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Research Article

On Reverses of Some Inequalities in n-Inner Product Spaces

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We present some new reverses of Cauchy-Bunyakovsky-Schwarz inequality, and Triangle and Boas-Bellman Type inequalities in n-inner product spaces. The results obtained generalize the results of Dragomir(2003–2005) in n-inner product spaces. Also we provide some applications for determinantal integral inequalities.

1. Introduction

In 1964, Gähler [1] introduced the concept of 2-norm and 2-inner product spaces as generalization of norm and inner product spaces. A systematic presentation of the results related to the theory of 2-inner product spaces can be found in the book in [2, 3] and in list of references in it. Generalization of 2-inner product space for $n \geq 2$ was developed by Misiak [4] in 1989. Gunawan and Mashadi [5] in 2001 introduced the concept of n-normed linear spaces. Gunawan [6] obtained Cauchy-Bunyakovsky inequality in these spaces. Cho et al. [7] extended parallelogram law and proved Hlwaka's type inequality in n-inner product spaces. In this paper by, using same ideas of [8–13] we establish some results regarding new reverses of Cauchy-Bunyakovsky-Schwarz inequality, and Triangle and Boas-Bellman Type inequalities in n-inner product spaces and we also provide some applications for determinantal integral inequalities.

2. Preliminaries

Definition 2.1 (see [4]). Assume that n is a positive integer and X is a vector space over the field $K = R$ of real numbers or the field $K = C$ of complex numbers, such that $\dim X \geq n$ and $\langle \cdot, \cdot \mid \underbrace{\cdot, \ldots, \cdot}_{n-1} \rangle$ is a K valued function defined on $\underbrace{X \times X \times \cdots \times X}_{n+1}$ such that:

(nI$_1$) $\langle x_1, x_1 \mid x_2, \ldots, x_n \rangle \geq 0$, for any $x_1,\ x_2, \ldots,\ x_n \in X$ and $\langle x_1, x_1 \mid x_2, \ldots, x_n \rangle = 0$ if and only if $x_1,\ x_2, \ldots,\ x_n$ are linearly dependent vectors;

(nI$_2$) $\langle a, b \mid x_1, \ldots, x_{n-1} \rangle = \langle a, b \mid \pi(x_1), \ldots, \pi(x_{n-1}) \rangle$, for any $a, b, x_1, x_2, \ldots, x_{n-1} \in X$ and for any bijections $\pi : \{x_1,\ x_2, \ldots,\ x_{n-1}\} \to \{x_1,\ x_2, \ldots,\ x_{n-1}\}$;

(nI$_3$) If $n > 1$, then $\langle x_1, x_1 \mid x_2, \ldots, x_n \rangle = \langle x_2, x_2 \mid x_1, x_3, \ldots, x_n \rangle$, for any $x_1, x_2, \ldots, x_n \in X$;

(nI$_4$) $\langle a, b \mid x_1, \ldots, x_{n-1} \rangle = \overline{\langle b, a \mid x_1, \ldots, x_{n-1} \rangle}$

(nI$_5$) $\langle \alpha a, b \mid x_1, \ldots, x_{n-1} \rangle = \alpha \langle a, b \mid x_1, \ldots, x_{n-1} \rangle$, for any $a, b, x_1, \ldots, x_{n-1} \in X$ and any scalar $\alpha \in R$;

(nI$_6$) $\langle a + a_1, b \mid x_1, \ldots, x_{n-1} \rangle = \langle a, b \mid x_1, \ldots, x_{n-1} \rangle + \langle a_1, b \mid x_1, \ldots, x_{n-1} \rangle$, for any $a,\ b,\ a_1, x_1, \ldots,\ x_{n-1} \in X$.

Then $\langle \cdot, \cdot \mid \underbrace{\cdot, \ldots, \cdot}_{n-1} \rangle$ is called the n-inner product and $(X,\ \langle \cdot, \cdot \mid \underbrace{\cdot, \ldots, \cdot}_{n-1} \rangle)$ is called the n-prehilbert space. If $n = 1$, then Definition 2.1 reduces to the ordinary inner product. In any given n-inner product space $(X \cdot \langle \cdot, \cdot \mid \cdot, \cdot, \ldots, \cdot \rangle)$, we can define a function $\|\cdot, \cdot, \ldots, \cdot\|$ on $X^n = \underbrace{X \times X \times \cdots \times X}_{n\text{-times}}$ as

$$\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_1 \mid x_2, \ldots, x_n \rangle} \tag{2.1}$$

for any $x_1,\ x_2, \ldots,\ x_n \in X$. It is easy to see that this function satisfies the following conditions:

(nN1) $\|x_1, x_2, \ldots, x_n\| \geq 0$ and $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(nN2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,

(nN3) $\|x_1, x_2, \ldots, a x_n\| = |a| \|x_1, x_2, \ldots, x_n\|$, for any $a \in K$,

(nN4) $\|x_1, x_2, \ldots, x_n - 1,\ y\ +\ z\| \leq \|x_1, x_2, \ldots,\ x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$.

A function $\|\cdot, \cdot, \ldots, \cdot\|$ defined on X^n and satisfying the above conditions is called an n-norm on X and the pair $(X,\ \|\cdot, \ldots, \cdot\|)$ is called n-normed linear space. Whenever an n-inner product space $(X, \langle \cdot, \cdot \mid \cdot, \cdot, \ldots, \cdot \rangle)$ is given, we consider it as an n-normed space $(X, \|\cdot, \cdot, \ldots, \cdot\|)$ with the n-norm defined by (2.1).

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field$K$. The following inequality is known as Cauchy-Schwarz's inequality:

$$|\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2, \quad x,\ y \in H, \tag{2.2}$$

where $\|z\|^2 = \langle z, z \rangle,\ \ z \in H$. The equality occurs in (2.2) if and only if x and y are linearly dependent.

In [8], Dragomir obtained the following reverse of Cauchy-Schwarz's inequality:

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4, \tag{2.3}$$

provided $x,\ y \in H$ and $a, A\ \in K$ are so that either

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0, \tag{2.4}$$

or, equivalently

$$\left\| x - \frac{a+A}{2}, y \right\| \leq \frac{1}{2}|A - a| \|y\|, \tag{2.5}$$

holds. The constant $1/4$ is best possible in (2.3) in the sense that it cannot be replaced by a smaller quantity.

If $x,\ y,\ A, a$ satisfy either (2.4) or (2.5), then the following reverse of Cauchy-Schwarz's inequality also holds:

$$\|x\|\|y\| \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[A\overline{\langle x, y\rangle} + \overline{a}\langle x, y\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{1/2}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{[\operatorname{Re}(\overline{a}A)]^{1/2}} |\langle x, y\rangle| \tag{2.6}$$

provided that, the complex numbers a and A satisfy the condition $\operatorname{Re}(\overline{a}A) > 0$. In both inequalities in (2.6), the constant $1/2$ is best possible.

The following reverse of the triangle inequality in inner product space was also obtained by Dragomir [9]

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over$K$ and $x,\ y \in H$, $M \geq m > 0$ such that either $\operatorname{Re}\langle Ay - x, x - ay\rangle \geq 0$ or, equivalently $\|x - (a + A)/2, y\| \leq 1/2|A - a|\|y\|$, holds.

$$0 \leq \|x\| + \|y\| - \|x + y\| \leq \frac{\sqrt{M} - \sqrt{m}}{\sqrt{mM}} \sqrt{\operatorname{Re}\langle x, y\rangle} \tag{2.7}$$

holds.

