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*Research Article*

# Abnormal Curves on the Goursat Systems of $\mathbf{R}^n$

**Mohamad H. Cheaito and Hassan Zeineddine**

*Department of Mathematics, Faculty of Sciences, Lebanese University, Hadath, Lebanon*

Correspondence should be addressed to Mohamad H. Cheaito, mohamad_cheaito@yahoo.com

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We study the abnormal and the rigid curves of the 2-distributions of $\mathbf{R}^n$ satisfying everywhere the Goursat condition. We give the directions for the rigid and the abnormal curves when the systems satisfy the strong Goursat condition or when they have a singularity of order 2 in each dimension.

## 1. Introduction

Let $E$ be a 2-distribution on $\mathbf{R}^n$. We denote by

$$E^1 = E_1 = E, \qquad E^i = \left[E^{i-1}, E^{i-1}\right], \qquad E_i = [E, E_{i-1}]. \tag{1.1}$$

A *small growth vector* (sgv) of $E$, at a point $p \in \mathbf{R}^n$, is the sequence

$$[r_1(p), r_2(p), \dots]_S, \tag{1.2}$$

where $r_i(p) = \dim E_i(p)$, for every $i \geq 1$.

The *great growth vector*, at $p$, is the sequence

$$[m_1(p), m_2(p), \dots]_G, \tag{1.3}$$

where $m_j(p) = \dim E^j(p)$, for every $j \geq 1$.

If the dimensions of $E_i$ (resp., $E^j$) are independent of $p$, then the distribution is called *regular* (resp., *totally regular*).

If the great growth vector, at a point $p \in \mathbf{R}^n$, is $[2, 3, 4, \dots, n]_G$, then the distribution is called *distribution satisfying the Goursat condition at p*. Moreover, if $E$ satisfies, on a neighborhood of $p$, the Goursat condition, then its annihilator, $E^{\perp}$, is called *Goursat system* and denoted by (GS).

The classification of the distributions, with respect to the small and great growth vectors, was the object of many articles. The beginning was by Engel [1], where he gave the normal form of the (GS) in dimension 4.

In an article written in 1910, Cartan [2] studied the case of dimension 5. In 1978 Giaro et al. completed the work of Cartan about the systems of dimension 5 [3]. In such a case 2 nonequivalent models are presented. In 1981, Kumpera and Ruiz [4] gave the different normal forms in dimension $n \leq 6$.

The classification, of models, in dimensions 7 and 8 are given by [5]. The study of the models in dimension $n$ is also open. We say that [6], when the small and the great growth vector are the same, we have the system (GNF).

Zhitomirskiĭ [7] gave the asymptotic normal forms of the regular distributions and the generic case studied in many articles, for example [8].

The normal form of the model, satisfying at a neighborhood of a point the small growth vector $[2,3,4,4,5,5,\ldots,n-1,n-1,n]_S$, is given in [9].

## 2. Rigid and Abnormal Line Subdistributions of the Goursat Systems Satisfying the Strong Condition of Goursat

The *Goursat systems* are given by the following theorem.

**Theorem 2.1** (see [4, 5]). *Let $E$ be a 2-distribution on $\mathbf{R}^n$, satisfying in each point, the condition of Goursat, then*

$$E^{\perp} = \begin{cases} \omega_1 = dx_2 + x_3 dx_1, \\ \omega_2 = dx_3 + x_4 dx_1, \\ \omega_3 = dx_{i_3} + x_5 dx_{j_3}, \quad (i_3, j_3) \in \{(4,1),(1,4)\}, \\ \omega_4 = dx_{i_4} + X_6 dx_{j_4}, \quad (i_4, j_4) \in \{(5, j_3),(j_3, 5)\}, \\ \quad\vdots \\ \omega_{n-2} = dx_{i_{n-2}} + X_n dx_{j_{n-2}}, \quad (i_{n-2}, j_{n-2}) \in \{(n-1, j_{n-3}),(j_{n-3}, n-1)\}, \end{cases} \tag{2.1}$$

*where*

$$X_l = \begin{cases} x_l, & \text{if } (i_{l-2}, j_{l-2}) = (j_{l-3}, l-1), \\ x_l + c_l, & \text{if } (i_{l-2}, j_{l-2}) = (l-1, j_{l-3}), \end{cases} \tag{2.2}$$

*for $6 \leq l \leq n$ and $c_6, c_7, \ldots, c_{n-2}$ are real arbitrary constants.*

This theorem gives the different Goursat systems denoted by (GS).

