$$x_0 = u \in E, \qquad x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0. \tag{1.6}$$

They proved that the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (1.6) converges to a zero of m-accretive operator A under the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$ (equivalently, $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$),

(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$; $r_n \geq \epsilon$ for some $\epsilon > 0$ and for all $n \geq 0$,

(iv) $\sum_{n=1}^{\infty}|1 - r_{n-1}/r_n| < \infty$,

(v) $r_n \geq \epsilon$ for some $\epsilon > 0$ and for all $n \geq 0$ and $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$.

In 2007, Qin and Su [22] also considered the following iterative scheme in either a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping:

$$\begin{aligned} x_0 = u \in C, \qquad y_n &= \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n) y_n \quad n \geq 0, \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $(0,1)$. They proved that the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (1.7) converges strongly to a zero of m-accretive operator A provided that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ satisfy conditions (i), (ii), and (iii), and $\{r_n\}_{n=0}^{\infty}$ satisfies condition (v).

Chen and Zhu [23] considered the following viscosity iterative scheme for resolvent J_{r_n} of m-accretive mapping A:

$$x_0 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0, \tag{1.8}$$

where f is a contraction mapping defined on C. Under the assumption that $\{r_n\}_{n=0}^{\infty}$ satisfies condition (v), Chen and Zhu [23] proved in a reflexive Banach space having weakly sequentially continuous duality mapping that the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (1.8) converges strongly to a zero of A, which solves a certain variational inequality.

Recently, Jung [24] introduced the following viscosity iterative method:

$$\begin{aligned} x_0 \in C, \qquad y_n &= \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) y_n \quad n \geq 0. \end{aligned} \tag{1.9}$$

Under certain appropriate conditions on the parameters $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$ and the sequence $\{x_n\}_{n=0}^{\infty}$; Jung [24] established strong convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (1.9) to a zero of A, which is a unique solution of a certain variational inequality problem, in either a reflexive Banach space having a weakly sequentially continuous duality mapping or a reflexive Banach space having a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings.

In [5], Zegeye and Shahzad proved the following theorem.

Theorem ZS. *Let E be a strictly convex reflexive real Banach space which has uniformly Gâteaux differentiable norm and let K be a nonempty closed convex subset of E. Assume that every nonempty closed convex and bounded subset of E has the fixed point property for nonexpansive mappings.*

Let $A_i : K \to E$, $i = 1, 2, \ldots, r$ *be a finite family of* m*-accretive mappings with* $\bigcap_{i=1}^{r} A_i^{-1}(0) \neq \emptyset$*. For given* $u, x_1 \in K$*, let* $\{x_n\}_{n\geq 1}$ *be generated by the algorithm*

$$x_{n+1} = \theta_n u + (1 - \theta_n) S_r x_n, \quad \forall n \geq 1, \tag{1.10}$$

where $S_r = a_0 I + a_1 J_{A_1} + \cdots + a_r J_{A_r}$*, with* $J_{A_i} = (I + A_i)^{-1}$, $0 < a_i < 1$, $i = 1, \ldots, r$, $\sum_{i=1}^{r} a_i = 1$*, and* $\{\theta_n\}_{n\geq 1}$ *is a sequence in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\theta_n = 0$;

(ii) $\sum_{n=1}^{\infty} \theta_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\theta_n - \theta_{n-1}| < \infty$ *or* $\lim_{n\to\infty}(|\theta_n - \theta_{n-1}|/\theta_n) = 0$.

Then, $\{x_n\}_{n\geq 1}$ *converges strongly to a common solution of the equation* $A_i x = 0$ *for* $i = 1, 2, \ldots, r$.

Motivated by the results of the authors mentioned above, it is our purpose in this paper to prove new path convergnce theorems and introduce a new iteration process for a countably infinite family of m-accretive mappings and prove strong convergence of the sequence to a common zero of these operators in uniformly convex real Banach spaces. As a result, we obtain strong convergence theorems for a countably infinite family of pseudocontractive mappings. Our theorems extend and improve some important results which are announced recently by various authors.

2. Preliminaries

Let E be a real normed linear space. Let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm (and E is called *smooth*) if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Furthermore, E is said to be *uniformly smooth* if the limit exists uniformly for $(x, y) \in S \times S$.

Let E be a real normed linear space. The modulus of convexity of E is the function $\delta_E : [0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1,\ \epsilon = \|x - y\|\right\}. \tag{2.2}$$

The space E is said to be uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. E is said to be *strictly convex* if for all $x, y \in E$ such that $\|x\| = \|y\| = 1$ and for all $\lambda \in (0,1)$ we have $\|\lambda x + (1 - \lambda)y\| < 1$. It is well known that every uniformly convex Banach space is strictly convex.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$, then $Tx = p$.

A mapping $T : D(T) \subseteq E \rightarrow E$ is said to be *demicompact* at h if for any bounded sequence $\{x_n\}$ in $D(T)$ such that $(x_n - Tx_n) \rightarrow h$ as $n \rightarrow \infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ and $x^* \in D(T)$ such that $\{x_{n_j}\}$ converges strongly to x^* and $x^* - Tx^* = h$.

We need the following lemmas in the sequel.

Lemma 2.1. *Let E be a real normed space, then*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \tag{2.3}$$

for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$.