Gunawan [6] generalized the Cauchy-Bunyakovsky-Schwarz inequality (shortly, the CBS inequality) for inner product space to n-inner product space and obtained the following:

$$|\langle x, y \mid z_2, \ldots, z_n\rangle|^2 \leq \langle x, x \mid z_2, \ldots, z_n\rangle \langle y, y \mid z_2, \ldots, z_n\rangle. \tag{2.8}$$

Moreover, the equality holds if and only if $x,\ y,\ z_2, \ldots, z_n$ are linearly dependent. In terms of the n-norms, the (CBS)-inequality (2.8) can be written as

$$|\langle x, y \mid z_2, \ldots, z_n\rangle|^2 \leq \|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2. \tag{2.9}$$

The equality holds if and only if $x,\ y,\ z_2, \ldots, z_n$ are linearly dependent.

3. Main Results

The aim of the present paper is to generalize the above mentioned results of Dragomir [8, 9], that is, reverses of CBS inequality, Triangle inequality and Boas-bellman type inequalities in inner product space to n-inner product spaces.

3.1. Reverses of the CBS Inequality

Theorem 3.1. *Let A, $a \in K(K = C, R)$ and x, y, $z_2, \ldots, z_n \in X$, where $(X, \langle \cdot, \cdot \mid \cdot, \ldots, \cdot \rangle)$ is n-inner product space over K. If*

$$\operatorname{Re}(\langle Ay - x,\ x - ay \mid z_2, \ldots, z_n \rangle) \geq 0. \tag{3.1}$$

or, equivalently,

$$\left\| x - \frac{a + A}{2} y, z_2, \ldots, z_n \right\| \leq \frac{1}{2} |A - a| \| y, z_2, \ldots, z_n \|. \tag{3.2}$$

holds, then one has

$$0 \leq \| x, z_2, \ldots, z_n \|^2 \| y, z_2, \ldots, z_n \|^2 - |\langle x, y \mid z_2, \ldots, z_n \rangle|^2 \leq \frac{1}{4} |A - a|^2 \| y, z_2, \ldots, z_n \|^4. \tag{3.3}$$

The constant $1/4$ *in* (3.3) *cannot be replaced by a smaller constant.*

Proof. Using (nI_2)–(nI_6), we get

$$\begin{aligned} \langle z_2, z_2 \mid x \pm y, \ldots, z_n \rangle &= \langle x \pm y, x \pm y \mid z_2, \ldots, z_n \rangle \\ &= \langle x, x \mid z_2, \ldots, z_n \rangle + \langle y, y \mid z_2, \ldots, z_n \rangle \pm 2 \operatorname{Re} \langle x, y \mid z_2, \ldots, z_n \rangle, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\operatorname{Re} \langle x, y \mid z_2, \ldots, z_n \rangle \\ &\quad = \frac{1}{4} \left[\langle z_2, z_2 \mid x + y, \ldots, z_n \rangle - \langle z_2, z_2 \mid x - y, \ldots, z_n \rangle \right]. \end{aligned} \tag{3.5}$$

Considering the vectors x, u, U, $z_2, \ldots, z_n \in X$ and using (3.5) and (nN1–nN6), we have

$$\begin{aligned} &\operatorname{Re} \langle U - x,\ x - u \mid z_2, \ldots, z_n \rangle \\ &\quad = \frac{1}{4} \left[\langle z_2, z_2 \mid U - u, \ldots, z_n \rangle - \langle z_2, z_2 \mid -2x + U + u, \ldots, z_n \rangle \right] \\ &\quad = \frac{1}{4} \left[\langle U - u, U - u \mid z_2, \ldots, z_n \rangle - 4 \left\langle x - \frac{u + U}{2},\ x - \frac{u + U}{2} \mid z_2, \ldots, z_n \right\rangle \right] \\ &\quad = \frac{1}{4} \| U - u,\ z_2, \ldots, z_n \|^2 - \left\| x - \frac{u + U}{2},\ z_2, \ldots, z_n \right\|^2. \end{aligned} \tag{3.6}$$

Therefore, $\operatorname{Re} \langle U - x,\ x - u \mid z_2, \ldots, z_n \rangle \geq 0$ if and only if

$$\left\| x - \frac{u + U}{2},\ z_2, \ldots, z_n \right\| \leq \frac{1}{2} \| U - u,\ z_2, \ldots, z_n \|. \tag{3.7}$$

Applying this to the vectors $U = Ay$ and $u = ay$, we obtain that the (3.1) and (3.2) are equivalent. If we consider the real numbers

$$\begin{aligned} I_1 &= \operatorname{Re}\Big[\Big(A\|y, z_2, \dots, z_n\|^2 - \langle x, y \mid z_2, \dots, z_n\rangle\Big) \\ &\qquad \times\Big(\overline{\langle x, y \mid z_2, \dots, z_n\rangle} - \overline{a}\|y, z_2, \dots, z_n\|^2\Big)\Big], \\ I_2 &= \|y, z_2, \dots, z_n\|^2 \operatorname{Re}\langle Ay - x,\ x - ay \mid z_2, \dots, z_n\rangle. \end{aligned} \tag{3.8}$$

Then we obtain, by using properties of n-inner product space,

$$\begin{aligned} I_1 &= \|y, z_2, \dots, z_n\|^2 \operatorname{Re}\Big[A\overline{\langle x, y \mid z_2, \dots, z_n\rangle} + \overline{a}\ \langle x, y \mid z_2, \dots, z_n\rangle\Big] \\ &\quad - \big|\langle x, y \mid z_2, \dots, z_n\rangle\big|^2 - \|y, z_2, \dots, z_n\|^4 \operatorname{Re}(A\overline{a}). \\ I_2 &= \|y, z_2, \dots, z_n\|^2 \operatorname{Re}\Big[A\overline{\langle x, y \mid z_2, \dots, z_n\rangle} + \overline{a}\langle x, y \mid z_2, \dots, z_n\rangle\Big] \\ &\quad - \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - \|y, z_2, \dots, z_n\|^4 \operatorname{Re}(A\overline{a}) \end{aligned} \tag{3.9}$$

which gives

$$I_1 - I_2 = \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - \big|\langle x, y \mid z_2, \dots, z_n\rangle\big|^2, \tag{3.10}$$

for any $x,\ y,\ z_2, \dots, z_n \in X$ and $a,\ A \in K$. If (3.1) holds, then $I_2 \geq 0$ and thus $I_1 - I_2 \leq I_1$

$$\begin{aligned} &\Longrightarrow \|x, z_2, \dots, z_n\|^2 \|y, z_2, \dots, z_n\|^2 - \big|\langle x, y \mid z_2, \dots, z_n\rangle\big|^2 \\ &\quad \leq \operatorname{Re}\Big[\Big(A\|y, z_2, \dots, z_n\|^2 - \langle x, y \mid z_2, \dots, z_n\rangle\Big) \\ &\qquad \times\Big(\overline{\langle x, y \mid z_2, \dots, z_n\rangle} - \overline{a}\|y, z_2, \dots, z_n\|^2\Big)\Big]. \end{aligned} \tag{3.11}$$

Now using the elementary inequality $\operatorname{Re}(\alpha\overline{\beta}) \leq 1/4|\alpha + \beta|^2$ for any $\alpha, \beta \in K$ $(K = R,\ C)$, one yields

$$\begin{aligned} &\operatorname{Re}\Big[\Big(A\|y, z_2, \dots, z_n\|^2 - \langle x, y \mid z_2, \dots, z_n\rangle\Big)\Big(\overline{\langle x, y \mid z_2, \dots, z_n\rangle} - \overline{a}\|y, z_2, \dots, z_n\|^2\Big)\Big] \\ &\quad \leq \frac{1}{4}|A - a|^2 \|y, z_2, \dots, z_n\|^4. \end{aligned} \tag{3.12}$$

If we combine (3.11) and (3.12), we get the required inequality.

To prove the sharpness of the constant $1/4$, assume that (3.3) holds with a constant $C > 0$, that is,

$$\|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 - |\langle x, y \mid z_2, \ldots, z_n\rangle|^2 \le C|A-a|^2 \|y, z_2, \ldots, z_n\|^4, \tag{3.13}$$

where $x, y, z_2, \ldots, z_n, A$, and a satisfy the hypothesis of the theorem.