*Definition 2.2.* Let $E$ be a 2-distribution on $\mathbf{R}^n$, $p \in \mathbf{R}^n$. $E$ satisfies the strong condition of Goursat, at $p$, if the small and the big growth vectors, at this point, are $[2,3,\ldots,n]_S$ and $[2,3,\ldots,n]_B$.

**Theorem 2.3** (see [6]). *Let $E$ be a 2-distribution on $\mathbf{R}^n$ satisfying, in each point, the condition of Goursat. Suppose that $E$ satisfies the strong condition of Goursat, at a point $p \in \mathbf{R}^n$, then there exists a local coordinate system $(x, U)$, around $p$, such that*

$$E^{\perp} = \begin{cases} \omega_1 = dx_2 + x_3 dx_1, \\ \omega_2 = dx_3 + x_4 dx_1, \\ \omega_3 = dx_4 + x_5 dx_1, \\ \omega_4 = dx_5 + x_6 dx_1, \\ \quad \vdots \\ \omega_{n-2} = dx_{n-1} + x_n dx_1, \end{cases} \tag{2.3}$$

*it means that $E$ is spanned by $v_1 = \partial/\partial x_n$ and*

$$v_2 = \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} - \cdots - x_n \frac{\partial}{\partial x_{n-1}}. \tag{2.4}$$

Remark that, in this theorem, $E$ satisfies the strong condition of Goursat, at a point $p$. Such property can be extended without difficulty to a neighborhood of $p$. For the definitions of abnormal and rigid curves, see [10].

*Definition 2.4.* Let $E$ be a 2-distribution on $M$; a $C^1$-curve $\gamma : [\alpha, \beta] \to M$ is said to be horizontal (or $E$-curve) if $\gamma^{\cdot}(t) \in E_{(\gamma(t))}$, for any $t \in [\alpha, \beta]$.

The set of horizontal curves connecting two points $a$ and $b$ of $M$, will be denoted by $\Omega_{a,b}([\alpha, \beta])$. The theorem of Chow [11] certified that $\Omega_{a,b}([\alpha, \beta]) \neq \phi$, for any $a, b \in M$.

*Definition 2.5.* Let $E$ be a 2-distribution on $M$, *a* $C^1$-curve $\gamma : [\alpha, \beta] \to M$ is said to be rigid, if $\gamma$ is an isolated point of $\Omega_{a,b}([\alpha, \beta])$ for the $C^1$-topology.

*Definition 2.6.* Let $E$ be a 2-distribution on $M$. A line subdistribution (i.e., distribution of dimension one) of $L$ is said to be rigid, if any $L$-curve is rigid. $L$ is said to be local rigid, if for any $p \in M$, there exists a neighborhood $U$ of $p$, such that any $L_U$-curve is rigid.

If $E$ is a 2-distribution on $M$, we denote $\Omega_a([\alpha, \beta])$ the set of $E$-curves $\gamma : [\alpha, \beta] \to M$, starting from the point $a$.

*Definition 2.7.* A curve $\gamma \in \Omega_a([\alpha, \beta])$, is said to be abnormal, if the mapping end: $\Omega_a([\alpha, \beta]) \to M$, defined by end $(\gamma) = \gamma(\beta)$, is not a submersion at $\gamma$.

**Proposition 2.8** (see [10]). *Let $E$ be a $k$-distribution on $M$. If $v_1, v_2, \ldots, v_k$ form a basis of $E$ and if $\gamma \in \Omega_a([\alpha, \beta])$, such that $\gamma^{\cdot}(t) = u_1(t)v_1 + \cdots + u_k(t)v_k|_{\gamma(t)}$, then the following propositions are equivalent.*

(1) *$\gamma$ is abnormal.*

(2) *There exists a lift curve $\Gamma : [\alpha, \beta] \to T^{\star}M$, absolutely continuous, of coordinates $(q_1, q_2, \ldots, q_n)$, such that*

  (a) *$\Gamma(t) \neq 0$, for any $t \in [\alpha, \beta]$,*

  (b) *$\Gamma(t) \in E^{\perp}$,*

  (c) *$\Gamma$ satisfies the equation $(q_1^{\cdot}, q_2^{\cdot}, \ldots, q_n^{\cdot}) = u_1(t)(q_1, q_2, \ldots, q_n)dv_1 + \cdots + u_k(t)(q_1, q_2, \ldots, q_n)dv_k|_{\gamma(t)}$.*

*Definition 2.9.* Let $E$ be a 2-distribution on $M$; a line subdistribution $L$ is said to be abnormal, if any $L$-curve is abnormal. $L$ is said to be local abnormal, if for any $p \in M$, there exists a neighborhood $U$ of $p$, such that any $L_U$-curve is abnormal.