Lemma 2.2 (Lemma 3 of Bruck [25]). *Let K be a nonempty closed convex subset of a strictly convex real Banach space E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings from K to E such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, then a mapping G on K defined by $Gx := \sum_{i=1}^{\infty} \lambda_i T_i x$ for all $x \in K$ is well defined, nonexpansive, and $F(G) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.3 (Xu [26]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n \sigma_n, \quad n \ge 1, \tag{2.4}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ and $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ satisfying (i) $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \le 0$. *Then, $a_n \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 2.4 (Suzuki [27]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \ge 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.5 (Cioranescu [28]). *Let A be a continuous accretive operator defined on a real Banach space E with $D(A) = E$. Then, A is m-accretive.*

Lemma 2.6 (C. E. Chidume and C. O. Chidume [29]). *Let K be a nonempty closed convex subset of a real Banach space E. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \le 1\}$. Then, there exists a continuous strictly increasing function $g : [0,\infty) \rightarrow [0,\infty)$, $g(0) = 0$ such that for every $x, y \in B_r(0)$ and for $p \in (1,\infty)$, the following inequality holds:*

$$4.2^p g\left(\frac{1}{2}\|x + y\|\right) \le (p.2^p - 4)\|x\|^p + p.2^p\langle y, j_p(x)\rangle + 4\|y\|^p. \tag{2.5}$$

3. Path Convergence Theorems

We begin with the following lemma.

Lemma 3.1. *Let K be a nonempty closed convex subset of a reflexive strictly convex Banach space E. Let $T : K \rightarrow K$ be a nonexpansive mapping. Let $\{x_n\}_{n=1}^{\infty}$, a bounded sequence in K, be an approximate fixed point sequence of T, that is $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Let $\varphi(x) = \mu_n\|x_n - x\|^2$, for all $x \in K$ and* *let* $\Gamma = \{x \in K \cap B : \varphi(x) = \min_{z\in K}\varphi(z)\}$, *where* B *is any bounded closed convex nonempty subset of* E *such that* $x_n \in B$ *for all* $n \in \mathbb{N}$. *Then* T *has a fixed point in* Γ, *provided that* $F(T) \neq \emptyset$.

Proof. Since E is a reflexive Banach space, then Γ is a bounded closed convex nonempty subset of E. Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, we have that for all $x \in \Gamma$,

$$\begin{aligned}\varphi(Tx) = \mu_n\|x_n - Tx\|^2 &\le \mu_n(\|x_n - Tx_n\| + \|Tx_n - Tx\|)^2 \\ &\le \mu_n\|x_n - x\|^2 = \varphi(x).\end{aligned} \tag{3.1}$$

Hence, $Tx \in \Gamma$, for all $x \in \Gamma$, that is, Γ is invariant under T. Let $x^* \in F(T)$. Then since every closed convex nonempty subset of a reflexive strictly convex Banach space is a Chebyshev set (see, e.g., [30, Corollary 5.1.19]), there exists a unique $u^* \in \Gamma$ such that

$$\|x^* - u^*\| = \inf_{z\in\Gamma}\|x^* - z\|, \tag{3.2}$$

but $x^* = Tx^*$ and $Tu^* \in \Gamma$. Thus,

$$\|x^* - Tu^*\| = \|Tx^* - Tu^*\| \le \|x^* - u^*\|. \tag{3.3}$$

So, $Tu^* = u^*$. Hence, $F(T) \cap \Gamma \neq \emptyset$. This completes the proof. □

Proposition 3.2. *Let* K *be a nonempty closed convex subset of a real Banach space* E. *Let* $A_i : K \to E$, $i = 1,2,\ldots$, *be a countably infinite family of* m*-accretive mappings and define* $J_{A_i} := (I + A_i)^{-1}$, $i = 1,2,\ldots$. *Let* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\sigma_{i,n}\}_{n=1}^{\infty}$, $i = 1,2,\ldots$ *be sequences in* $(0,1)$ *such that* $\sum_{i=1}^{\infty}\sigma_{i,n} = (1 - \alpha_n)$. *Fix* $\delta \in [\gamma_1,\gamma_2]$, *for some* $\gamma_1,\gamma_2 \in (0,1)$. *For arbitrary fixed* $u \in K$, *define a map* $T_n : K \to K$ *by*

$$T_n x = \alpha_n u + \sum_{i=1}^{\infty}\sigma_{i,n}((1-\delta)x + \delta J_{A_i}x), \quad \forall x \in K. \tag{3.4}$$

Then, T_n *is a strict contraction on* K.