Consider $y \in X$ with $\|y, z_2, \ldots, z_n\| = 1$, $a \neq A$, $m \in X$ with $\|m, z_2, \ldots, z_n\| = 1$ and $\langle y, m \mid z_2, \ldots, z_n\rangle = 0$ and define

$$x = \frac{A+a}{2} y + \frac{A-a}{2} m \tag{3.14}$$

then we have

$$\operatorname{Re}\langle Ay - x,\ x - ay \mid z_2, \ldots, z_n\rangle = \frac{|A-a|^2}{4} \operatorname{Re}\langle y - m, y + m \mid z_2, \ldots, z_n\rangle = 0 \tag{3.15}$$

and then the condition (3.1) is fulfilled. From (3.13) we deduce

$$\left\| \frac{A+a}{2} y + \frac{A-a}{2} m, z_2, \ldots, z_n \right\| - \left| \left\langle \frac{A+a}{2} y + \frac{A-a}{2} m, y \mid z_2, \ldots, z_n \right\rangle \right|^2 \le C|A-a|^2 \tag{3.16}$$

and, since

$$\left\| \frac{A+a}{2} y + \frac{A-a}{2} m,\ z_2, \ldots, z_n \right\|^2 = \left| \frac{A+a}{2} \right|^2 + \left| \frac{A-a}{2} \right|^2, \quad \left| \left\langle \frac{A+a}{2} y + \frac{A-a}{2} m,\ y \mid z_2, \ldots, z_n \right\rangle \right|^2 = \left| \frac{A+a}{2} \right|^2. \tag{3.17}$$

By (3.16), we have

$$\left| \frac{A-a}{2} \right|^2 \le C|A-a|^2, \tag{3.18}$$

for any A, $a \in K$ with $a \neq A$, which implies $C \ge 1/4$. This completes the proof. □

Theorem 3.2. *Assume that x, y, $z_2, \ldots, z_n$, a and A are the same as in above Theorem 3.1. If* $\mathrm{Re}(\overline{a}A) > 0$, *then one has*

$$\begin{aligned} \|x, z_2, \ldots, z_n\| \|y, z_2, \ldots, z_n\| &\le \frac{1}{2} \frac{\mathrm{Re}\left[\left(\overline{A} + \overline{a}\right)\langle x, y \mid z_2, \ldots, z_n\rangle\right]}{[\mathrm{Re}(\overline{a}A)]^{1/2}} \\ &\le \frac{1}{2} \frac{|A + a|}{[\mathrm{Re}(\overline{a}A)]^{1/2}} \left|\langle x, y \mid z_2, \ldots, z_n\rangle\right|. \end{aligned} \tag{3.19}$$

The constant $1/2$ is best possible in both inequalities in the sense that it cannot be replaced by a smaller constant.

Proof. Define

$$\begin{aligned} I &= \mathrm{Re}\langle Ay - x,\ x - ay \mid z_2, \ldots, z_n\rangle \\ &= \mathrm{Re}\left[A\overline{\langle x, y \mid z_2, \ldots, z_n\rangle} + \overline{a}\langle x, y \mid z_2, \ldots, z_n\rangle\right] \\ &\quad - \|x, z_2, \ldots, z_n\|^2 - \|y, z_2, \ldots, z_n\|^2 \mathrm{Re}(A\overline{a}). \end{aligned} \tag{3.20}$$

We know that, for a complex number $\alpha \in C$, $\mathrm{Re}(\alpha) = \mathrm{Re}(\overline{\alpha})$ and thus

$$\mathrm{Re}\left[A\overline{\langle x, y \mid z_2, \ldots, z_n\rangle}\right] = \mathrm{Re}\left[\overline{A}\ \langle x, y \mid z_2, \ldots, z_n\rangle\right] \tag{3.21}$$

which implies

$$I = \mathrm{Re}\left[\left(\overline{A} + \overline{a}\right)\langle x, y \mid z_2, \ldots, z_n\rangle\right] - \|x, z_2, \ldots, z_n\|^2 - \|y, z_2, \ldots, z_n\|^2 \mathrm{Re}(A\overline{a}). \tag{3.22}$$

Since x, $y, z_2, \ldots, z_n, a$ and A are assumed to satisfy the condition (3.1), by (3.22), we deduce

$$\|x, z_2, \ldots, z_n\|^2 + \|y, z_2, \ldots, z_n\|^2 \mathrm{Re}(A\overline{a}) \le \mathrm{Re}\left[\left(\overline{A} + \overline{a}\right)\langle x, y \mid z_2, \ldots, z_n\rangle\right], \tag{3.23}$$

which gives

$$\begin{aligned} &\frac{1}{[\mathrm{Re}(\overline{a}A)]^{1/2}} \|x, z_2, \ldots, z_n\|^2 + [\mathrm{Re}(\overline{a}A)]^{1/2} \|y, z_2, \ldots, z_n\|^2 \\ &\quad \le \frac{\mathrm{Re}\left[\left(\overline{A} + \overline{a}\right)\langle x, y \mid z_2, \ldots, z_n\rangle\right]}{[\mathrm{Re}(\overline{a}A)]^{1/2}} \end{aligned} \tag{3.24}$$

since $\mathrm{Re}(A\overline{a}) > 0$.

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq \tag{3.25}$$

for $p,\ q \geq 0$ and $\alpha > 0$, we have

$$\begin{aligned} &2\|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| \\ &\quad \leq \frac{1}{[\operatorname{Re}(\overline{a}A)]^{1/2}} \|x, z_2, \dots, z_n\|^2 + [\operatorname{Re}(\overline{a}A)]^{1/2} \|y, z_2, \dots, z_n\|^2. \end{aligned} \tag{3.26}$$

Using (3.24) and (3.26), we deduce the first inequality in (3.22). The last part is obvious by the fact, for $z \in C$, $|\operatorname{Re}(z)| \leq |z|$.

To prove the sharpness of the constant $1/2$ in the first inequality in (3.22), we assume that (3.22) holds with a constant $c > 0$, that is,

$$\|x, z_2, \dots, z_n\| \|y, z_2, \dots, z_n\| \leq c \frac{\operatorname{Re}\left[\left(\overline{A} + \overline{a}\right)\langle x, y \mid z_2, \dots, z_n\rangle\right]}{[\operatorname{Re}(\overline{a}A)]^{1/2}} \tag{3.27}$$

provided $x,\ y,\ z_2, \dots, z_n, a$ and A satisfy (2.1). If we take $a = A = 1,\ y = x \neq 0$, then obviously (2.1) holds and from (3.27), we obtain

$$\|x, z_2, \dots, z_n\|^2 \leq 2c\|x, z_2, \dots, z_n\|^2 \tag{3.28}$$

for any linearly independent vectors $x, z_2, \dots, z_n \in X$, which implies $c \geq 1/2$. This completes the proof. □

When the constants involved are assumed to be positive, then we may state the following result.

Corollary 3.3. *Let $M \geq m > 0$ and assume that, for $x,\ y,\ z_2, \dots, z_n \in X$, one has*

$$\operatorname{Re}\langle My - x,\ x - my \mid z_2, \dots, z_n\rangle \geq 0 \tag{3.29}$$

or, equivalently,

$$\left\| x - \frac{M+m}{2}, z_2, \dots, z_n \right\| \leq \frac{1}{2}(M-m) \|y, z_2, \dots, z_n\|. \tag{3.30}$$

Then one has the following reverse of CBS inequality:

$$\begin{aligned} \|x,z_2,\dots,z_n\|\|y,z_2,\dots,z_n\| &\le \frac{1}{2}\frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x,y\mid z_2,\dots,z_n\rangle \\ &\le \frac{1}{2}\frac{M+m}{\sqrt{mM}}\left|\langle x,y\mid z_2,\dots,z_n\rangle\right|. \end{aligned} \tag{3.31}$$

The constant $1/2$ *is sharp in* (3.31).

