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Research Article

Approach Merotopological Spaces and their Completion

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This paper introduces the concept of an approach merotopological space and studies its category-theoretic properties. Various topological categories are shown to be embedded into the category whose objects are approach merotopological spaces. The order structure of the family of all approach merotopologies on a nonempty set is discussed. Employing the theory of bunches, bunch completion of an approach merotopological space is constructed. The present study is a unified look at the completion of metric spaces, approach spaces, nearness spaces, merotopological spaces, and approach merotopological spaces.

1. Introduction

Some of the applications of nearness-like structures within topology are unification, extensions, homology, and connectedness. The categories of R_0-topological spaces, uniform spaces [1, 2], proximity spaces [2, 3], and contiguity spaces [4, 5] are embedded into the category of nearness spaces. The study of proximity, contiguity, and merotopic spaces in the more generalized setting of L-fuzzy theory can be seen in [6–13]. In [14], the notion of an approach space was introduced via different equivalent set of axioms to measure the degree of nearness between a set and a point. While developing the theory of approach spaces, Lowen et al. many a time employed tools from nearness-like structures. The notion of "distance" in approach spaces is closely related to the notion of nearness; further proximity and nearness concepts arise naturally in the context of approach spaces as can be seen in [15–18]. Hence it became mandatory looking into the nearness-like concepts in approach theory, more clearly. With the same spirit, Lowen and Lee [19] made an attempt to measure how near a collection of sets is and, in the process, axiomatized the two equivalent concepts: approach merotopic and approach seminearness structures, respectively, to measure the degree of smallness and nearness of an arbitrary collection of sets, and therefore generalized approach spaces in a sense. In 2004, Bentley and Herrlich [20] gave the idea of merotopological spaces as a supercategory of many of the above mentioned categories. They also constructed the functorial completion of merotopological spaces employing the theory of bunches in merotopological spaces. In [21], we axiomatized the notion of approach nearness by adding to the axioms of an approach merotopy the axiom relating a collection of sets and the closure induced by the respective approach merotopy; and we analogously obtained cluster completion of an approach nearness space.

Prerequisites for the paper are collected in Section 2. In Section 3, we axiomatize approach merotopological spaces. The category **AMT** of approach merotopological spaces and their respective morphisms is shown to be a topological construct and a supercategory of some of the known topological categories, including the category of topological spaces and continuous maps. Order structure of the family of all approach merotopologies on X is also discussed. In Section 4, bunch completion of an approach merotopological space is constructed, employing the theory of bunches. The concept of regularity in an approach merotopic space is introduced to obtain a relationship between cluster and bunch completion of an approach nearness space. Indeed, it is shown that cluster completion is a retract of the bunch completion of a regular approach nearness space (X, ν).

2. Preliminaries and Basic Results

Let X be a nonempty ordinary set. The power set of X is denoted by $\mathcal{P}(X)$ and the family of all subsets of $\mathcal{P}(X)$ is denoted by $\mathcal{P}^2(X)$. We denote by $\aleph_0$ the first infinite cardinal number, by $|A|$ the cardinality of A where $A \subseteq X$, and by J an arbitrary index set. For $\mathcal{A}, \mathcal{B}$ subsets of $\mathcal{P}(X)$, $\mathcal{A} \vee \mathcal{B} \equiv \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$; $\mathcal{A}$ *corefines* $\mathcal{B}$ (written as $\mathcal{A} \prec \mathcal{B}$) if and only if for all $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $B \subseteq A$. For $\mathcal{A} \subseteq \mathcal{P}(X)$, $\operatorname{stack}(\mathcal{A}) = \{A \subseteq X : B \subseteq A$ for some $B \in \mathcal{A}\}$ and $\sec \mathcal{A} = \{B \subseteq X : A \cap B \neq \emptyset$, for all $A \in \mathcal{A}\} = \{B \subseteq X : X - B \notin \operatorname{stack}(\mathcal{A})\}$. Observe that $\sec^2 \mathcal{A} = \operatorname{stack}(\mathcal{A})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$. A *grill* on X is a subset $\mathcal{G}$ of $\mathcal{P}(X)$ satisfying $\emptyset \notin \mathcal{G}$; if $A \in \mathcal{G}$ and $A \subseteq B$, then $B \in \mathcal{G}$; and if $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$. For basic definitions and results of merotopic spaces and nearness spaces, we refer to [1].