*Definition 2.10.* Let $E$ be a 2-distribution on $M$; a distribution $D$ on $M$ is said to be nice with respect to $E$ if $D$ is an involutive distribution of codimension 2 such that $E_p \not\subset D_p$ and $\dim(E_p^2 \cap D_p) = 2$, for any point $p \in M$.

**Proposition 2.11** (see [10]). *Let $E$ be a 2-distribution on $\mathbf{R}^n$ and $L$ be a line subdistribution on $E$. Consider the following properties.*

(a) *$L$ is locally rigid.*

(b) *$L$ is locally abnormal.*

(c) *Locally $L$ is the intersection of $E$ and a nice distribution.*

(d) *$\dim(ad_L^{\infty})_p < n$, for every $p \in \mathbf{R}^n$.*

*Then, one has the following implication:*

$$\begin{array}{ccccc} & & \text{(a)} & & \\ & & \Downarrow & & \\ \text{(c)} & \Longrightarrow & \text{(b)} & \Longrightarrow & \text{(d)} \end{array} \tag{2.5}$$

Zhitomirskiĭ, in [10], conjectured that (d) $\Rightarrow$ (b), and he proved that (a), (b), (c), and (d) are not equivalent in general. Now we prove that, The properties are equivalent if the distribution satisfies the strong condition of Goursat.

**Theorem 2.12.** *Let $E$ be a 2-distribution on $\mathbf{R}^n$, $n \geq 4$, satisfying in each point the strong condition of Goursat, then the properties (a), (b), (c), and (d) are equivalent.*

*Proof.* By Theorem 2.3, $E$ is spanned, on a neighborhood $U$, by

$$v_1 = \frac{\partial}{\partial x_n}, \qquad v_2 = \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} - \cdots - x_n \frac{\partial}{\partial x_{n-1}}. \tag{2.6}$$

Let $L$ be a line subdistribution satisfying (d), and let $u = av_1 + bv_2$ be a generator of $L$. We have

$$[v_2, v_1] = \frac{\partial}{\partial x_{n-1}}, \qquad [v_2, [v_2, v_1]] = \frac{\partial}{\partial x_{n-2}}. \tag{2.7}$$

Easily, by induction, we say that

$$\left[ad^i_{v_2}, v_1\right] = \frac{\partial}{\partial x_{n-i}}, \qquad \left[v_1, \left[ad^i_{v_2}, v_1\right]\right] = \left[\frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_{n-i}}\right] = 0, \tag{2.8}$$

for every $i = 1, 2, \ldots, n-2$.

A simple induction shows that

$$ad^i_u(v_1) = \alpha^i_1 v_1 + \alpha^i_2 v_2 \sum_{j=1}^{i-1} \alpha^i_j ad^j_{v_2}(v_1) + b^i ad^i_{v_2}(v_1), \tag{2.9}$$

where ${\alpha_j}^i$, for $j = 1, 2, \ldots, i$, are $C^\infty$ functions on $U$ to $\mathbf{R}$. Because $\dim(ad^\infty_L)_p < n$, for every $p \in \mathbf{R}^n$, we have necessarily $b = 0$ and by consequently $L$ is spanned by $v_1$.

Prove now (d) $\Rightarrow$ (c). Let $Z = \ker(dx_1 \wedge dx_2)$, we say easily $Z$ is a nice distribution (see [10]). In fact: $v_1(x_1) = v_1(x_2) = 0$, then $v_1 \in E \cap Z$. Otherwise $[v_1, v_2] = -\partial/\partial x_{n-1}$, then $[v_1, v_2](x_1) = [v_1, v_2](x_2) = 0$, we deduce that $E^2 \cap Z = \operatorname{span}\{v_1, [v_1, v_2]\}$ and consequently $\dim(E^2 \cap Z)_p = 2$.

Now $\operatorname{cod}(Z) = 2$ and $Z$ is integrable. Because $v_2(x_1) = 0$, we obtain $E_p$ is not a subset of $Z_p$, for every $p \in \mathbf{R}^n$, then $Z$ is a nice distribution. Moreover $L = E \cap Z$, then (d) $\Rightarrow$ (c), by [10].