Proof. Let $x, y \in K$, then

$$\begin{aligned}\|T_n x - T_n y\| &= \left\|\sum_{i=1}^{\infty}\sigma_{i,n}\left((1-\delta)(x-y) + \delta\left(J_{A_i}x - J_{A_i}y\right)\right)\right\| \\ &\le \sum_{i=1}^{\infty}\sigma_{i,n}\left((1-\delta)\|x-y\| + \delta\|J_{A_i}x - J_{A_i}y\|\right) \\ &\le (1-\alpha_n)\|x-y\|.\end{aligned} \tag{3.5}$$

□

Thus, for each $n \in \mathbb{N}$, there is a unique $z_n \in K$ satisfying

$$z_n = \alpha_n u + \sum_{i=1}^{\infty}\sigma_{i,n}((1-\delta)z_n + \delta J_{A_i}z_n). \tag{3.6}$$

Lemma 3.3. *Let K be a nonempty closed convex subset of a real Banach space E. For each $i \geq 1$, let $A_i : K \to E$ be a countably infinite family of m-accretive mappings. For $n \in \mathbb{N}$, let $\{z_n\}$ be a sequence satisfying* (3.6) *and assume $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Then, $\{z_n\}$ is bounded.*

Proof. Let $x^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0) = \bigcap_{i=1}^{\infty} F(J_{A_i})$. Then, using (3.6), we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &= \left\langle \alpha_n(u - x^*) + \sum_{i=1}^{\infty}\sigma_{i,n}((1-\delta)z_n + \delta J_{A_i} z_n - x^*), j(z_n - x^*) \right\rangle \\ &\leq \alpha_n\langle u - x^*, j(z_n - x^*)\rangle + \sum_{i=1}^{\infty}\sigma_{i,n}\|z_n - x^*\|^2 \\ &= \alpha_n\langle u - x^*, j(z_n - x^*)\rangle + (1-\alpha_n)\|z_n - x^*\|^2,\end{aligned} \tag{3.7}$$

which implies that $\|z_n - x^*\| \leq \|u - x^*\|$. Thus, $\{z_n\}$ is bounded. □

Lemma 3.4. *Let K be a nonempty closed convex subset of a uniformly convex real Banach space E. For each $i \geq 1$, let $A_i : K \to E$ be a countably infinite family of m-accretive mappings such that $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}(\alpha_n/\sigma_{i,n}) = 0$, for all $i \geq 1$, $\sum_{i=1}^{\infty}\sigma_{i,n} = (1-\alpha_n)$. Let $\{z_n\}$ be a sequence satisfying* (3.6). *Then, $\lim_{n\to\infty}\|z_n - J_{A_i} z_n\| = 0$, for all $i \geq 1$. Furthermore, if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\lambda_i = 1$; $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\sigma_{i,n} - \lambda_i| = 0$ and define $G := (1-\delta)I + \delta T$, where $T := \sum_{i=1}^{\infty}\lambda_i J_{A_i}$, then $\lim_{n\to\infty}\|z_n - Gz_n\| = 0$.*

Proof. We start by showing that $\lim_{n\to\infty}\|z_n - J_{A_i} z_n\| = 0$, for all $i \geq 1$. For this, let $S_i := (1-\delta)I + \delta J_{A_i}$, where I is the identity operator on K. Since $\{z_n\}_{n=1}^{\infty}$ is bounded, then for each $i \geq 1$ and $x^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$, we have the following using (2.5):

$$\begin{aligned}4.2^p g\left(\frac{1}{2}\|S_i z_n - z_n\|\right) &= 4.2^p g\left(\frac{1}{2}\|S_i z_n - x^* + x^* - z_n\|\right) \\ &\leq (p.2^p - 4)\|x^* - z_n\|^p + p.2^p\langle S_i z_n - x^*, j_p(x^* - z_n)\rangle + 4\|S_i z_n - x^*\|^p \\ &\leq (p.2^p - 4)\|x^* - z_n\|^p + p.2^p\langle S_i z_n - z_n + z_n - x^*, j_p(x^* - z_n)\rangle \\ &\quad + 4\|S_i z_n - x^*\|^p \\ &= (p.2^p - 4)\|x^* - z_n\|^p + p.2^p\langle S_i z_n - z_n, j_p(x^* - z_n)\rangle \\ &\quad - p.2^p\langle x^* - z_n, j_p(x^* - z_n)\rangle + 4\|S_i z_n - x^*\|^p \\ &\leq p.2^p\langle z_n - S_i z_n, j_p(z_n - x^*)\rangle.\end{aligned} \tag{3.8}$$

Hence,

$$\frac{4}{p} g\left(\frac{1}{2}\|S_i z_n - z_n\|\right) \leq \langle z_n - S_i z_n, j_p(z_n - x^*)\rangle, \tag{3.9}$$

and so,

$$\frac{4}{p}\sum_{i=1}^{\infty}\sigma_{i,n}g\left(\frac{1}{2}\|S_iz_n - z_n\|\right) \le \frac{4}{p}\sum_{i=1}^{\infty}\sigma_{i,n}\left\langle z_n - S_iz_n, j_p(z_n - x^*)\right\rangle. \tag{3.10}$$

Using (3.6), we have

$$\begin{aligned}\left\langle z_n - x^*, j_p(z_n - x^*)\right\rangle &= \alpha_n\langle u - x^*, j_p(z_n - x^*)\rangle \\ &\quad + \sum_{i=1}^{\infty}\sigma_{i,n}\left\langle S_iz_n - z_n + z_n - x^*, j_p(z_n - x^*)\right\rangle \\ &= \alpha_n\langle u - x^*, j_p(z_n - x^*)\rangle + \sum_{i=1}^{\infty}\sigma_{i,n}\left\langle S_iz_n - z_n, j_p(z_n - x^*)\right\rangle \\ &\quad + (1 - \alpha_n)\left\langle z_n - x^*, j_p(z_n - x^*)\right\rangle,\end{aligned} \tag{3.11}$$

which implies

$$\sum_{i=1}^{\infty}\sigma_{i,n}\left\langle z_n - S_iz_n, j_p(z_n - x^*)\right\rangle = \alpha_n\left\langle u - z_n, j_p(z_n - x^*)\right\rangle. \tag{3.12}$$