Definition 2.1 (see [20]). A *merotopological space* is the triple (X, ξ, cl), where ξ is a merotopy and cl is a Kuratowski closure operator on X such that $\{\mathrm{cl}(A) : A \in \mathcal{A}\} \in \xi \Rightarrow \mathcal{A} \in \xi$, for all $\mathcal{A} \in \mathcal{P}^2(X)$.

Definition 2.2 (see [19, 21]). A function $\nu : \mathcal{P}^2(X) \to [0, \infty]$ is called an *approach merotopy* on X if for any $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$ the following conditions are satisfied:

(AM1) $\mathcal{A} \prec \mathcal{B} \Rightarrow \nu(\mathcal{A}) \leq \nu(\mathcal{B})$,

(AM2) $\bigcap \mathcal{A} \neq \emptyset \Rightarrow \nu(\mathcal{A}) = 0$,

(AM3) $\emptyset \in \mathcal{A} \Rightarrow \nu(\mathcal{A}) = \infty$,

(AM4) $\nu(\mathcal{A} \vee \mathcal{B}) \geq \nu(\mathcal{A}) \wedge \nu(\mathcal{B})$.

The pair (X, ν) is called an *approach merotopic space*. For an approach merotopic space (X, ν), we define $\mathrm{cl}_\nu(A) = \{x \in X : \nu(\{\{x\}, A\}) = 0\}$, for all $A \subseteq X$. Then cl_ν is a Čech closure operator on X.

An approach merotopy ν on X is called an *approach nearness* on X [21] if the following condition is satisfied:

(AN5) $\nu(\{\mathrm{cl}_\nu(A) : A \in \mathcal{A}\}) \geq \nu(\mathcal{A})$.

In this case, cl_ν is a Kuratowski closure operator on X. Denote $\mathrm{cl}(\mathcal{A}) = \{\mathrm{cl}(A) : A \in \mathcal{A}\}$.

Definition 2.3 (see [21]). Let $\mathcal{A} \in \mathcal{P}^2(X)$ and ν be an approach merotopy on X. Then we say that $\mathcal{A}$ is

(i) *near* in ν if $\nu(\mathcal{A}) = 0$,

(ii) *ν-clan* if $\mathcal{A}$ is a near grill,

(iii) *ν-closed* if $\nu(\{\{A\} \cup \mathcal{B}\}) = 0$, for all $\mathcal{B} \subseteq \mathcal{A} \Rightarrow A \in \mathcal{A}$,

(iv) *ν-cluster* if $\mathcal{A}$ is a ν-closed ν-clan.

For any approach merotopic spaces (X, ν_X) and (Y, ν_Y), a map $f : X \to Y$ is called a *contraction* if $\nu_Y(f(\mathcal{A})) \leq \nu_X(\mathcal{A})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$, or equivalently $\nu_X(f^{-1}(\mathcal{B})) \geq \nu_Y(\mathcal{B})$, for all $\mathcal{B} \in \mathcal{P}^2(Y)$. For any approach merotopies ν_1 and ν_2 on X, $\nu_2 \leq \nu_1$ (ν_1 is *finer* than ν_2, or ν_2 is *coarser* than ν_1) if the identity mapping $1_X : (X, \nu_1) \to (X, \nu_2)$ is a contraction (see [19]). For standard definitions in the theory of categories we refer to [22], for approach spaces we refer to [14], and for lattices see [23].

3. Approach Merotopological Spaces

In this section, we introduce approach merotopological spaces and establish some category-theoretic results for them. Lattice structure of the family of all approach merotopologies on X is also discussed.

Definition 3.1. An *approach merotopological space* is a triple (X, ν, cl), where ν is an approach merotopy on X and cl is a Kuratowski closure operator on X such that the following condition is satisfied:

(AM5) $\nu(\mathrm{cl}(\mathcal{A})) \geq \nu(\mathcal{A})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$.

We call ν to be an *approach merotopology with respect to the closure operator* cl *on* X.

Example 3.2. (i) Let (X, ν, cl) be an approach merotopological space and let $\mathfrak{G}(X)$ be the family of all grills on X. Define $\nu_g : \mathcal{P}^2(X) \to [0, \infty]$ as follows:

$$\nu_g(\mathcal{A}) = \inf\{\nu(\mathcal{G}) : \mathcal{A} \prec \mathcal{G},\ \mathcal{G} \in \mathfrak{G}(X)\}, \quad \forall \mathcal{A} \in \mathcal{P}^2(X). \tag{3.1}$$

Then ν_g is an approach merotopology with respect to cl on X (see [19, 24]).