Corollary 3.4. *With the assumptions of the Theorem 3.2, one has*

$$\begin{aligned} 0 &< \|x,z_2,\dots,z_n\|^2\|y,z_2,\dots,z_n\|^2 - \left|\langle x,y\mid z_2,\dots,z_n\rangle\right|^2 \\ &\le \frac{1}{4}\frac{|A-a|^2}{\operatorname{Re}(\overline{a}A)}\left|\langle x,y\mid z_2,\dots,z_n\rangle\right|^2. \end{aligned} \tag{3.32}$$

The constant $1/4$ *is best possible in* (3.32).

Corollary 3.5. *With the assumption of Corollary 3.3, one has*

$$\begin{aligned} 0 &\le \|x,z_2,\dots,z_n\|\|y,z_2,\dots,z_n\| - \left|\langle x,y\mid z_2,\dots,z_n\rangle\right| \\ &\le \|x,z_2,\dots,z_n\|\|y,z_2,\dots,z_n\| - \operatorname{Re}(\langle x,y\mid z_2,\dots,z_n\rangle) \\ &\le \frac{1}{2}\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}}\operatorname{Re}(\langle x,y\mid z_2,\dots,z_n\rangle) \le \frac{1}{2}\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}}\left|\langle x,y\mid z_2,\dots,z_n\rangle\right|, \end{aligned} \tag{3.33}$$

$$\begin{aligned} 0 &\le \|x,z_2,\dots,z_n\|^2\|y,z_2,\dots,z_n\| - \left|\langle x,y\mid z_2,\dots,z_n\rangle\right|^2 \\ &\le \|x,z_2,\dots,z_n\|^2\|y,z_2,\dots,z_n\|^2 - \left[\operatorname{Re}(\langle x,y\mid z_2,\dots,z_n\rangle)\right]^2 \\ &\le \frac{1}{4}\frac{(M-m)^2}{mM}\left[\operatorname{Re}(\langle x,y\mid z_2,\dots,z_n\rangle)\right]^2 \le \frac{1}{4}\frac{(M-m)^2}{mM}\left|\langle x,y\mid z_2,\dots,z_n\rangle\right|^2. \end{aligned} \tag{3.34}$$

The constant $1/2$ *in* (3.33) *and the constant* $1/4$ *in* (3.34) *are best possible.*

3.2. Reverse of the Triangle Inequality

Corollary 3.6. *Assume* $x,y,z_2,\dots,z_n$, m,M *are the same as in Corollary 3.3 and* $\operatorname{Re}\langle x,y\mid z_2,\dots,z_n\rangle \ge 0$. *Then one has the following reverse of the triangle inequality:*

$$\begin{aligned} 0 &\le \|x,z_2,\dots,z_n\| + \|y,z_2,\dots,z_n\| - \|x+y,z_2,\dots,z_n\| \\ &\le \frac{\sqrt{M}-\sqrt{m}}{(mM)^{1/4}}\sqrt{\operatorname{Re}\langle x,y\mid z_2,\dots,z_n\rangle}. \end{aligned} \tag{3.35}$$

Proof. It is easy to see that

$$
\begin{aligned}
0 &\le \left(\|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\|\right)^2 - \|x + y, z_2, \ldots, z_n\|^2 \\
&= 2\left[\|x, z_2, \ldots, z_n\|\|y, z_2, \ldots, z_n\| - \operatorname{Re}\left(\langle x, y \mid z_2, \ldots, z_n\rangle\right)\right]
\end{aligned}
\tag{3.36}
$$

for any $x,\ y,\ z_2, \ldots, z_n \in X$. If the assumption of Corollary 3.3 holds, then (3.33) is valid and, by (3.36), we deduce

$$
\begin{aligned}
0 &\le \left(\|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\|\right)^2 - \|x + y, z_2, \ldots, z_n\|^2 \\
&\le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{mM}} \operatorname{Re}\left(\langle x, y \mid z_2, \ldots, z_n\rangle\right)
\end{aligned}
\tag{3.37}
$$

which gives

$$
\begin{aligned}
&\left(\|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\|\right)^2 \\
&\quad \le \|x + y, z_2, \ldots, z_n\|^2 + \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{mM}} \operatorname{Re}\left(\langle x, y \mid z_2, \ldots, z_n\rangle\right).
\end{aligned}
\tag{3.38}
$$

Taking square root in (3.38), we have

$$
\begin{aligned}
&\|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\| \\
&\quad \le \sqrt{\|x + y, z_2, \ldots, z_n\|^2 + \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{mM}} \operatorname{Re}\left(\langle x, y \mid z_2, \ldots, z_n\rangle\right)} \\
&\quad \le \|x + y, z_2, \ldots, z_n\| + \frac{\sqrt{M} - \sqrt{m}}{(mM)^{1/4}} \sqrt{\operatorname{Re}\langle x, y \mid z_2, \ldots, z_n\rangle}.
\end{aligned}
\tag{3.39}
$$

From where we deduce the desire inequality (3.35). This completes the proof. □

Let $(X, \langle \cdot, \cdot \mid \cdot, \ldots\rangle)$ be a n-inner product space over real or complex number field K. If $(e_i)_{1<i<m}$ are linearly independent vector in the n-inner product space X, and, for given $z_2, \ldots, z_n \in X$, $\langle e_i, e_j \mid z_2, \ldots, z_n\rangle = \delta_{ij}$ for all $i, j \in \{1, 2, \ldots, m\}$ where δ_{ij} is the Kronecker

delta,then the following inequality is the Bessel's inequality for $z_2, \dots, z_n$ orthonormal family $(e_i)_{1<i<m}$ in n-inner product space $(X, \langle \cdot, \cdot \mid \cdot, \dots \rangle)$:

$$\sum_{i=1}^{m} |\langle x, e_i \mid z_2, \dots, z_n \rangle|^2 \leq \|x, z_2, \dots, z_n\|^2 \quad \text{for any } x \in X. \tag{3.40}$$

Theorem 3.7. *Let $x_1, x_2, \dots, x_m,\ z_2, \dots, z_n \in X$ and $\alpha_1, \dots, \alpha_m \in K$. then one has*

$$\begin{aligned}
&\left\| \sum_{i=1}^{m} \alpha_i x_i, z_2, \dots, z_n \right\|^2 \\
&\quad \leq \begin{cases} \max\limits_{1\leq i\leq m} |\alpha_i|^2 \sum\limits_{i=1}^{m} \|x_i, z_2, \dots, z_n\|^2, \\ \left(\sum\limits_{i=1}^{m} |\alpha_i|^{2a} \right)^{1/a} \left(\sum\limits_{i=1}^{m} \|x_i, z_2, \dots, z_n\|^{2b} \right)^{1/b}, & \text{where } a > 1,\ \dfrac{1}{a} + \dfrac{1}{b} = 1, \\ \sum\limits_{i=1}^{m} |\alpha_i|^2 \max\limits_{1\leq i\leq m} \|x_i, z_2, \dots, z_n\|^2 \end{cases} \\
&\quad + \begin{cases} \max\limits_{1\leq i\neq j\leq m} \{|\alpha_i \alpha_j|\} \sum\limits_{1\leq i\neq j\leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|, \\ \left[\left(\sum\limits_{i=1}^{m} |\alpha_i|^c \right)^2 - \left(\sum\limits_{i=1}^{m} |\alpha_i|^{2c} \right) \right]^{1/c} \\ \quad \times \left(\sum_{1\leq i\neq j\leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|^d \right)^{1/d}, & \text{where } c > 1,\ \dfrac{1}{c} + \dfrac{1}{d} = 1, \\ \left[\left(\sum\limits_{i=1}^{m} |\alpha_i| \right)^2 - \sum\limits_{i=1}^{m} |\alpha_i|^2 \right] \max\limits_{1\leq i\neq j\leq m} |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|. \end{cases}
\end{aligned} \tag{3.41}$$