Prove now (d) $\Rightarrow$ (a). Consider the form $\omega_{n-2}$ of the system $E^\perp$. We have $(\omega_{n-2})_0 = (dx_{n-2})_0 \neq 0$ and

$$i_{v_1} d\omega_{n-2} = i_{v_1}(dx_{n-1} \wedge dx_1) = i_{\partial/\partial x_n}(dx_{n-1} \wedge dx_1) = 0. \tag{2.10}$$

Otherwise $[v_2, [v_1, v_2]] = -\partial/\partial x_{n-2}$ then $\omega([v_2, [v_1, v_2]]) = -1 \neq 0$ and $\omega_0$ is not in $E^3|_0$. By Theorem 5.7 of [10], $L$ is locally rigid. □

Let $E$ be a 2-distribution of $\mathbf{R}^n$, spanned by $v_1$ and $v_2$. $L_E$ is the line subdistribution spanned by a vector field in the form $av_1 + bv_2$, where $a$ and $b$ are such that $a[v_1, [v_1, v_2]] + b[v_2, [v_1, v_2]]$ is in $E^2$ and $(a^2 + b^2 \neq 0)$. We say that $L_E$ is independent of the choice of $v_1$ and $v_2$. Zhitomirskiĭ [10] proved that $L_E$ is a line subdistribution locally rigid, also by a conjecture, it is unique, in the case where $E$ is regular and satisfying the condition

$$\dim E^2 = 3, \qquad \dim E^3 = 4, \tag{2.11}$$

this is the case of $(GS_1)$.

## 3. Rigid and Abnormal Line Subdistributions of the Goursat Systems Presenting in Each Dimension a Singularity of Order 2

*Definition 3.1.* Let $S$ be a Goursat system. $S$ is called presenting a transposition of order $l$, $l \in \{3, 4, \ldots, n-2\}$ if

$$\begin{aligned} \omega_{l-1} &= dx_{i_{l-1}} + X_{l+1} dx_{j_{l-1}}, \\ \omega_l &= dx_{j_{l-1}} + x_{l+2} dx_{l+1}. \end{aligned} \tag{3.1}$$

*Definition 3.2.* If the small growth vector of a 2-distribution $E$ on $\mathbf{R}^n$, at a point $p$ of $\mathbf{R}^n$, has the form $[2, 3, \ldots, \underbrace{s, s, \ldots, s}_{k \text{ times}}, \ldots, n]$ (denoted by $[2, 3, \ldots, s_k, \ldots, n]$), the distribution is called a distribution presenting, in the dimension $s$, a singularity of order $k$.

*Remark 3.3.* If the distribution satisfies the condition of Goursat the dimensions 2, 3 and $n$ are of order 1 at every point.

*Notation.* The system of Goursat satisfying, at every point $x \in \mathbf{R}^n$, the condition $[2, 3, 4_k, 5_k, \ldots, (n-1)_k, n]_S$ is denoted by $(\mathrm{GS}_k)$.

**Theorem 3.4** (see [9]). *Let $E$ be a 2-distribution on $\mathbf{R}^n$, satisfying at every point the Goursat condition, such that at $x_0 \in \mathbf{R}^n$, we have $[2, 3, 4_2, 5_2, \ldots, (n-1)_2, n]_S$. Then there exists a local system of coordinates $(x, U)$, around $x_0$, such that*

$$E^{\perp} = \begin{cases} \omega_1 = dx_2 + x_3 dx_1, \\ \omega_2 = dx_3 + x_4 dx_1, \\ \omega_3 = dx_4 + x_5 dx_1, \\ \omega_4 = dx_5 + x_6 dx_1, \\ \quad \vdots \\ \omega_{n-3} = dx_{n-2} + x_{n-1} dx_1, \\ \omega_{n-2} = dx_1 + x_n dx_{n-1}, \end{cases} \tag{3.2}$$

*it means that E is spanned by*

$$v_1 = \frac{\partial}{\partial x_n}, \qquad v_2 = -x_n \frac{\partial}{\partial x_1} + x_n x_3 \frac{\partial}{\partial x_2} + x_n x_4 \frac{\partial}{\partial x_3} + \cdots + x_n x_{n-1} \frac{\partial}{\partial x_{n-2}} + \frac{\partial}{\partial x_{n-1}}. \tag{3.3}$$

Now we want to study the rigid and the abnormal line subdistributions (directions) for the Goursat systems $(\mathrm{GS}_2)$.

*Definition 3.5.* Let $E$ be a 2-distribution spanned by $v_1$ and $v_2$. The line subdistribution $L_E$, is the line subdistribution spanned by a vector field in the form $av_1 + bv_2$, where $a$ and $b$ are such that $a[v_1, [v_1, v_2]] + b[v_2, [v_1, v_2]] \in E^2$ and $a^2 + b^2 \neq 0$.