Using this and (3.10), we get

$$\frac{4}{p}\sum_{i=1}^{\infty}\sigma_{i,n}g\left(\frac{1}{2}\|S_iz_n - z_n\|\right) \le \alpha_n\left\langle u - z_n, j_p(z_n - x^*)\right\rangle. \tag{3.13}$$

Since $\{z_n\}$ is bounded, we have that

$$\sum_{i=1}^{\infty}\sigma_{i,n}g\left(\frac{1}{2}\|S_iz_n - z_n\|\right) \le \alpha_n M, \tag{3.14}$$

for some constant $M > 0$. This yields

$$g\left(\frac{1}{2}\|S_iz_n - z_n\|\right) \le \frac{\alpha_n}{\sigma_{i,n}}M. \tag{3.15}$$

Thus, since g is continuous, strictly increasing, $g(0) = 0$, and $\lim_{n\to\infty}(\alpha_n/\sigma_{i,n}) = 0$, for all $i \ge 1$, we have

$$2g\left(\frac{1}{2}\lim_{n\to\infty}\|S_iz_n - z_n\|\right) = 0. \tag{3.16}$$

So, $\lim_{n\to\infty}\|S_i z_n - z_n\| = 0$, for all $i \geq 1$, but

$$\begin{aligned}\|S_i z_n - z_n\| &= \|(1-\delta)z_n + \delta J_{A_i} z_n - z_n\| \\ &= \|\delta(J_{A_i} z_n - z_n)\| \\ &= \delta\|J_{A_i} z_n - z_n\|.\end{aligned} \tag{3.17}$$

Thus,

$$\lim_{n\to\infty}\|J_{A_i} z_n - z_n\| = 0, \quad \forall i \geq 1. \tag{3.18}$$

Next, we show that $\lim_{n\to\infty}\|z_n - Gz_n\| = 0$. Observe that

$$z_n - Gz_n = \alpha_n u + \sum_{i=1}^{\infty}(\sigma_{i,n} - \lambda_i)[(1-\delta)z_n + \delta J_{A_i} z_n]. \tag{3.19}$$

So,

$$\|z_n - Gz_n\| \leq \alpha_n\|u\| + M\sum_{i=1}^{\infty}|\sigma_{i,n} - \lambda_i| \tag{3.20}$$

for some $M > 0$. Hence,

$$\lim_{n\to\infty}\|z_n - Gz_n\| = 0. \tag{3.21}$$

This completes the proof. □

Theorem 3.5. *Let K be a nonempty closed convex subset of a uniformly convex real Banach space E with uniformly Gâteaux differentiable norm. Let $A_i : K \to E$, $i = 1, 2, \ldots$, be a countably infinite family of m-accretive mappings such that $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let $\{z_n\}$ be a sequence satisfying (3.6). Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\lambda_i = 1$ and $\lim_{n\to\infty}\sum_{i=1}^{\infty}|\sigma_{i,n} - \lambda_i| = 0$. Let $G := (1-\delta)I + \delta T$, where $T := \sum_{i=1}^{\infty}\lambda_i J_{A_i}$. Then, $\{z_n\}$ converges strongly to an element in $\bigcap_{i=1}^{\infty} A_i^{-1}(0)$.*

Proof. Observe that by Lemma 2.2, $T := \sum_{i=1}^{\infty}\lambda_i J_{A_i}$ is well defined, nonexpansive, and $F(T) = \bigcap_{i=1}^{\infty} F(J_{A_i}) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Furthermore, it is easy to see that G is nonexpansive and that $F(G) = F(T) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Now, since $\{z_n\}$ is bounded and $\lim_{n\to\infty}\|Gz_n - z_n\| = 0$, we have by Lemma 3.1 that there exists a unique z^* in the set $\Omega^* := \{x \in K \cap B^* : \mu_n\|z_n - x\|^2 = \min_{y\in K}\|z_n - y\|\}$ such that $Gz^* = z^*$, where B^* is a bounded closed convex nonempty subset of E such that $u, z_n \in B^*$ for all $n \in \mathbb{N}$. Thus, $z^* \in F(G) = \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Let $t \in (0,1)$, then by convexity of $K \cap B^*$, we have that $(1-t)z^* + tu \in K \cap B^*$. Thus, $\mu_n\|z_n - z^*\|^2 \leq \mu_n\|z_n - ((1-t)z^* + tu)\|^2 = \mu_n\|z_n - z^* - t(u - z^*)\|^2$. Moreover, we have, by Lemma 2.1 that

$$\|z_n - z^* - t(u - z^*)\|^2 \leq \|z_n - z^*\|^2 - 2t\langle u - z^*, j(z_n - z^* - t(u - z^*))\rangle. \tag{3.22}$$

This implies that $\mu_n\langle u - z^*, j(z_n - z^* - t(u - z^*))\rangle \leq 0$. Furthermore, since E has uniformly Gâteaux differentiable norm, we obtain that

$$\lim_{t\to 0}\left(\left\langle u - z^*, j(z_n - z^*)\right\rangle - \left\langle u - z^*, j(z_n - z^* - t(u - z^*))\right\rangle\right) = 0. \tag{3.23}$$