Proof. We observe that

$$\begin{aligned}
\left\| \sum_{i=1}^{m} \alpha_i x_i, z_2, \dots, z_n \right\|^2 &= \left\langle \sum_{i=1}^{m} \alpha_i x_i, \sum_{j=1}^{m} \alpha_j x_j, \mid z_2, \dots, z_n \right\rangle \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \overline{\alpha_j} \langle x_i, x_j \mid z_2, \dots, z_n \rangle \\
&= \left| \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \overline{\alpha_j} \langle x_i, x_j \mid z_2, \dots, z_n \rangle \right| \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{m} |\alpha_i| |\overline{\alpha_j}| |\langle x_i, x_j \mid z_2, \dots, z_n \rangle| \\
&= \sum_{i=1}^{m} |\alpha_i|^2 \|x_i, z_2, \dots, z_n\|^2 + \sum_{1\leq i\neq j\leq m} |\alpha_i| |\alpha_j| |\langle x_i, x_j \mid z_2, \dots, z_n \rangle|.
\end{aligned} \tag{3.42}$$

Using Holder's inequality, we may write that

$$\sum_{i=1}^{m}|\alpha_i|^2\|x_i,z_2,\ldots,z_n\|^2$$
$$\leq\begin{cases}\max\limits_{1\leq i\leq m}|\alpha_i|^2\sum\limits_{i=1}^{m}\|x_i,z_2,\ldots,z_n\|^2,\\ \left(\sum\limits_{i=1}^{m}|\alpha_i|^{2a}\right)^{1/a}\left(\sum\limits_{i=1}^{m}\|x_i,z_2,\ldots,z_n\|^{2b}\right)^{1/b}, \quad \text{where } a>1,\ \dfrac{1}{a}+\dfrac{1}{b}=1,\\ \sum\limits_{i=1}^{m}|\alpha_i|^2\max\limits_{1\leq i\leq m}\|x_i,z_2,\ldots,z_n\|^2.\end{cases} \tag{3.43}$$

By Holder's inequality for double sum, we also have

$$\sum_{1\leq i\neq j\leq m}|\alpha_i||\alpha_j||\langle x_i,x_j\mid z_2,\ldots,z_n\rangle|$$
$$\leq\begin{cases}\max\limits_{1\leq i\neq j\leq m}|\alpha_i\alpha_j|\sum\limits_{1\leq i\neq j\leq m}|\langle x_i,x_j\mid z_2,\ldots,z_n\rangle|,\\ \left(\sum\limits_{1\leq i\neq j\leq m}|\alpha_i|^c|\alpha_j|^c\right)^{1/c}\\ \quad\times\left(\sum\limits_{1\leq i\neq j\leq m}|\langle x_i,x_j\mid z_2,\ldots,z_n\rangle|^d\right)^{1/d}, \quad \text{where } c>1,\ \dfrac{1}{c}+\dfrac{1}{d}=1,\\ \sum\limits_{1\leq i\neq j\leq m}|\alpha_i||\alpha_j|\max\limits_{1\leq i\neq j\leq m}|\langle x_i,x_j\mid z_2,\ldots,z_n\rangle|\end{cases}$$
$$=\begin{cases}\max\limits_{1\leq i\neq j\leq m}\{|\alpha_i\alpha_j|\}\sum\limits_{1\leq i\neq j\leq m}|\langle x_i,x_j\mid z_2,\ldots,z_n\rangle|,\\ \left[\left(\sum\limits_{i=1}^{m}|\alpha_i|^c\right)^2-\left(\sum\limits_{i=1}^{m}|\alpha_i|^{2c}\right)\right]^{1/c}\\ \quad\times\left(\sum\limits_{1\leq i\neq j\leq m}|\langle x_i,x_j\mid z_2,\ldots,z_n\rangle|^d\right)^{1/d}, \quad \text{where } c>1,\ \dfrac{1}{c}+\dfrac{1}{d}=1,\\ \left[\left(\sum\limits_{i=1}^{m}|\alpha_i|\right)^2-\sum\limits_{i=1}^{m}|\alpha_i|^2\right]\max\limits_{1\leq i\neq j\leq m}|\langle x_i,x_j\mid z_2,\ldots,z_n\rangle|.\end{cases} \tag{3.44}$$

Using (3.43) and (3.44) in (3.42), we may deduce the desired result. □

Corollary 3.8. *With the assumption in above Theorem 3.7, one has*

$$\begin{aligned}
&\left\|\sum_{i=1}^{m}\alpha_i x_i, z_2, \dots, z_n\right\|^2 \\
&\leq \sum_{i=1}^{n}|\alpha_i|^2\left\{\max_{1\leq i\leq n}\|x_i, z_2, \dots, z_n\|^2 + \frac{\left[\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^2 - \sum_{i=1}^{n}|\alpha_i|^4\right]^{1/2}}{\sum_{i=1}^{n}|\alpha_i|^2}\right. \\
&\qquad\left.\times\left(\sum_{1\leq i\neq j\leq n}\left|(x_i, x_j \mid z_2, \dots, z_n)\right|^2\right)^{1/2}\right\} \\
&\leq \sum_{i=1}^{m}|\alpha_i|^2\left\{\max_{1\leq i\leq m}\|x_i, z_2, \dots, z_n\|^2 + \left(\sum_{1\leq i\neq j\leq m}\left|\langle x_i, x_j \mid z_2, \dots, z_n\rangle\right|^2\right)^{1/2}\right\}.
\end{aligned} \tag{3.45}$$

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for $c = d = 2$. The second inequality in (3.45) *follows by the fact that*

$$\left[\left(\sum_{i=1}^{m}|\alpha_i|^2\right)^2 - \sum_{i=1}^{m}|\alpha_i|^4\right]^{1/2} \leq \sum_{i=1}^{m}|\alpha_i|^2. \tag{3.46}$$

By applying the following Cauchy-Bunyakovsky-Schwarz inequality:

$$\left(\sum_{i=1}^{m}a_i\right)^2 \leq m\sum_{i=1}^{m}a_i^2, \quad a_i \in R_+,\ 1 \leq i \leq m. \tag{3.47}$$

one may write that

$$\left(\sum_{i=1}^{m}|\alpha_i|^c\right)^2 - \sum_{i=1}^{m}|\alpha_i|^{2c} \leq (m-1)\sum_{i=1}^{m}|\alpha_i|^{2c}, \quad (m \geq 1), \tag{3.48}$$

$$\left(\sum_{i=1}^{m}|\alpha_i|\right)^2 - \sum_{i=1}^{m}|\alpha_i|^2 \leq (m-1)\sum_{i=1}^{m}|\alpha_i|^2, \quad (m \geq 1). \tag{3.49}$$

It is obvious that

$$\max_{1\leq i\neq j\leq m}\left\{\left|\alpha_i\alpha_j\right|\right\} \leq \max_{1\leq i\leq m}|\alpha_i|^2. \tag{3.50}$$