(ii) Let (X, cl) be a topological space. Then (X, ν_g, cl) is an approach merotopological space, where ν_g is defined as in the above example in which $\nu : \mathfrak{G}(X) \to [0, \infty]$ can be defined in the following ways for $\mathcal{G} \in \mathfrak{G}(X)$.

(a) $\nu(\mathcal{G}) = 0$, if $\bigcap \mathrm{cl}(\mathcal{G}) \neq \emptyset$; and $\nu(\mathcal{G}) = \sup\{|\mathrm{cl}(G)| : G \in \mathcal{G} \text{ and } |\mathrm{cl}(G)| < \aleph_0\}$, otherwise.

(b) $\nu(\mathcal{G}) = 0$, if $\bigcap \mathrm{cl}(\mathcal{G}) \neq \emptyset$; and $\nu(\mathcal{G}) = \sup_{G \in \mathcal{G}} |\mathrm{stack}(\mathrm{cl}(G))|$, otherwise.

(c) $\nu(\mathcal{G}) = 0$, if $\bigcap \mathrm{cl}(\mathcal{G}) \neq \emptyset$; and $\nu(\mathcal{G}) = |\,\mathrm{cl}(\mathcal{G})|$, otherwise.

(d) $\nu(\mathcal{G}) = 0$, if $\bigcap \mathrm{cl}(\mathcal{G}) \neq \emptyset$; and $\nu(\mathcal{G}) = \inf_{G \in \sec \mathrm{cl}(\mathcal{G})} |G|$, otherwise.

(e) $\nu(\mathcal{G}) = 0$, if $\bigcap \mathrm{cl}(\mathcal{B}) \neq \emptyset$ for every finite subset $\mathcal{B}$ of $\mathcal{G}$; and $\nu(\mathcal{G}) = \inf\{|G| : G \in \sec \mathrm{cl}(\mathcal{G})\}$, otherwise.

(f) $\nu(\mathcal{G}) = 0$, if $\bigcap \mathrm{cl}(\mathcal{G}) \neq \emptyset$ or each element of $\mathcal{G}$ is infinite; and $\nu(\mathcal{G}) = \sup\{|G| : G \in \mathcal{G}$ and $|G| < \aleph_0\}$, otherwise. (In this case, cl is a T_1-closure operator on X.)

Observe that if (X, cl) is a symmetrical topological space (i.e., $x \in \mathrm{cl}(\{y\}) \Rightarrow y \in \mathrm{cl}(\{x\})$, for all $x, y \in X$), then $\mathrm{cl}_\nu = \mathrm{cl}$ in all of the above cases (note that a T_1-space is already a symmetrical topological space).

(iii) Let $f : (X, \mathrm{cl}_X) \to (Y, \mathrm{cl}_Y)$ be a closed and continuous map. Define $\nu : \mathfrak{G}(X) \to [0, \infty]$ as follows: for $\mathcal{G} \in \mathfrak{G}(X)$, $\nu(\mathcal{G}) = 0$, if $\bigcap\{\mathrm{cl}_Y(f(G)) : G \in \mathcal{G}\} \neq \emptyset$; and $\nu(\mathcal{G}) = \inf_{G \in \sec \mathrm{cl}(\mathcal{G})} |G|$, otherwise. Then $(X, \nu_g, \mathrm{cl}_X)$ is an approach merotopological space. Further if $y \in \mathrm{cl}_Y(f(x)) \Rightarrow f(x) \in \mathrm{cl}_Y(\{y\})$ for all $y \in Y$ and $x \in X$, then $\mathrm{cl}_\nu = \mathrm{cl}_X$.

(iv) Let (X, ν, cl) be an approach merotopological space. Define $\nu_c : \mathcal{P}^2(X) \to [0, \infty]$ as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$,

$$\nu_c(\mathcal{A}) = \sup\{\nu(\mathcal{B}) : \mathcal{B} \subseteq \mathcal{A},\ |\mathcal{B}| < \aleph_0\}. \tag{3.2}$$

Then ν_c is an approach merotopology with respect to the closure cl on X.

