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Research Article
Heterogeneous Riemannian Manifolds

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We solve Ambrose's Problem for a generic class of Riemannian metrics on a smooth manifold, namely, the class of heterogeneous metrics.

1. Introduction

We define a Riemannian metric g on a manifold M to be heterogeneous if no two distinct points of M have isometric neighborhoods. Intuitively, a heterogeneous metric is as far as possible from being homogeneous. Heterogeneity can be reformulated in terms of a multijet transversality condition so that by an application of the standard transversality theorems, the genericity of heterogeneous metrics is established.

Theorem 1.1. *The set of heterogeneous metrics on a smooth manifold M of dimension $n \geq 2$ is residual in the space of Riemannian metrics on M with the strong C^∞ topology.*

A version of Theorem 1.1 for compact manifolds was stated and proved by Sunada [1, Proposition 1].

Ambrose [2] asked whether or not a complete simply connected Riemannian manifold is determined up to isometry by the behavior of curvature under parallel transport along geodesics emanating from a point. For heterogeneous metrics the answer is always yes.

Proposition 1.2. *If (M, g) is a complete, connected, simply connected, heterogeneous Riemannian manifold of dimension $n \geq 2$, then, for every $p \in M$, (M, g) is determined up to isometry by the behavior of curvature under parallel transport along geodesics emanating from p.*

Proposition 1.2 combines with Theorem 1.1 to produce a generic dense family of metrics on a manifold which answers Ambrose's Problem in the affirmative.

Theorem 1.3. *The set of complete metrics on a connected, simply connected smooth manifold M of dimension $n \geq 2$ which, for every point p in M, are determined up to isometry by the behavior of curvature under parallel transport along geodesics emanating from p, is residual in the space of complete Riemannian metrics on M with the strong C^∞ topology.*

Although Ambrose's Problem has been completely settled in dimension 2 [3, 4], Theorem 1.3 does give a significant advance on the problem in higher dimensions, since earlier partial results in [5, 6] apply only to metrics with rather special properties.

2. Ambrose's Problem and Heterogeneity

Let us recall what it means for a complete, connected, simply connected, n-dimensional Riemannian manifold (M, g) to be determined up to isometry by the behavior of curvature under parallel transport along geodesics emanating from a point p in M. Let $(\overline{M}, \overline{g})$ be another complete, connected, simply connected, n-dimensional Riemannian manifold. Let $\overline{p} \in \overline{M}$, and let $I : T_pM \to T_{\overline{p}}\overline{M}$ be a linear isometry between the tangent spaces. For each geodesic $\gamma : [0,1] \to M$ satisfying $\gamma(0) = p$, there is a corresponding geodesic $\overline{\gamma} : [0,1] \to \overline{M}$ satisfying $\overline{\gamma}(0) = \overline{p}$ characterized by the initial condition $\overline{\gamma}'(0) = I(\gamma'(0))$. Given such a geodesic γ, $I_\gamma = P_{\overline{\gamma}} \circ I \circ P_\gamma^{-1}$ defines a linear isometry from $T_{\gamma(1)}M$ onto $T_{\overline{\gamma}(1)}\overline{M}$ where P_γ and $P_{\overline{\gamma}}$ denote parallel transport along γ and $\overline{\gamma}$, respectively. Consider the hypothesis

$$I_\gamma(R(X,Y)Z) = \overline{R}(I_\gamma(X), I_\gamma(Y))I_\gamma(Z) \tag{$*$}$$

for every such geodesic γ and for all vectors $X, Y, Z \in T_{\gamma(1)}M$, where R and $\overline{R}$ are the respective Riemann curvature tensors. Then (M, g) is determined up to isometry by the behavior of curvature under parallel transport along geodesics emanating from p if the hypothesis $(*)$ implies that there exists an isometry $\Phi : M \to \overline{M}$ with $\Phi(p) = \overline{p}$ and $d\Phi = I$ at p (cf., [2, 5, 6]).

In order to prove Proposition 1.2, suppose that (M, g) is heterogeneous, and assume the hypothesis $(*)$. We will first find an isometry $\overline{\Phi} : \overline{M} \to M$ with $\overline{\Phi}(\overline{p}) = p$ and $d\overline{\Phi} = I^{-1}$ at $\overline{p}$. Thus we obtain the desired isometry Φ by setting $\Phi = \overline{\Phi}^{-1}$.