Corollary 3.9. *With the assumption in above Theorem 3.7, one has*

$$\left\|\sum_{i=1}^{m}\alpha_i x_i, z_2,\dots,z_n\right\|^2 \leq \begin{cases} \max\limits_{1\le i\le m}|\alpha_i|^2 \sum\limits_{i=1}^{m}\|x_i, z_2,\dots,z_n\|^2, \\ \left(\sum\limits_{i=1}^{m}|\alpha_i|^{2a}\right)^{1/a}\left(\sum\limits_{i=1}^{m}\|x_i, z_2,\dots,z_n\|^{2b}\right)^{1/b}, \quad \text{where } a>1,\ \dfrac{1}{a}+\dfrac{1}{b}=1, \\ \sum\limits_{i=1}^{m}|\alpha_i|^2 \max\limits_{1\le i\le m}\|x_i, z_2,\dots,z_n\|^2 \end{cases}$$
$$+\begin{cases} \max\limits_{1\le i\le m}|\alpha_i|^2 \sum\limits_{1\le i\le m}|\langle x_i, x_i \mid z_2,\dots,z_n\rangle|, \\ (m-1)^{1/c}\left(\sum\limits_{i=1}^{m}|\alpha_i|^{2c}\right)^{1/c}\left(\sum\limits_{1\le i\neq j\le m}\left|\langle x_i, x_j \mid z_2,\dots,z_n\rangle\right|^{d}\right)^{1/d}, \quad \text{where } c>1,\ \dfrac{1}{c}+\dfrac{1}{d}=1, \\ (m-1)\sum\limits_{i=1}^{m}|\alpha_i|^2 \max\limits_{1\le i\le m}\|x_i, z_2,\dots,z_n\|^2. \end{cases} \tag{3.51}$$

The proof is obvious by Theorem 3.7 on applying (3.48)–(3.50).

Theorem 3.10. *Let $x, y_1,\dots,y_m, z_2,\dots,z_n$ be vectors of an n-inner product space $(X, \langle\cdot,\cdot\mid\cdot,\dots\rangle)$ and $\alpha_1,\alpha_2,\dots,\alpha_n \in K(K=R,C)$. Then*

$$\left|\sum_{i=1}^{m}\alpha_i\langle x, y_i \mid z_2,\dots,z_n\rangle\right|^2 \leq \|x, z_2,\dots,z_n\|^2 \begin{cases} \max\limits_{1\le i\le m}|\alpha_i|^2 \sum\limits_{i=1}^{m}\left\|y_i, z_2,\dots,z_n\right\|^2, \\ \left(\sum\limits_{i=1}^{m}|\alpha_i|^{2a}\right)^{1/a}\left(\sum\limits_{i=1}^{m}\left\|y_i, z_2,\dots,z_n\right\|^{2b}\right)^{1/b}, \quad \text{where } a>1,\ \dfrac{1}{a}+\dfrac{1}{b}=1, \\ \sum\limits_{i=1}^{m}|\alpha_i|^2 \max\limits_{1\le i\le m}\left\|y_i, z_2,\dots,z_n\right\|^2 \end{cases}$$
$$+\|x, z_2,\dots,z_n\|^2 \begin{cases} \max\limits_{1\le i\neq j\le m}\left\{\left|\alpha_i\alpha_j\right|\right\} \sum\limits_{1\le i\neq j\le m}\left|\langle y_i, y_j \mid z_2,\dots,z_n\rangle\right|, \\ \left[\left(\sum\limits_{i=1}^{m}|\alpha_i|^{c}\right)^2-\left(\sum\limits_{i=1}^{m}|\alpha_i|^{2c}\right)\right]^{1/c}\left(\sum\limits_{1\le i\neq j\le m}\left|\langle y_i, y_j \mid z_2,\dots,z_n\rangle\right|^{d}\right)^{1/d}, \\ \left[\left(\sum\limits_{i=1}^{m}|\alpha_i|\right)^2-\left(\sum\limits_{i=1}^{m}|\alpha_i|^{2}\right)\right]\sum\limits_{i=1}^{m}|\alpha_i|^2 \max\limits_{1\le i\le m}\|x_i, z_2,\dots,z_n\|^2. \end{cases} \tag{3.52}$$

Proof. Since

$$\sum_{i=1}^{m} \alpha_i \langle x, y_i \mid z_2, \ldots, z_n \rangle = \left\langle x, \sum \overline{\alpha_i} y_i \mid z_2, \ldots, z_n \right\rangle \tag{3.53}$$

and by Schwarz's inequality in n-inner product spaces,

$$\left| \sum_{i=1}^{m} \alpha_i \langle x, y_i \mid z_2, \ldots, z_n \rangle \right|^2 \le \|x, z_2, \ldots, z_n\|^2 \left\| \sum_{i=1}^{m} \overline{\alpha_i} y_i, z_2, \ldots, z_n \right\|^2. \tag{3.54}$$

Using $\alpha_i = \overline{\alpha_i}$, $z_i = y_i$ $(i = 1, 2, \ldots, n)$ in the Theorem 3.7, we get the desired inequality (3.52). □

Corollary 3.11. *With the assumptions in Theorem 3.10, the following holds:*

$$\begin{aligned}
&\left| \sum_{i=1}^{m} \alpha_i \langle x, y_i \mid z_2, \ldots, z_n \rangle \right|^2 \\
&\quad \le \|x, z_2, \ldots, z_n\|^2 \\
&\qquad \times \begin{cases}
\displaystyle\sum_{i=1}^{m} |\alpha_i|^2 \left\{ \max_{1\le i\le m} \|y_i, z_2, \ldots, z_n\|^2 + \left(\sum_{1\le i\ne j\le m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle|^2 \right)^{1/2} \right\}, \\
\displaystyle\max_{1\le i\le m} |\alpha_i|^2 \left\{ \sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^2 + \sum_{1\le i\ne j\le m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle| \right\}, \\
\displaystyle\left(\sum_{i=1}^{m} |\alpha_i|^{2p} \right)^{1/p} \\
\displaystyle\quad \times \left\{ \left(\sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^{2q} \right)^{1/q} + (m-1) \left(\sum_{1\le i\ne j\le m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle|^q \right)^{1/q} \right\}, \\
\displaystyle\sum_{i=1}^{m} |\alpha_i|^2 \left\{ \max_{1\le i\le m} \|y_i, z_2, \ldots, z_n\|^2 + (m-1) \max_{1\le i\ne j\le m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle| \right\}.
\end{cases}
\end{aligned} \tag{3.55}$$

3.3. Some Boas-Bellman Type Inequalities in n-Inner Product Spaces

If we put $\alpha_i = \overline{\langle x, y_i \mid z_2, \ldots, z_n \rangle}$ $(i = 1, 2, \ldots, m)$, in the first inequality of (3.55)

$$\begin{aligned}\left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2\right)^2 &\\ \leq \|x, z_2, \ldots, z_n\|^2 \sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2 &\\ \times \left\{\max_{1\leq i\leq m}\left\|y_i, z_2, \ldots, z_n\right\|^2 + \left(\sum_{1\leq i\neq j\leq m} \left|\langle y_i, y_j \mid z_2, \ldots, z_n\rangle\right|^2\right)^{1/2}\right\}&\end{aligned} \tag{3.56}$$

which is equivalent to the following Boass-Bellman type inequality for n-inner products:

$$\begin{aligned}\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2 &\\ \leq \|x, z_2, \ldots, z_n\|^2 \left\{\max_{1\leq i\leq m}\left\|y_i, z_2, \ldots, z_n\right\|^2 + \left(\sum_{1\leq i\neq j\leq m} \left|\langle y_i, y_j \mid z_2, \ldots, z_n\rangle\right|^2\right)^{1/2}\right\}.&\end{aligned} \tag{3.57}$$

Now, if we take $\alpha_i = \overline{\langle x, y_i \mid z_2, \ldots, z_n \rangle}$ $(i = 1, 2, \ldots, m)$, in second inequality of (3.55), we have

$$\begin{aligned}\left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2\right)^2 &\\ \leq \|x, z_2, \ldots, z_n\|^2 \max_{1\leq i\leq m}\left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2 &\\ \times \left\{\sum_{i=1}^{m} \left\|y_i, z_2, \ldots, z_n\right\|^2 + \sum_{1\leq i\neq j\leq m} \left|\langle y_i, y_j \mid z_2, \ldots, z_n\rangle\right|\right\}.&\end{aligned} \tag{3.58}$$