(v) Let (X, cl) be a topological space and $r \in (0, \infty]$. Define $\nu^r : \mathcal{P}^2(X) \to [0, \infty]$ as follows: for $\mathcal{A} \in \mathcal{P}^2(X), \nu^r(\mathcal{A}) = 0$, if $\bigcap \mathrm{cl}(\mathcal{A}) \neq \emptyset; \nu^r(\mathcal{A}) = \infty$, if $\emptyset \in \mathcal{A}$; and $\nu^r(\mathcal{A}) = r$, otherwise. Then ν^r is an approach merotopology with respect to the closure cl on X.

(vi) The function $\nu_d : \mathcal{P}^2(X) \to [0, \infty]$ this is defined as follows: for $\mathcal{A} \in \mathcal{P}^2(X), \nu_d(\mathcal{A}) = 0$, if $\bigcap \mathcal{A} \neq \emptyset$, and $\nu_d(\mathcal{A}) = \infty$, otherwise, is an approach merotopology with respect to the discrete closure operator cl_d on X and $\mathrm{cl}_\nu = \mathrm{cl}_d$. We call $(X, \nu_d, \mathrm{cl}_d)$ the *discrete approach merotopological space*.

(vii) The function $\nu_i : \mathcal{P}^2(X) \to [0, \infty]$ this is defined as follows: for $\mathcal{A} \in \mathcal{P}^2(X), \nu_i(\mathcal{A}) = 0$, if $\emptyset \notin \mathcal{A}$, and $\nu_i(\mathcal{A}) = \infty$, otherwise, is an approach merotopology with respect to the indiscrete closure operator cl_i on X and $\mathrm{cl}_\nu = \mathrm{cl}_i$. We call $(X, \nu_i, \mathrm{cl}_i)$ the *indiscrete approach merotopological space*.

Having established the existence of an adequate number of approach merotopologies on X (taking several values in $[0, \infty]$), it is now relevant to study the category **AMT** having objects as approach merotopological spaces. A morphism in **AMT** is a map $f : X \to Y$ which is both continuous (with respect to the topologies) and contraction (with respect to the approach merotopies). We denote by **TOP** the category of topological spaces and continuous maps. Also let T : **AMT** $\to$ **TOP** and M : **AMT** $\to$ **AMER** denote the forgetful functors that keep the topology but forget the approach merotopy, and keep the approach merotopy but forget the topology, respectively.

Proposition 3.3. *Let* (X, ν, cl) *be an approach merotopological space. Then* $\mathrm{cl}(A) \subseteq \mathrm{cl}_v(A)$ *for all* $A \subseteq X$, *where* cl_v *denotes the closure operator induced by the approach merotopy* ν *on* X.

Proof. Let $A \subseteq X$ and $x \in \mathrm{cl}(A)$. Then $\nu(\{\mathrm{cl}(\{x\}), \mathrm{cl}(A)\}) = 0$ which yields $\nu(\{\{x\}, A\}) = 0$ and hence $x \in \mathrm{cl}_\nu(A)$. □

Theorem 3.4. *The category **AMT** is a topological construct.*

Proof. Let $((Y_j, \nu_j, \mathrm{cl}_j))_{j\in J}$ be a family of approach merotopological spaces indexed by J, and let $(f_j : T(X) \to T(Y_j))_{j\in J}$ be a source in **TOP**. Define $\nu : \mathcal{P}^2(X) \to [0,\infty]$ as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$,

$$\nu(\mathcal{A}) = \sup\left\{ \inf_{i=1}^{n} \sup_{j\in J} \nu_j\left(f_j(\mathcal{A}_i)\right) : (\mathcal{A})_{i=1}^{n} \in \mathfrak{C}(\mathcal{A}) \right\}, \tag{3.3}$$