To prove the existence of $\overline{\Phi}$, let $\overline{q}$ in $\overline{M}$ be any nonconjugate cut point to $\overline{p}$ of order two, and let $\overline{\gamma}_1$ and $\overline{\gamma}_2$ be the two minimizing geodesics joining $\overline{p}$ to $\overline{q}$, whose respective initial tangent vectors are $\overline{X}_1 = \overline{\gamma}_1'(0)$ and $\overline{X}_2 = \overline{\gamma}_2'(0)$. Thus $\exp_{\overline{p}}(\overline{X}_1) = \exp_{\overline{p}}(\overline{X}_2) = \overline{q}$. Since $\overline{q}$ is not conjugate to $\overline{p}$ along $\overline{\gamma}_1$ or $\overline{\gamma}_2$, there are neighborhoods V_1 of $\overline{X}_1$ and V_2 of $\overline{X}_2$ in $T_{\overline{p}}\overline{M}$ for which $\exp_{\overline{p}}$ carries both V_1 and V_2 diffeomorphically onto the same neighborhood W of $\overline{q}$. As noted in [6, page 561], for $i \in \{1,2\}$, the map $\overline{f}_i = \exp_p \circ I^{-1} \circ (\exp_{\overline{p}} \mid V_i)^{-1} : W \to M$ is an isometry (after possibly cutting down W) onto a neighborhood U_i of $\gamma_i(1)$. It follows that $\gamma_1(1) = \gamma_2(1)$, because otherwise $\overline{f}_2 \circ \overline{f}_1^{-1} : U_1 \to U_2$ would be an isometry between neighborhoods of two distinct points of M, contradicting the assumed heterogeneity of M. This verifies the hypothesis of Lemma 2.1 in [6], with the roles of M and $\overline{M}$ interchanged. Hence there exists an isometric immersion $\overline{\Phi} : \overline{M} \to M$ with $\overline{\Phi}(\overline{p}) = p$ and $d\overline{\Phi} = I^{-1}$. Actually $\overline{\Phi}$ is an isometry because (1) an isometric immersion between two complete Riemannian manifolds of the same dimension is a covering map by Theorem IV.4.6 in [7, page 176], and (2) a smooth covering map between two simply connected manifolds is a diffeomorphism.

This completes the proof of Proposition 1.2.

3. Heterogeneity and Transversality

Let us say that two k-jets, $J_p^k g$ and $J_q^k h$, of germs of Riemannian metrics g and h at points p and q in M are equivalent if there is a germ of a diffeomorphism f with $f(p) = q$ such that $J_p^k(f^*(h)) = J_p^k g$. We then define a Riemannian metric g on M to be heterogeneous of order k if the k-jets of g at any two distinct points of M are not equivalent. Obviously, being heterogeneous of order k for some $k \geq 2$ is a stronger condition than being simply heterogeneous. Let us proceed to explain how to express heterogeneity of order k in terms of transversality when k is sufficiently large.

Let $\pi : X \to M$ denote the the bundle of positive definite symmetric covariant 2-tensors over M. Thus sections of X are just Riemannian metrics on M. Let $\pi^k : X^k \to M$ denote the bundle of k-jets of Riemannian metrics on M. Following [8, 9], let ${}_2\pi^k : {}_2X^k \to {}_2M$ denote the bundle of multijets of Riemannian metrics of order k and multiplicity 2. Thus

$$ {}_2X^k = \left\{ \left(J_p^k g, J_q^k h \right) \in X^k \times X^k : p \neq q \right\}, \qquad {}_2M = \{ (p,q) \in M \times M : p \neq q \}, \tag{3.1} $$

and ${}_2\pi^k(J_p^k g, J_q^k h) = (\pi^k(J_p^k g), \pi^k(J_q^k h)) = (p,q)$. Given a Riemannian metric g, the multijet extension

$$ {}_2J^k g : {}_2M \longrightarrow {}_2X^r \tag{3.2} $$

is defined by the formula ${}_2J^k g(p,q) = (J_p^k g, J_q^k g)$ for $(p,q) \in {}_2M$.

Since ${}_2X^k$ is just the set of ordered pairs of k-jets of Riemannian metrics over distinct points of M, the equivalence relation on k-jets of Riemannian metrics on M defines a subset $S \subset {}_2X^k$ consisting of pairs of equivalent k-jets. Obviously, g is heterogeneous of order k if and only if the image of its multijet extension ${}_2J^k g({}_2M)$ misses the set S. In the next section we investigate the structure of the set S and prove the following proposition.

Proposition 3.1. *Let n be the dimension of M. Then the set S is a union of finitely many submanifolds of codimension at least $\binom{n+1}{2}\binom{n+k}{k} - n\binom{n+k+1}{k+1} + n$. In particular* $\operatorname{codim}(S) > 2n$ *in any of the three cases* (i) $n = 2$ *and* $k \geq 4$, (ii) $n = 3$ *and* $k \geq 3$, *or* (iii) $n \geq 4$ *and* $k \geq 2$.