By taking the square root in

$$\begin{aligned}\left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2\right) &\\ \leq \|x, z_2, \ldots, z_n\| \max_{1\leq i\leq m}\left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right| &\\ \times \left\{\sum_{i=1}^{m} \left\|y_i, z_2, \ldots, z_n\right\|^2 + \sum_{1\leq i\neq j\leq m} \left|\langle y_i, y_j \mid z_2, \ldots, z_n\rangle\right|\right\}^{1/2}&\end{aligned} \tag{3.59}$$

for any $x, y_1, \ldots, y_m, z_2, \ldots, z_n$ be vectors of an n-inner product space $(X, \langle \cdot, \cdot \mid \cdot, \ldots \rangle)$. If we assume that $(e_i)_{1\le i\le m}$ is an orthonormal family in X with respect to the vector $z_2, \ldots, z_n$, $(e_i, e_j \mid z_2, \ldots, z_n) = \delta_{ij}$ for all $i, j \in \{1, \ldots, m\}$ then by (3.57) we deduce Bessel's inequality $\sum_{i=1}^m |\langle x, e_i \mid z_2, \ldots, z_n\rangle|^2 \le \|x, z_2, \ldots, z_n\|^2$, and (3.59) implies

$$\sum_{i=1}^{m} |\langle x, e_i \mid z_2, \ldots, z_n\rangle|^2 \le \sqrt{m}\|x, z_2, \ldots, z_n\| \max_{1\le i\le m} |\langle x, e_i \mid z_2, \ldots, z_n\rangle|^2. \tag{3.60}$$

For the third inequality in (3.55) $\alpha_i = \overline{\langle x, y_i \mid z_2, \ldots, z_n\rangle}$ $(i = 1, 2, \ldots, m)$, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2\right)^2 \\
&\quad\le \|x, z_2, \ldots, z_n\|^2 \left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^{2p}\right)^{1/p} \\
&\qquad\times \left\{\left(\sum_{i=1}^{m} \left\|y_i, z_2, \ldots, z_n\right\|^{2q}\right)^{1/q} + (m-1)\left(\sum_{1\le i\ne j\le m} \left|\langle y_i, y_j \mid z_2, \ldots, z_n\rangle\right|^{q}\right)^{1/q}\right\}
\end{aligned} \tag{3.61}$$

for $p > 1$, $(1/p) + (1/q) = 1$. Taking the square root in this inequality, we get

$$\begin{aligned}
&\left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2\right) \\
&\quad\le \|x, z_2, \ldots, z_n\| \left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^{2p}\right)^{1/2p} \\
&\qquad\times \left\{\left(\sum_{i=1}^{m} \left\|y_i, z_2, \ldots, z_n\right\|^{2q}\right)^{1/q} + (m-1)\left(\sum_{1\le i\ne j\le m} \left|\langle y_i, y_j \mid z_2, \ldots, z_n\rangle\right|^{q}\right)^{1/q}\right\}^{1/2}.
\end{aligned} \tag{3.62}$$

For any $x, y_1, \ldots, y_m, z_2, \ldots, z_n \in X$, and $p > 1$, with $(1/p) + (1/q) = 1$, then the above inequality (3.62) becomes, for an orthonormal family with respect to the vector $z_2, \ldots, z_n$,

$$\left(\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2\right) \le m^{1/q}\|x, z_2, \ldots, z_n\| \left(\sum_{i=1}^{m} \left|\langle x, e_i \mid z_2, \ldots, z_n\rangle\right|^{2p}\right)^{1/2p}. \tag{3.63}$$

We take $\alpha_i = \overline{\langle x, y_i \mid z_2, \ldots, z_n \rangle}$ $(i = 1, 2, \ldots, m)$, in the last inequality of (3.55)

$$\begin{aligned}
&\left(\sum_{i=1}^{m} \left| \langle x, y_i \mid z_2, \ldots, z_n \rangle \right|^2 \right)^2 \\
&\quad \leq \| x, z_2, \ldots, z_n \|^2 \sum_{i=1}^{m} \left| \langle x, y_i \mid z_2, \ldots, z_n \rangle \right|^2 \\
&\qquad \times \left\{ \max_{1 \leq i \leq m} \left\| y_i, z_2, \ldots, z_n \right\|^2 + (m-1) \max_{1 \leq i \neq j \leq m} \left| \langle y_i, y_j \mid z_2, \ldots, z_n \rangle \right| \right\}
\end{aligned} \tag{3.64}$$

which implies

$$\begin{aligned}
&\left(\sum_{i=1}^{m} \left| \langle x, y_i \mid z_2, \ldots, z_n \rangle \right|^2 \right) \\
&\quad \leq \| x, z_2, \ldots, z_n \|^2 \left\{ \max_{1 < i < m} \left\| y_i, z_2, \ldots, z_n \right\|^2 + (m-1) \max_{1 < i \neq j < m} \left| \langle y_i, y_j \mid z_2, \ldots, z_n \rangle \right| \right\}
\end{aligned} \tag{3.65}$$

for any $x, y_1, \ldots, y_m,\ z_2, \ldots, z_n \in X$.

4. Applications for Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra Σ of subsets of Ω and a countably additive measure μ on Σ with values in $R \cup \{\infty\}$. Denote by $L^2\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are n-ρ-integrable on Ω, that is, $\int_\Omega \rho(s)|f(s)|^n d\mu(s) < \infty$ where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following n-inner product on $L^2\rho(\Omega)$:

$$\begin{aligned}
\langle f, g \mid h_2, \ldots, h_n \rangle_\rho = \frac{1}{n!} &\underbrace{\int_\Omega \int_\Omega \cdots \int_\Omega}_{n\text{-times}} \rho(s_1)\rho(s_2) \cdots \rho(s_n) \\
&\times \begin{vmatrix} f(s_1) & f(s_2) \ldots & f(s_n) \\ h_2(s_1) & h_2(s_2) \ldots & h_2(s_n) \\ \vdots & \ldots & \vdots \\ h_n(s_1) & h_n(s_2) \ldots & h_n(s_n) \end{vmatrix} \\
&\times \begin{vmatrix} g(s_1) & g(s_2) \ldots & g(s_n) \\ h_2(s_1) & h_2(s_2) \ldots & h_2(s_n) \\ \vdots & \ldots & \vdots \\ h_n(s_1) & h_n(s_2) \ldots & h_n(s_n) \end{vmatrix} d\mu(s_1) d\mu(s_2) \cdots d\mu(s_n),
\end{aligned} \tag{4.1}$$

where by

$$\begin{vmatrix} f(s_1) & f(s_2)\ldots & f(s_n) \\ h_2(s_1) & h_2(s_2)\ldots & h_2(s_n) \\ \vdots & \cdots & \vdots \\ h_n(s_1) & h_n(s_2)\ldots & h_n(s_n) \end{vmatrix} \tag{4.2}$$

we denote the determinant of matrix

$$\begin{bmatrix} f(s_1) & f(s_2) & \cdots & \cdots & f(s_n) \\ h_2(s_1) & h_2(s_2) & \cdots & \cdots & h_2(s_n) \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h_n(s_1) & h_n(s_2) & \cdots & \cdots & h_n(s_n) \end{bmatrix}. \tag{4.3}$$

Generating the n-norm on $L^2\rho(\Omega)$ is expressed by

$$\begin{aligned} &\|f, h_2, \ldots, h_n\|_p \\ &= \left(\frac{1}{n!} \underbrace{\int_\Omega \int_\Omega \cdots \int_\Omega}_{n\text{-times}} \rho(s_1)\rho(s_2)\cdots\rho(s_n) \right. \\ &\quad \left. \times \begin{vmatrix} f(s_1) & f(s_2) & \cdots & \cdots & f(s_n) \\ h_2(s_1) & h_2(s_2) & \cdots & \cdots & h_2(s_n) \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ h_n(s_1) & h_n(s_2) & \cdots & \cdots & h_n(s_n) \end{vmatrix}^2 d\mu(s_1)d\mu(s_2)\cdots d\mu(s_n) \right)^{1/2}. \end{aligned} \tag{4.4}$$

A simple calculation with integrals shows that

$$\langle f, g \mid h_2, \ldots, h_n \rangle_\rho = \begin{vmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h_2 d\mu & \cdots & \cdots & \int_\Omega \rho f h_n d\mu \\ \int_\Omega \rho f h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots & \cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \int_\Omega \rho g h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots & \cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix}, \tag{4.5}$$

$$\|f, h_2, \ldots, h_n\|_p = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h_2 d\mu & \cdots & \cdots & \int_\Omega \rho f h_n d\mu \\ \int_\Omega \rho f h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots & \cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \int_\Omega \rho g h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots & \cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix}, \tag{4.6}$$

where, for simplicity, instead of $\int_\Omega \rho(s)f(s)g(s)d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$.