where $\mathfrak{C}(\mathcal{A})$ is the collection of all finite families $(\mathcal{A}_i)_{i=1}^{n} \subseteq \mathcal{P}^2(X)$ such that $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A}$. Then ν is an approach merotopy on X (see [19, Theorem 3.8]). Let cl denote the closure induced by the initial topology on X and $(\mathcal{A}_i)_{i=1}^{n} \in \mathfrak{C}(\mathcal{A})$. Then $(\mathrm{cl}(\mathcal{A}))_{i=1}^{n} \in \mathfrak{C}(\mathrm{cl}(\mathcal{A}))$. Also since f_j is continuous for all $j \in J$, therefore $\nu_j(f_j(\mathcal{A}_i)) \le \nu_j(f_j(\mathrm{cl}(\mathcal{A}_i)))$, for all $(\mathcal{A}_i)_{i=1}^{n} \in \mathfrak{C}(\mathcal{A})$ which in turn yields $\nu(\mathcal{A}) \le \nu(\mathrm{cl}(\mathcal{A}))$. Hence (X, ν, cl) is an approach merotopological space. To show that the source is initial, let $(Z, \nu_Z, \mathrm{cl}_Z)$ be an approach merotopological space and $g : T(Z) \to T(X)$ be a continuous map such that, for every $j \in J$, the map $f_j \circ g : Z \to Y_j$ is an **AMT**-morphism. Then $g : M(Z) \to M(X)$ is a contraction since $(f_j : M(X) \to M(Y_j))_{j\in J}$ is initial in **AMER** (see [19, Theorem 3.8]). Thus (X, ν, cl) is the initial source in **AMT** and hence **AMT** is a topological construct. □

Corollary 3.5. *Let $((Y_j, \nu_j, cl_j))_{j\in J}$ be a family of approach merotopological spaces. Then $(f_j : X \to Y_j)_{j\in J}$ is an initial source in **AMT** if and only if $(f_j : T(X) \to T(Y_j))_{j\in J}$ is initial in **TOP** and $(f_j : M(X) \to M(Y_j))_{j\in J}$ is initial in **AMER**.*

Proposition 3.6. *Let $((X_j, \nu_j, cl_j))_{j\in J}$ be a family of approach merotopological spaces and $(f_j : X_j \to X)_{j\in J}$ be a sink in **AMT**. Then $(f_j : X_j \to X)_{j\in J}$ is a final sink in **AMT** if and only if both of the following conditions hold:*

(i) $(f_j : T(X_j) \to T(X))_{j\in J}$ *is final in **TOP**,*

(ii) *the approach merotopology with respect to the closure operator* cl *induced by the final topology on X is defined as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$,*

$$\nu(\mathcal{A}) = \begin{cases} 0, & \text{if } \bigcap \mathrm{cl}(\mathcal{A}) \neq \emptyset, \\ \inf_{j\in J} \nu_j\left(f_j^{-1}(\mathrm{cl}(\mathcal{A}))\right), & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. To prove that ν is an approach merotopy with respect to cl on X, we will prove only (AM5). For this, let $\bigcap \mathrm{cl}(\mathcal{A}) \neq \emptyset$. Then there exists $x \in X$ such that $\mathcal{A} \subseteq \{A \subseteq X : x \in \mathrm{cl}(A)\}$. Hence $\nu(\mathrm{cl}(\mathcal{A})) = 0 = \nu(\mathcal{A})$. To show that $(f_j : X_j \to X)_{j\in J}$ is final, let $(Z, \nu_Z, \mathrm{cl}_Z)$ be an approach merotopological space and let $g : X \to Z$ be a map such that, for every $j \in J$, $g \circ f_j : X_j \to Z$ is an **AMT**-morphism. Then $g : X \to Z$ is continuous since $(f_j : T(X_j) \to T(X))_{j\in J}$

is final in **TOP**. Also for $\mathcal{A} \subseteq \mathcal{P}(X)$ with $\bigcap \mathrm{cl}(\mathcal{A}) = \emptyset$,

$$\begin{aligned}\nu(\mathcal{A}) &= \inf_{j\in J} \nu_j\left(f_j^{-1}(\mathrm{cl}(\mathcal{A}))\right)\\ &\geq \inf_{j\in J} \nu_Z\left(g\left(f_j\left(f_j^{-1}(\mathrm{cl}(\mathcal{A}))\right)\right)\right)\\ &\geq \nu_Z(g(\mathrm{cl}(\mathcal{A})))\\ &\geq \nu_Z(g(\mathcal{A})).\end{aligned} \tag{3.5}$$

On the other hand, if $\bigcap \mathrm{cl}(\mathcal{A}) \neq \emptyset$, then there exists $x \in X$ such that $g(x) \in g(\mathrm{cl}(A)) \subseteq \mathrm{cl}_Z(g(A))$, for all $A \in \mathcal{A}$. Hence $\nu_Z(g(\mathcal{A})) = 0$. Thus, $g : (X, \nu) \to (Z, \nu_Z)$ is a contraction implying that (X, ν, cl) is the final approach merotopological space. □