As a corollary we obtain the following stronger version of Theorem 1.1.

Theorem 3.2. *The set of Riemannian metrics on a manifold M of dimension n, which are heterogeneous of order k, is residual in the space of Riemannian metrics on M with the strong C^∞ topology as long any one of the three cases listed in Proposition 3.1 for which* $\operatorname{codim}(S) > 2n$ *holds.*

Proof. Since S is a union of submanifolds, application of the multijet transversality theorem (Corollary 3.4 [8] or [9, page 739]), shows that the set of g for which ${}_2J^k g$ is transverse to S forms a residual set in the space of Riemannian metrics with the strong C^∞ topology. But because $\dim({}_2M) = 2n$ and $\operatorname{codim}(S) > 2n$, it follows that ${}_2J^k g$ is transverse to S if and only if its image misses S, that is, if and only g is heterogeneous of order k.

This completes the proof of Theorem 3.2. Theorem 1.1 is an immediate consequence. □

4. The Structure of S

Consider the collection $\mathcal{M}^k(n)$ of k-jets of Riemannian metrics on $\mathbf{R}^n$ at 0 and the so-called jet group $\mathcal{G}^{k+1}(n)$, consisting of the $(k+1)$-jets of diffeomorphisms $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfying $f(0) = 0$ ([12, page 128]). The jet group acts upon $\mathcal{M}^k(n)$ on the left by the formula $f \cdot g = J_0^k((f^{-1})^* g)$ for $f \in \mathcal{G}^{k+1}(n)$ and $g \in \mathcal{M}^k(n)$. For any subgroup $\mathcal{H}$ of $\mathcal{G}^{k+1}(n)$, let $\mathcal{M}^k_{[\mathcal{H}]}(n)$ denote the set of points on orbits of type $\mathcal{G}^{k+1}(n)/\mathcal{H}$, that is,

$$\mathcal{M}^k_{[\mathcal{H}]}(n) = \left\{ g \in \mathcal{M}^k(n) : \text{The isotropy group of } g \text{ is conjugate to } \mathcal{H} \right\}. \tag{4.1}$$

Finally, let $\overline{\mathcal{M}}^k_{[\mathcal{H}]}(n)$ denote the orbit space $\mathcal{G}^{k+1}(n) \setminus \mathcal{M}^k_{[\mathcal{H}]}(n)$, and let Q denote the quotient projection $Q : \mathcal{M}^k_{[\mathcal{H}]}(n) \to \overline{\mathcal{M}}^k_{[\mathcal{H}]}(n)$.

Proposition 4.1. *$\mathcal{M}^k_{[\mathcal{H}]}(n)$ and $\overline{\mathcal{M}}^k_{[\mathcal{H}]}(n)$ are smooth manifolds, and Q is a smooth fibration with fibers $\mathcal{G}^{k+1}(n)/\mathcal{H}$. Moreover, there are only finitely many distinct orbit types, and these finitely many submanifolds $\mathcal{M}^k_{[\mathcal{H}]}(n)$ stratify $\mathcal{M}^k(n)$.*

Proof. If $\mathcal{G}^{k+1}(n)$ was compact, this proposition would follow from Theorem IV.3.3 in [10, page 182]. Although $\mathcal{G}^{k+1}(n)$ is not compact, its action on $\mathcal{M}^k(n)$ reduces to a compatible action of the orthogonal group $O(n)$, which canonically includes as a subgroup of $\mathcal{G}^{k+1}(n)$, on the subset $\mathcal{N}^k(n)$ of $\mathcal{M}^k(n)$. Here $\mathcal{N}^k(n)$ denotes the set of k-jets of Riemannian metrics $\mathbf{R}^n$ at 0 for which the standard coordinates on $\mathbf{R}^n$ form a normal coordinate system. These are the jets that satisfy the conditions (2.5.1-3) in [11] for $1 \le r \le k$. If one carries out the proof of Theorem 2.6 in [11] for k-jets, rather than for ∞-jets, one concludes that each $\mathcal{G}^{k+1}(n)$ orbit in $\mathcal{M}^k(n)$ meets $\mathcal{N}^k(n)$ in an $O(n)$ orbit. Thus the inclusion of the orbit space $O(n) \setminus \mathcal{N}^k(n)$ into $\mathcal{G}^{k+1}(n) \setminus \mathcal{M}^k(n)$ is a one-to-one correspondence. This also implies that every isotropy subgroup for the $\mathcal{G}^{k+1}(n)$ action is conjugate to an isotropy subgroup of the $O(n)$ action. Thus there is a one-to-one correspondence between the orbit types of the two actions. In addition, we see that the isotropy subgroups of the $\mathcal{G}^{k+1}(n)$ action are compact. It also follows that every slice for the $O(n)$ action on $\mathcal{N}^k_n$ is a slice for the $\mathcal{G}^{k+1}(n)$ action on $\mathcal{M}^k(n)$ which proves that slices exist for the latter action. Because of these observations, the conclusions of Theorem IV.3.3 in [10] which apply to the action of $O(n)$ on $\mathcal{N}^k(n)$ also apply to the action of $\mathcal{G}^{k+1}(n)$ on $\mathcal{M}^k(n)$. That there are only finitely many of orbit types follows in the same way from the well-known finiteness of orbit types for orthogonal actions [10, page 112]. This completes the proof of Proposition 4.1. □