Proposition 4.1. *Let* $f, g_1, \ldots, g_m, h_2, \ldots, h_n \in L^2\rho(\Omega)$, *where* $\rho : \Omega \to [0, \infty)$ *a measure function is on* Ω. *Then one has*

$$\sum_{i=1}^{m}\begin{vmatrix} \int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h_2 d\mu & \cdots\cdots & \int_\Omega \rho f h_n d\mu \\ \int_\Omega \rho g_i h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho g_i h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix}^2$$
$$\leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h_2 d\mu & \cdots\cdots & \int_\Omega \rho f h_n d\mu \\ \int_\Omega \rho f h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho f h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix}$$
$$\times \left\{ \max_{1\leq i\leq m}\begin{vmatrix} \int_\Omega \rho {g_i}^2 d\mu & \int_\Omega \rho g_i h_2 d\mu & \cdots\cdots & \int_\Omega \rho g_i h_n d\mu \\ \int_\Omega \rho g_i h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho g_i h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix} + \left(\sum_{1\leq i\neq j\leq m}\begin{vmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h_2 d\mu & \cdots\cdots & \int_\Omega \rho g_j h_n d\mu \\ \int_\Omega \rho g_i h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho g_i h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix}^2 \right)^{1/2} \right\}. \tag{4.7}$$

Proof. Applying the n-inner product and n-norm defined in (4.1) and (4.4), and using (4.5) and (4.6) in (3.57)

$$\sum_{i=1}^{m} \left|\langle x, y_i \mid z_2, \ldots, z_n\rangle\right|^2$$
$$\leq \|x, z_2, \ldots, z_n\|^2 \tag{4.8}$$
$$\times \left\{ \max_{1\leq i\leq m}\left\|y_i, z_2, \ldots, z_n\right\|^2 + \left(\sum_{1\leq i\neq j\leq m} \left|\langle y_i, y_j \mid z_2, \ldots, z_n\rangle\right|^2 \right)^{1/2} \right\},$$

we get the required proof of the proposition. □

Proposition 4.2. *Let $f, g_1, \dots, g_m, h_2, \dots, h_n \in L^2\rho(\Omega)$, where $\rho{:}\Omega \to [0,\infty)$ a measure function is on Ω. Then one has*

$$\begin{aligned}
&\sum_{i=1}^{m}\begin{vmatrix} \int_\Omega \rho f g_i d\mu & \int_\Omega \rho f h_2 d\mu & \cdots\cdots & \int_\Omega \rho f h_n d\mu \\ \int_\Omega \rho g_i h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho g_i h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix}^2 \\
&\leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h_2 d\mu & \cdots\cdots & \int_\Omega \rho f h_n d\mu \\ \int_\Omega \rho f h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho f h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix} \\
&\times \left\{ \max_{1\leq i\leq m} \begin{vmatrix} \int_\Omega \rho {g_i}^2 d\mu & \int_\Omega \rho g_i h_2 d\mu & \cdots\cdots & \int_\Omega \rho g_i h_n d\mu \\ \int_\Omega \rho g_i h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho g_i h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix} \right. \\
&\quad \left. +(m-1)\max_{1\leq i\neq j\leq m} \left| \begin{vmatrix} \int_\Omega \rho g_j g_i d\mu & \int_\Omega \rho g_j h_2 d\mu & \cdots\cdots & \int_\Omega \rho g_j h_n d\mu \\ \int_\Omega \rho g_i h_2 d\mu & \int_\Omega \rho {h_2}^2 d\mu & \cdots\cdots & \int_\Omega \rho h_2 h_n d\mu \\ \vdots & \vdots & \cdots\cdots & \vdots \\ \int_\Omega \rho g_i h_n d\mu & \int_\Omega \rho h_2 h_n d\mu & \cdots\cdots & \int_\Omega \rho {h_n}^2 d\mu \end{vmatrix} \right| \right\}.
\end{aligned} \tag{4.9}$$

Proof. By (3.65),

$$\begin{aligned}
&\left(\sum \left|\langle x, y_i \mid z_2, \dots, z_n\rangle\right|^2\right) \\
&\quad\leq \|x, z_2, \dots, z_n\|^2 \\
&\qquad\times \left\{ \max_{1\leq i\leq m} \|y_i, z_2, \dots, z_n\|^2 + (m-1) \max_{1\leq i\neq j\leq m} \left|\langle y_i, y_j \mid z_2, \dots, z_n\rangle\right| \right\},
\end{aligned} \tag{4.10}$$

and using (4.1) and (4.4), we get the proof of the proposition. □

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References

[1] S. Gähler, "Unter Suchungen Uber Veralla gemeinerte m-etrische Raume I," *Mathematische Nachrichten*, vol. 40, pp. 165–189, 1969.
[2] R. W. Freese and Y. J. Cho, *Geometry of Linear 2-Normed Spaces*, Nova Science, Hauppauge, NY, USA, 2001.
[3] Y. J. Cho, P. C. S. Lin, S. S. Kim, and A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science, Huntington, NY, USA, 2001.
[4] A. Misiak, "n-inner product spaces," *Mathematische Nachrichten*, vol. 140, pp. 299–319, 1989.
[5] H. Gunawan and M. Mashadi, "On n-normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 10, pp. 631–639, 2001.
[6] H. Gunawan, "On n-inner products, n-norms, and the Cauchy-Schwarz inequality," *Scientiae Mathematicae Japonicae*, vol. 55, no. 1, pp. 53–60, 2002.
[7] Y. J. Cho, M. Matić, and J. Pečarić, "Inequalities of Hlawka's type in n-inner product spaces," *Korean Mathematical Society*, vol. 17, no. 4, pp. 583–592, 2002.
[8] S. S. Dragomir, "A counterpart of Schwarz's inequality in inner product spaces," *RGMIA Research Report Collection*, vol. 6, supplement 18, 2003.
[9] S. S. Dragomir, "A generalization of the Cassels and Greub-Reinboldt inequalities in inner product spaces," *Nonlinear Analysis Forum*, vol. 8, no. 2, pp. 169–178, 2003.
[10] S. S. Dragomir, "New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces," *The Australian Journal of Mathematical Analysis and Applications*, vol. 1, no. 1, pp. 1–18, 2004.
[11] S. S. Dragomir, "Refinements of Buzano's and Kurepa's inequalities in inner product spaces," *Facta Universitatis. Series: Mathematics and Informatics*, no. 20, pp. 65–73, 2005.
[12] S. S. Dragomir, "Reverses of Schwarz, triangle and Bessel inequalities in inner product spaces," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 3, article 76, 18 pages, 2004.
[13] S. S. Dragomir, "Some companions of the Grüss inequality in inner product spaces," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 5, article 87, 10 pages, 2003.