Thus, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $t \in (0, \delta_\epsilon)$ and for all $n \in \mathbb{N}$,

$$\left\langle u - z^*, j(z_n - z^*)\right\rangle < \epsilon + \left\langle u - z^*, j(z_n - z^* - t(u - z^*))\right\rangle. \tag{3.24}$$

Taking Banach limit on both sides of this inequality, we obtain

$$\mu_n\left\langle u - z^*, j(z_n - z^*)\right\rangle \leq \epsilon; \tag{3.25}$$

and since $\epsilon > 0$ is arbitrary, we have that

$$\mu_n\left\langle u - z^*, j(z_n - z^*)\right\rangle \leq 0. \tag{3.26}$$

Now, using (3.6), we have that

$$\begin{aligned}\|z_n - z^*\|^2 &= \left\langle \alpha_n(u - z^*) + \sum_{i=1}^{\infty}\sigma_{i,n}(((1-\delta)z_n + \delta J_{A_i}z_n) - z^*), j(z_n - z^*)\right\rangle \\ &\leq \alpha_n\left\langle u - z^*, j(z_n - z^*)\right\rangle + (1-\alpha_n)\|z_n - z^*\|^2.\end{aligned} \tag{3.27}$$

So,

$$\|z_n - z^*\|^2 \leq \left\langle u - z^*, j(z_n - z^*)\right\rangle. \tag{3.28}$$

Again, taking Banach limit, we obtain

$$\mu_n\|z_n - z^*\|^2 \leq \mu_n\left\langle u - z^*, j(z_n - z^*)\right\rangle \leq 0, \tag{3.29}$$

so that $\mu_n\|z_n - z^*\|^2 = 0$. Hence, there exists a subsequence $\{z_{n_l}\}_{l=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{l\to\infty} z_{n_l} = z^*$. We now show that $\{z_n\}_{n=1}^{\infty}$ actually converges to z^*. Suppose there is another subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} z_{n_k} = u^*$. Then, since $\lim_{n\to\infty}\|J_{A_i}z_n - z_n\| = 0$ and J_{A_i} is continuous for all $i \in \mathbb{N}$, we have that $u^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$.

Claim 1 ($u^* = z^*$). Suppose for contradiction that $u^* \neq z^*$, then $\|u^* - z^*\| > 0$, but using (3.6), we have that

$$\begin{aligned}
\|z_{n_l} - u^*\|^2 &= \left\langle \alpha_{n_l}(u - u^*) + \sum_{l=1}^{\infty} \sigma_{i,n_l}(((1-\delta)z_{n_l} + \delta J_{A_i} z_{n_l}) - u^*), j(z_{n_l} - u^*) \right\rangle \\
&= \alpha_{n_l}\langle u - z^*, j(z_{n_l} - u^*)\rangle + \alpha_{n_l}\langle z^* - z_{n_l}, j(z_{n_l} - u^*)\rangle \\
&\quad + \alpha_{n_l}\|z_{n_l} - u^*\|^2 + \sum_{i=1}^{\infty} \sigma_{i,n_l}\langle (1-\delta)z_{n_l} + \delta J_{A_i} z_{n_l} - u^*, j(z_{n_l} - u^*)\rangle \\
&\le \alpha_{n_l}\langle u - z^*, j(z_{n_l} - u^*)\rangle + \alpha_{n_l}\langle z^* - z_{n_l}, j(z_{n_l} - u^*)\rangle \\
&\quad + \alpha_{n_l}\|z_{n_l} - u^*\|^2 + (1-\delta)(1-\alpha_{n_l})\|z_{n_l} - u^*\|^2 + \delta(1-\alpha_{n_l})\|z_{n_l} - u^*\|^2 \\
&= \alpha_{n_l}\langle u - z^*, j(z_{n_l} - u^*)\rangle + \alpha_{n_l}\langle z^* - z_{n_l}, j(z_{n_l} - u^*)\rangle + \|z_{n_l} - u^*\|^2.
\end{aligned} \tag{3.30}$$

Thus,

$$\langle u - z^*, j(u^* - z_{n_l})\rangle \le \|z_{n_l} - u^*\| \|z_{n_l} - z^*\|. \tag{3.31}$$

Using the fact that $\{z_n\}_{n=1}^{\infty}$ is bounded and that E has a uniformly Gâteaux differentiable norm, we obtain from (3.31) that

$$\langle u - z^*, j(u^* - z^*)\rangle \le 0. \tag{3.32}$$

Similarly, we also obtain that $\langle u - u^*, j(z^* - u^*)\rangle \le 0$ or rather

$$\langle u^* - u, j(u^* - z^*)\rangle \le 0. \tag{3.33}$$

Adding (3.32) and (3.33), we have that $\|z^* - u^*\| \le 0$, a contradiction. Thus, $z^* = u^*$. Hence, $\{z_n\}_{n=1}^{\infty}$ converges strongly to $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. This completes the proof. □

Theorem 3.6. *Let K be a nonempty closed convex subset of a uniformly convex real Banach space E with uniformly Gâteaux differentiable norm. Let $A_i : K \to E$, $i = 1, 2, \ldots$, be a countably infinite family of m-accretive mappings such that $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let $\{z_n\}$ be a sequence satisfying* (3.6). *If at least one of the mappings J_{A_i} is demicompact, then $\{z_n\}$ converges strongly to an element of $\bigcap_{i=1}^{\infty} A_i^{-1}(0)$.*