Let $\mathcal{P}^{k+1}(M)$ denote the principal $\mathcal{G}^{k+1}(n)$ bundle of the $(k+1)$th order frames on the n-dimensional manifold M [12, page 122]. Clearly, the bundle X^k of k-jets of Riemannian metrics on M is the associated bundle $\mathcal{P}^{k+1}(M) \times_{\mathcal{G}^{k+1}(n)} \mathcal{M}^k(n)$. Since the stratification by orbit types of $\mathcal{M}^k(n)$ is invariant under $\mathcal{G}^{k+1}(n)$, it induces a stratification of X^k where the typical stratum takes the form of the associated bundle

$$X^k_{[\mathcal{H}]} = \mathcal{P}^{k+1}(M) \times_{\mathcal{G}^{k+1}(n)} \mathcal{M}^k_{[\mathcal{H}]}(n). \tag{4.2}$$

Moreover, the quotient map Q induces a smooth submersion $X^k_{[\mathcal{H}]} \to \overline{\mathcal{M}}^k_{[\mathcal{H}]}(n)$.

Because $S \subset X^k \times X^k$, we may set

$$S_{[\mathscr{H}]} = S \cap \left(X^k_{[\mathscr{H}]} \times X^k_{[\mathscr{H}]} \right). \tag{4.3}$$

Since two equivalent jets automatically have the same orbit type, we have

$$S = \bigcup_{[\mathscr{H}]} S_{[\mathscr{H}]}. \tag{4.4}$$

Clearly, $S_{[\mathscr{H}]}$ is just the inverse image of the diagonal in $\overline{\mathcal{M}}^k_{[\mathscr{H}]}(n) \times \overline{\mathcal{M}}^k_{[\mathscr{H}]}(n)$ under the product submersion $X^k_{[\mathscr{H}]} \times X^k_{[\mathscr{H}]} \to \overline{\mathcal{M}}^k_{[\mathscr{H}]}(n) \times \overline{\mathcal{M}}^k_{[\mathscr{H}]}(n)$ induced by Q. Therefore, $S_{[\mathscr{H}]}$ is a smooth submanifold of $X^k_{[\mathscr{H}]} \times X^k_{[\mathscr{H}]}$ whose codimension satisfies

$$\operatorname{codim}\left(S_{[\mathscr{H}]} \subset X^k_{[\mathscr{H}]} \times X^k_{[\mathscr{H}]} \right) = \dim\left(\overline{\mathcal{M}}^k_{[\mathscr{H}]}(n) \right). \tag{4.5}$$

We may now compute the codimension of $S_{[\mathscr{H}]}$ in ${}_2X^k$:

$$\begin{aligned}
\operatorname{codim}(S_{[\mathscr{H}]}) &= \operatorname{codim}\left(S_{[\mathscr{H}]} \subset X^k \times X^k \right) \\
&= \operatorname{codim}\left(S_{[\mathscr{H}]} \subset X^k_{[\mathscr{H}]} \times X^k_{[\mathscr{H}]} \right) + 2\operatorname{codim}\left(X^k_{[\mathscr{H}]} \subset X^k \right) \\
&= \dim\left(\overline{\mathcal{M}}^k_{[\mathscr{H}]}(n) \right) + 2\operatorname{codim}\left(\mathcal{M}^k_{[\mathscr{H}]}(n) \subset \mathcal{M}^k(n) \right) \\
&= \dim\left(\mathcal{M}^k_{[\mathscr{H}]}(n) \right) - \dim\left(\mathcal{G}^{k+1}(n)/\mathscr{H} \right) \\
&\quad + 2\left(\dim\left(\mathcal{M}^k(n) \right) - \dim\left(\mathcal{M}^k_{[\mathscr{H}]}(n) \right) \right) \\
&= \dim\left(\mathcal{M}^k(n) \right) - \dim\left(\mathcal{G}^{k+1}(n) \right) + \dim(\mathscr{H}) \\
&\quad + \left(\dim\left(\mathcal{M}^k(n) \right) - \dim\left(\mathcal{M}^k_{[\mathscr{H}]}(n) \right) \right) \\
&\geq \dim\left(\mathcal{M}^k(n) \right) - \dim\left(\mathcal{G}^{k+1}(n) \right) \\
&= \binom{n+1}{2}\binom{n+k}{k} - \left(n\binom{n+k+1}{k+1} - n \right).
\end{aligned} \tag{4.6}$$