**Theorem 3.6.** *In the Goursat systems ($GS_2$), $L_E$ is the unique direction of abnormal and rigid curves.*

*Proof.* $E$ is spanned by

$$v_1 = \frac{\partial}{\partial x_n}, \qquad v_2 = -x_n \frac{\partial}{\partial x_1} + x_n x_3 \frac{\partial}{\partial x_2} + x_n x_4 \frac{\partial}{\partial x_3} + \cdots + x_n x_{n-1} \frac{\partial}{\partial x_{n-2}} + \frac{\partial}{\partial x_{n-1}}, \tag{3.4}$$

and $E^2$ is spanned by $v_1$, $v_2$, and $[v_1, v_2]$, where $[v_1, v_2] = -\partial/\partial x_1 + x_3(\partial/\partial x_2) + x_4(\partial/\partial x_3) + \cdots + x_{n-1}(\partial/\partial x_{n-2})$.

Prove now $L_E$ is spanned by $v_1$. In fact $[v_1, [v_1, v_2]] = 0$ and $[v_2, [v_1, v_2]] = \partial/\partial x_{n-2}$, then necessarily $b = 0$ and $L_E = \operatorname{span}\{v_1\}$. Recall that $L_E$ is a direction of rigid curves, then of abnormal curves.

Does exist another direction field of the abnormal curves?

Let $L = \operatorname{Vect}\{\alpha v_1 + \beta v_2\}$ be an arbitrary line subdistribution of $E$. Let $\gamma : I \to \mathbf{R}^n$ be a horizontal curve of $L$, (i.e., $\dot{\gamma}(t) \in L_{\gamma(t)}$). Suppose that $\gamma$ is an abnormal curve. There exists a lift curve $\Gamma : I \to T^\star\mathbf{R}^n$ satisfying the adjoint equation: $(\dot{p}_1, \dot{p}_2, \dots, \dot{p}_n) = -\alpha(p_1, p_2, \dots, p_n) dv_1 - \beta(p_1, p_2, \dots, p_n) dv_2$. In other hand:

$$(\dot{p}_1, \dots, \dot{p}_n) = -\beta(p_1, \dots, p_n) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & x_n & 0 & & 0 & \cdots & 0 & x_3 \\ 0 & 0 & 0 & x_n & 0 & 0 & \cdots & 0 & x_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots & x_n & x_{n-1} \\ 0 & 0 & \cdots & \cdots & & & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & & & \cdots & \cdots & 0 \end{pmatrix}. \tag{3.5}$$

We verify that $E^2 = \operatorname{Vect}\{v_1, v_2, \partial/\partial x_{n-1}\}$. But $\Gamma \subset (E^2)^\perp$, then we have

$$p_n = p_{n-1} = 0, \tag{3.6}$$

$$p_1 = x_3 p_2 + x_4 p_3 + \cdots + x_{n-1} p_{n-3}. \tag{I}$$

Suppose that $\beta \neq 0$. By the adjoint equation, we have $\dot{p}_{n-1} = -\beta x_n p_{n-2} = 0$, but $\beta \neq 0$, then $p_{n-2} = 0$. Similarly $\dot{p}_{n-2} = -\beta x_n p_{n-3} = 0$, then $p_{n-3} = 0$.

Show that by induction $p_{n-i} = 0$, for every $i = 1, 2, \dots, n-2$.

For $i = 1$, the property is true. Suppose that $p_{n-i-1} = 0$, prove that $p_{n-i} = 0$. By the adjoint equation

$$\dot{p}_i = -\beta x_n p_{i-1} = 0 \tag{3.7}$$

for every $i = 1, 2, \dots, n-3$. we deduce that $p_i = 0$. Finally, using $(I)$ we have $p_1 = 0$. We deduce $\Gamma = 0$, impossible, then we obtain $\beta = 0$ and $L$ is spanned by $v_1$, by consequently $L = L_E$ and $L$ is locally rigid. $\square$

**Corollary 3.7.** *With the same conditions of Theorem 2.3, the distribution $L_E$ is the unique line subdistribution locally rigid on $E$.*

*Proof.* In fact, $[v_1,[v_1,v_2]] = 0$ and $a[v_1,[v_1,v_2]] + b[v_2,[v_1,v_2]] = b[v_2,[v_1,v_2]] = -b(\partial/\partial x_{n-2}) \in E^2 = \text{span}\{v_1,v_2,\partial/\partial x_{n-1}\}$ if $b = 0$, then $L_E = \text{span}\{v_1\}$, but the distribution spanned by $v_1$ is the unique locally rigid subdistribution, on $E$, of dimension 1. □

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