Proof. For fixed $s \in \mathbb{N}$, let J_{A_s} be demicompact. Since $\lim_{n\to\infty}\|z_n - J_{A_s} z_n\| = 0$, there exists a subsequence say $\{z_{n_k}\}$ of $\{z_n\}$ that converges strongly to some point $z^* \in K$. By continuity of J_{A_i} and the fact that $\lim_{k\to\infty}\|z_{n_k} - J_{A_i} z_{n_k}\| = 0$, $i = 1, 2, \ldots$, we have that $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Assuming that there is another subsequence $\{z_{n_j}\}_{j=1}^{\infty}$ of $\{z_n\}$ that converges strongly to a point say, q^*, then following the argument of the last part of proof of Theorem 3.5, we obtain that $\{z_n\}_{n=1}^{\infty}$ converges strongly to $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. This completes the proof. □

Theorem 3.7. *Let K be a nonempty closed convex subset of a uniformly convex real Banach space E with uniformly Gâteaux differentiable norm and also admits weakly sequential continuous duality* *mapping. Let $A_i : K \to E$, $i = 1,2,\ldots$, be a countably infinite family of m-accretive mappings such that $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. Let $\{z_n\}$ be a sequence satisfying* (3.6)*. Then, $\{z_n\}$ converges strongly to an element of $\bigcap_{i=1}^{\infty} A_i^{-1}(0)$.*

Proof. Since $\{z_n\}$ is bounded, there exists a subsequence say $\{z_{n_k}\}$ of $\{z_n\}$ that converges weakly to some point $z^* \in K$. Using the demiclosedness property of $(I - J_{A_i})$ at 0 for each $i \geq 1$ (since J_{A_i} is nonexpansive for each $i \in \mathbb{N}$, see, e.g., [31]) and the fact that $\lim_{k\to\infty}\|z_{n_k} - J_{A_i} z_{n_k}\| = 0$, we get that $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. We also observe from (3.6) that

$$\begin{aligned}
\|z_{n_k} - z^*\|^2 &= \left\langle \alpha_{n_k} u + \sum_{i=1}^{\infty} \sigma_{i,n_k}\left((1-\delta) z_{n_k} + \delta J_{A_i} z_{n_k}\right) - z^*, j(z_{n_k} - z^*) \right\rangle \\
&\leq \alpha_{n_k} \langle u - z^*, j(z_{n_k} - z^*)\rangle + \sum_{i=1}^{\infty} \sigma_{i,n_k} \|z_{n_k} - z^*\|^2 \\
&= \alpha_{n_k} \langle u - z^*, j(z_{n_k} - z^*)\rangle + (1 - \alpha_{n_k})\|z_{n_k} - z^*\|^2.
\end{aligned} \tag{3.34}$$

This implies that $\|z_{n_k} - z^*\|^2 \leq \langle u - z^*, j(z_{n_k} - z^*)\rangle$. Using the fact that j is weakly sequential continuous, then we have from the last inequality that $\{z_{n_k}\}$ converges strongly to z^*. Then following the argument of the last part of proof of Theorem 3.5, we obtain that $\{z_n\}_{n=1}^{\infty}$ converges strongly to $z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. This completes the proof. □

4. Convergence Theorems for Countably Infinite Family of m-Accretive Mappings

For the rest of this paper, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\sigma_{i,n}\}_{n=1}^{\infty}$ are in $(0,1)$ satisfying the following additional conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty} \sum_{i=1}^{\infty} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$.

Theorem 4.1. *Let K be a nonempty closed convex subset of a uniformly convex real Banach space E with uniformly Gâteaux differentiable norm. Let $A_i : K \to E$, $i = 1,2,\ldots$, be a countably infinite family of m-accretive mappings such that $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \neq \emptyset$. For fixed $\delta \in [\gamma_1, \gamma_2]$, for some $\gamma_1, \gamma_2 \in (0,1)$, $u \in K$, let $\{x_n\}_{n=1}^{\infty}$ be generated by*

$$x_1 \in K, \quad x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty} \sigma_{i,n}\left((1-\delta) x_n + \delta J_{A_i} x_n\right), \quad n \geq 1, \tag{4.1}$$

then $\lim_{n\to\infty} \sum_{i=1}^{\infty} \sigma_{i,n}((1-\delta)x_n + \delta J_{A_i} x_n - x_n) = 0$. *Furthermore, if $\{\alpha_n\}_{n\geq 1}$ is such that*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{\infty} \sigma_{i,n}((1-\delta)x_n + \delta J_{A_i} x_n - x_n)}{\alpha_n} = 0, \tag{4.2}$$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common zero of $\{A_i\}_{i=1}^{\infty}$.

Proof. Using mathematical induction, it is easy to see that for $x^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$ fixed

$$\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}, \quad \forall n \geq 1. \tag{4.3}$$

Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded and so $\{J_{A_i} x_n\}_{n=1}^{\infty}$ is also bounded.