Let ρ be a metric on X. Define $\nu_\rho : \mathcal{P}^2(X) \to [0, \infty]$ as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$,

$$\nu_\rho(\mathcal{A}) = \inf_{x\in X} \sup_{A\in\mathcal{A}} \inf_{y\in A} \rho(x, y). \tag{3.6}$$

Then ν_ρ is an approach merotopology with respect to the closure cl induced by ρ on X and $\mathrm{cl}_{\nu_\rho} = \mathrm{cl}$. Also for any metric spaces (X, ρ) and (Y, ρ'), $f : (X, \rho) \to (Y, \rho')$ is a nonexpansive map if and only if $f : (X, \nu_\rho, \mathrm{cl}_{\nu_\rho}) \to (Y, \nu_{\rho'}, \mathrm{cl}_{\nu_{\rho'}})$ is an **AMT**-morphism. Thus the category **MET** of metric spaces and nonexpansive maps is embedded as a full subcategory into **AMT** by the functor F : **MET** $\to$ **AMT** defined as follows: $F((X, \rho)) = (X, \nu_\rho, cl_{\nu_\rho})$ and $F(f) = f$.

Proposition 3.7. *The category **TOP** is a bireflective full subcategory of **AMT**.*

Proof. Let (X, ν, cl) be any approach merotopological space. Set $r = \sup\{\nu(\mathcal{A}) : \mathcal{A} \in \mathcal{P}^2(X), \nu(\mathcal{A}) \neq 0 \text{ and } \emptyset \notin \mathcal{A}\}$ and ν^r is defined as in Example 3.2 (v). Then $\nu^r \wedge \nu$ is an approach merotopology on X with respect to the closure operator cl, which is coarser than ν. Therefore for any approach merotopological space (X, ν, cl), the identity mapping $1_X : (X, \nu, \mathrm{cl}) \to (X, \nu^r \wedge \nu, \mathrm{cl})$ is the **TOP**-bireflection of (X, ν, cl). □

Remark 3.8. (i) There is a full embedding of **TOP** into **AMT** that is defined by associating to a topological space X, the approach merotopy ν^r as defined in Example 3.2 (v). Thus a topological space can be viewed as an approach merotopological space having the approach merotopy ν^r. Thus following this convention, the forgetful functor T : **AMT** $\to$ **TOP** shall be viewed as keeping the topology but replacing the approach merotopy with that of ν^r.

(ii) It is known that an approach nearness space always induces a Kuratowski closure operator on the underlying space. Thus by adjoining to an approach nearness space (X, ν), the topology given by the closure operator cl_ν induced by ν, we can obtain a full embedding of **ANEAR** (the category of approach nearness spaces and contractions) into **AMT**. (Observe that every contraction is a continuous map with respect to the closures induced by the approach merotopies.) Therefore it is clear that an approach nearness space (X, ν) can be regarded as a special approach merotopological space (X, ν, cl) for which $\mathrm{cl} = \mathrm{cl}_\nu$. Hence approach merotopological spaces are generalization of approach nearness spaces. Also since the underlying topology of an approach nearness space is always symmetric, therefore if we restrict the category **TOP** to the category **STOP** having objects symmetrical topological spaces, then the full embedding of **TOP** into **AMT** is actually the full embedding of **STOP** into **ANEAR**. Moreover for every approach merotopic space (X, ν), if we associate the discrete closure operator cl on X, then we obtain an approach merotopological space (X, ν, cl). Thus the category **AMER** can also be embedded into **AMT**.

(iii) Let δ be an approach space. Then the function $\nu_\delta : \mathcal{P}^2(X) \to [0, \infty]$ that is defined as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$,

$$\nu_\delta(\mathcal{A}) = \inf_{x \in X} \sup_{A \in \mathcal{A}} \delta(x, A) \tag{3.7}$$

is an approach merotopy on X. Also for any approach spaces (X, δ) and (Y, δ'), $f : (X, \delta) \to (Y, \delta')$ is a contraction map if and only if $f : (X, \nu_\delta) \to (Y, \nu_{\delta'})$ is a contraction. So we get the functor $F : \mathbf{AP} \to \mathbf{AMER}$ defined as follows: $F((X, \delta)) = (X, \nu_\delta)$ and $F(f) = f$. Therefore **AP** is a full subcategory of **AMER** (cf. [17, 19]). As a consequence, **AMT** is also a supercategory of **AP**.