In view of Proposition 4.1, this completes the proof of first statement in Proposition 3.1. The second statement of Proposition 3.1 follows directly from the first.

5. The Space of Complete Metrics

Since every Riemannian metric on a compact manifold is complete, Theorem 1.3 is an immediate consequence of Theorem 1.1 and Proposition 1.2 when M is compact. If M is not compact, the space of complete metrics on M is a proper subspace of the space of all Riemannian metrics on M. Thus to prove Theorem 1.3 in general, we need to show that the subspace of complete metrics inherits the property of being a Baire space from the space of all metrics, and that an open dense subset of the space of all metrics intersects the subspace of complete metrics in an open dense subset of the subspace. Both of these statements are immediate consequences of the next proposition.

Proposition 5.1. *The set of all complete Riemannian metrics on a given smooth manifold M is both open and closed in the space of Riemannian metrics on M with the strong C^∞ topology.*

Proof. Fix a Riemannian metric g_0 on M. It suffices to show that there is a neighborhood $\mathcal{U}$ of g_0 in the strong C^∞ topology consisting either entirely of complete metrics if g_0 is complete or entirely of incomplete metrics if g_0 is incomplete.

To this end, let $\mathcal{U}$ be the set of all Riemannian metrics g such that

$$\frac{1}{2}g_0(X,X) < g(X,X) < 2g_0(X,X) \tag{5.1}$$

for all $X \neq 0$ in TM. Obviously, $\mathcal{U}$ is an open neighborhood of g_0 in the strong C^∞ topology. Clearly, if $g \in \mathcal{U}$, and if dist and dist_0 are the associated distance functions corresponding to g and g_0, respectively, then we have

$$\frac{1}{2}\text{dist}_0(p,q) \le \text{dist}(p,q) \le 2\,\text{dist}_0(p,q) \tag{5.2}$$

for all p and q in M. It follows that dist and dist_0 have the same Cauchy sequences and the same convergent sequences with the same limits. Thus g and g_0 are either both complete or both incomplete. This finishes the proof of Proposition 5.1. □

References

[1] T. Sunada, "Riemannian coverings and isospectral manifolds," *Annals of Mathematics*, vol. 121, no. 1, pp. 169–186, 1985.

[2] W. Ambrose, "Parallel translation of Riemannian curvature," *Annals of Mathematics, Second Series*, vol. 64, pp. 337–363, 1956.

[3] J. Hebda, "Metric structure of cut loci in surfaces and Ambrose's problem," *Journal of Differential Geometry*, vol. 40, no. 3, pp. 621–642, 1994.

[4] J. Itoh, "The length of a cut locus on a surface and Ambrose's problem," *Journal of Differential Geometry*, vol. 43, no. 3, pp. 642–651, 1996.

[5] J. Hebda, "Conjugate and cut loci and the Cartan-Ambrose-Hicks theorem," *Indiana University Mathematics Journal*, vol. 31, no. 1, pp. 17–26, 1982.

[6] J. Hebda, "Parallel translation of curvature along geodesics," *Transactions of the American Mathematical Society*, vol. 299, no. 2, pp. 559–572, 1987.

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry. Vol I*, John Wiley & Sons, New York, NY, USA, 1963.

[8] J. Mather, "Stability of C^∞ mappings. V. Transversality," *Advances in Mathematics*, vol. 4, pp. 301–336, 1970.

[9] C. Wall, "Geometric properties of generic differentiable manifolds," in *Geometry and Topology*, vol. 597 of *Lecture Notes in Mathematics*, pp. 707–774, Springer, Berlin, Germany, 1977.
[10] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, NY, USA, 1972.
[11] D. B. A. Epstein, "Natural tensors on Riemannian manifolds," *Journal of Differential Geometry*, vol. 10, no. 4, pp. 631–645, 1975.
[12] I. Kolář, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry*, Springer, Berlin, Germany, 1993.