Now, define the sequences $\{\beta_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ by $\beta_n := (1-\delta)\alpha_n + \delta$ and $y_n := (x_{n+1} - x_n + \beta_n x_n)/\beta_n$. Then,

$$y_n = \frac{\alpha_n u + \delta \sum_{i \geq 1} \sigma_{i,n}((1-\delta)x_n + \delta J_{A_i} x_n)}{\beta_n}. \tag{4.4}$$

Observe that $\{y_n\}_{n=1}^{\infty}$ is bounded and that

$$\begin{aligned} \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq& \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \|u\| \\ &+ \left| \frac{\delta(1-\alpha_{n+1})}{\beta_{n+1}} - 1 \right| \|x_{n+1} - x_n\| \\ &+ \frac{\delta M}{\beta_{n+1}\beta_n} \sum_{i=1}^{\infty} |\sigma_{i,n+1} - \sigma_{i,n}| + \frac{\delta M}{\beta_{n+1}\beta_n} |\beta_{n+1} - \beta_n|, \end{aligned} \tag{4.5}$$

for some $M > 0$. Thus,

$$\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{4.6}$$

Hence, by Lemma 2.4, we have $\lim_{n \to \infty} \|y_n - x_n\| = 0$. Consequently, we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \beta_n \|y_n - x_n\| = 0. \tag{4.7}$$

From (4.1), we have that

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \sum_{i=1}^{\infty} \sigma_{i,n}(((1-\delta)x_n + \delta J_{A_i} x_n) - x_n), \tag{4.8}$$

which implies that $\| \sum_{i=1}^{\infty} \sigma_{i,n}((1-\delta)x_n + \delta J_{A_i} x_n - x_n)\| \leq \|x_{n+1} - x_n\| + \alpha_n \|u - x_n\|$ and thus

$$\lim_{n \to \infty} \left\| \sum_{i=1}^{\infty} \sigma_{i,n}(((1-\delta)x_n + \delta J_{A_i} x_n) - x_n) \right\| = 0. \tag{4.9}$$

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence satisfying (3.6). Then, by Theorem 3.5, $z_n \to z^* \in \bigcap_{i=1}^{\infty} A_i^{-1}(0)$. Using Lemma 2.1, we have that

$$\begin{aligned}\|z_n - x_n\|^2 &\le \left\| \sum_{i=1}^{\infty} \sigma_{i,n}\left(((1-\delta)z_n + \delta J_{A_i} z_n) - ((1-\delta)x_n + \delta J_{A_i} x_n) + ((1-\delta)x_n + \delta J_{A_i} x_n) - x_n\right)\right\|^2 \\ &\quad + 2\alpha_n \langle u - x_n, j(z_n - x_n)\rangle \\ &\le \left((1-\alpha_n)\|z_n - x_n\| + \left\| \sum_{i=1}^{\infty} \sigma_{i,n}\left(((1-\delta)x_n + \delta J_{A_i} x_n) - x_n\right)\right\|\right)^2 \\ &\quad + 2\alpha_n \langle u - z_n, j(z_n - x_n)\rangle.\end{aligned} \tag{4.10}$$

This implies that

$$\begin{aligned}&\langle u - z_n, j(x_n - z_n)\rangle \\ &\quad \le \frac{\alpha_n}{2}\|z_n - x_n\|^2 + \frac{(1-\alpha_n)\|z_n - x_n\| \cdot \|\sum_{i=1}^{\infty} \sigma_{i,n}(((1-\delta)x_n + \delta J_{A_i} x_n) - x_n)\|}{\alpha_n} \\ &\quad + \frac{\|\sum_{i=1}^{\infty} \sigma_{i,n}(((1-\delta)x_n + \delta J_{A_i} x_n) - x_n)\|^2}{2\alpha_n}\end{aligned} \tag{4.11}$$

and hence,

$$\limsup_{n\to\infty} \langle u - z_n, j(x_n - z_n)\rangle \le 0. \tag{4.12}$$

Moreover, we have that

$$\begin{aligned}\langle u - z_n, j(x_n - z_n)\rangle &= \langle u - z^*, j(x_n - z^*)\rangle + \langle u - z^*, j(x_n - z_n) - j(x_n - z^*)\rangle \\ &\quad + \langle z^* - z_n, j(x_n - z_n)\rangle.\end{aligned} \tag{4.13}$$

Using the boundedness of $\{x_n\}_{n=1}^{\infty}$, we have

$$\langle z^* - z_n, j(x_n - z_n)\rangle \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4.14}$$

Also, since j is norm-to-weak* uniformly continuous on bounded subsets of E, we have that

$$\langle u - z^*, j(x_n - z_n) - j(x_n - z^*)\rangle \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4.15}$$

From (4.12) and (4.13), we obtain that

$$\limsup_{n\to\infty}\langle u-z^*, j(x_n-z^*)\rangle \le 0. \tag{4.16}$$

Finally, using Lemma 2.1 on (4.1), we get

$$\begin{aligned}\|x_{n+1}-z^*\|^2 &\le \left\|\sum_{i=1}^{\infty}\sigma_{i,n+1}(((1-\delta)x_n+\delta J_{A_i}x_n)-z^*)\right\|^2 \\ &\quad + 2\alpha_n\langle u-z^*, j(x_{n+1}-z^*)\rangle \\ &\le (1-\alpha_n)\|x_n-z^*\|^2+2\alpha_n\langle u-z, j(x_{n+1}-z^*)\rangle.\end{aligned} \tag{4.17}$$

Using (4.16) and Lemma 2.3 in (4.17), we get that $\{x_n\}_{n=1}^{\infty}$ converges strongly to common zero of the family $\{A_i\}_{i=1}^{\infty}$ of m-accretive operators. □

Remark 4.2. If K is replaced with E in Theorems 3.5, 3.6, 3.7, and 4.1, then by Lemma 2.5, the assumption that A_i is m-accretive for each $i \ge 1$ could be replaced with A_i is continuous for each $i \ge 1$.

Hence, we have the following theorem.

Theorem 4.3. *Let E be a uniformly convex real Banach space with uniformly Gâteaux differentiable norm. Let $A_i : E \to E$, $i = 1,2,\ldots$, be a countably infinite family of continuous accretive operators such that $\bigcap_{i=1}^{\infty} A_i^{-1}(0) \ne \emptyset$. For arbitrary but fixed $\delta \in (0,1)$, $u \in K$, let $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$,*

$$x_{n+1}=\alpha_n u+\sum_{i=1}^{\infty}\sigma_{i,n}((1-\delta)x_n+\delta J_{A_i}x_n), \quad n\ge 1, \tag{4.18}$$

then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common zero of $\{A_i\}_{i=1}^{\infty}$.

Proof. By Lemma 2.5, we have that A_i is m-accretive for each $i \ge 1$. Then, the rest of the proof follows from Theorem 4.1. □

We also have the following theorems.

Theorem 4.4. *Let K, E, A_i and $\{x_n\}_{n\ge1}$ be as an Theorem 4.1. Suppose that at least one of J_{A_i} is demicompact, then $\{x_n\}_{n\ge1}$ converges strongly to a common zero of A_i, $i = 1,2,\ldots$.*

Proof. The proof follows as in the proof of Theorem 4.1 but using Theorem 3.6. □

Theorem 4.5. *Let K, E, A_i and $\{x_n\}_{n\ge1}$ be as an Theorem 4.1. Suppose that, in addition, E admits weakly sequential continuous duality mapping, then $\{x_n\}_{n\ge1}$ converges strongly to a common zero of A_i, $i = 1,2,\ldots$.*

Proof. The proof follows as in the proof of Theorem 4.1 but using Theorem 3.7. □

5. Convergence Theorems for Countably Infinite Family of Pseudocontractive Mappings

Theorem 5.1. *Let K be a nonempty closed convex subset of a uniformly convex real Banach space E with uniformly Gâteaux differentiable norm. Let $T_i : K \to E$, $i = 1, 2, \ldots$, be a countably infinite family of pseudocontractive mappings such that for each $i \geq 1$, $(I - T_i)$ is m-accretive and $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $J_{T_i} = (2I - T_i)^{-1}$, $i \geq 1$. For fixed $\delta \in [\gamma_1, \gamma_2]$, for some $\gamma_1, \gamma_2 \in (0,1)$ and $u \in K$, let $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$:*

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty} \sigma_{i,n}((1-\delta)x_n + \delta J_{T_i} x_n), \quad n \geq 1, \tag{5.1}$$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Proof. Put $A_i := (I - T_i)$, $i \geq 1$. It is then obvious that $A_i^{-1}(0) = F(T_i)$, for all $i \in \mathbb{N}$ and hence $\bigcap_{i=1}^{\infty} A_i^{-1}(0) = \bigcap_{i=1}^{\infty} F(T_i)$. Furthermore, A_i is m-accretive for each $i \geq 1$. Thus, we obtain the conclusion from Theorem 4.1 with J_{A_i} in the definition of z_n replaced with J_{T_i}. □

Theorem 5.2. *Let E be a uniformly convex real Banach space with uniformly Gâteaux differentiable norm. For each $i \geq 1$, let $T_i : E \to E$ be a countably infinite family of continuous pseudocontractive mappings such that for each $i \geq 1$ and $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $J_{T_i} = (2I - T_i)^{-1}$, $i \geq 1$. For arbitrary but fixed $\delta \in [\gamma_1, \gamma_2]$, for some $\gamma_1, \gamma_2 \in (0,1)$ and $u \in K$, let $\{x_n\}_{n=1}^{\infty}$ be generated by $x_1 \in K$:*

$$x_{n+1} = \alpha_n u + \sum_{i=1}^{\infty} \sigma_{i,n}((1-\delta)x_n + \delta J_{T_i} x_n), \quad n \geq 1, \tag{5.2}$$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Remark 5.3. Theorems similar to Theorems 4.4 and 4.5 could also be obtained for countably infinite family of pseudocontractive mappings.

Remark 5.4. Prototypes for our iteration parameters are

$$\alpha_n = \frac{1}{n+1}, \qquad \sigma_{i,n} = \frac{n}{2^i(n+1)}, \qquad \lambda_i = \frac{1}{2^i}. \tag{5.3}$$

Remark 5.5. The addition of bounded error terms in any of our recursion formulas leads to no further generalization.

Remark 5.6. If $f : K \to K$ is a contraction map and we replace u by $f(x_n)$ in the recursion formulas of our theorems, we obtain what some authors now call *viscosity* iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces u by $f(x_n)$, repeats the argument of this paper, using the fact that f is a contraction map.

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