In Remark 3.8, we have established that a topological space can be regarded as an approach merotopological space by associating the approach merotopy ν^r with the topology on X. It should be clarified here that, in general for an approach merotopological space (X, ν, cl), the topology on X is not determined by its approach merotopy ν; it happens so for a topological space only if (X, cl) is a symmetrical topological space. Thus, in general, for an approach merotopological space (X, ν, cl), cl need not be equal to cl_ν. This fact can be supported by the following example: consider $\mathbb{R}$ with $\{\emptyset, \mathbb{R}\} \cup \{[0, n] : n \in \mathbb{N}\}$ as the collection of all closed sets. Then $(\mathbb{R}, \mathrm{cl})$ is a non-symmetric and T_0-space, where cl is the closure operator associated with the topology on $\mathbb{R}$ defined as above. Further let the approach merotopy ν_i on $\mathbb{R}$ be indiscrete. Then $(\mathbb{R}, \nu_i, \mathrm{cl})$ is an approach merotopological space. Also observe that $(\mathbb{R}, \nu_i)$ is an approach nearness space but $\mathrm{cl}_{\nu_i} \neq \mathrm{cl}$.

Remark 3.9. Let **MERTOP** denote the category of all merotopological spaces and their respective morphisms (see [20]). It is easy to verify that, for any merotopological space (X, ξ, cl), the pair $(X, \nu_\xi, \mathrm{cl})$ is an approach merotopological space (where ν_ξ is the induced approach merotopy on X defined as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$, $\nu(\mathcal{A}) = 0$, if $\mathcal{A} \in \xi$; and $\nu(\mathcal{A}) = \infty$, otherwise. Also for any merotopic spaces (X, ξ) and (Y, ξ'), $f : (X, \xi) \to (Y, \xi')$ is a merotopic map if and only if $f : (X, \nu_\xi) \to (Y, \nu_{\xi'})$ is a contraction. Thus **MERTOP** is embedded as a full subcategory in **AMT** by the functor $F : \mathbf{MERTOP} \to \mathbf{AMT}$ such that $F((X, \xi, \mathrm{cl})) = (X, \nu_\xi, \mathrm{cl})$ and $F(f) = f$.

An approach merotopological space (X, ν, cl) is induced by a merotopological space (X, ξ, cl) if and only if $\nu(\mathcal{P}^2(X)) \subseteq \{0, \infty\}$. Also for any approach merotopological space (X, ν, cl), the triples $(X, \xi_\nu, \mathrm{cl})$ and $(X, \xi^\nu, \mathrm{cl})$, where

$$\begin{aligned} \xi_\nu &= \left\{ \mathcal{A} \in \mathcal{P}^2(X) : \nu(\mathcal{A}) = 0 \right\}, \\ \xi^\nu &= \left\{ \mathcal{A} \in \mathcal{P}^2(X) : \nu(\mathcal{A}) < \infty \right\}, \end{aligned} \tag{3.8}$$

are merotopological spaces; and if $f : (X, \nu) \to (Y, \nu')$ is a contraction, then $f : (X, \xi^\nu) \to (Y, \xi^{\nu'})$ and $f : (X, \xi_\nu) \to (Y, \xi_{\nu'})$ are merotopic maps. So this defines the functors $G : \mathbf{AMT} \to \mathbf{MERTOP}$ by $G((X, \nu, \mathrm{cl})) = (X, \xi_\nu, \mathrm{cl})$ and $G(f) = f$ and $G' : \mathbf{AMT} \to \mathbf{MERTOP}$ by $G'((X, \nu, \mathrm{cl})) = (X, \xi^\nu, \mathrm{cl})$ and $G'(f) = f$. Hence we can conclude that the category **MERTOP** is a bicoreflective and bireflective subcategory of **AMT**: for any approach merotopological space (X, ν, cl), the identity mappings $1_X : (X, \nu_{\xi_\nu}, \mathrm{cl}) \to (X, \nu, \mathrm{cl})$ and $1_X : (X, \nu, \mathrm{cl}) \to (X, \nu_{\xi^\nu}, \mathrm{cl})$ are the **MERTOP**-bicoreflection and **MERTOP**-bireflection of (X, ν, cl), respectively. Further in [20], Bentley and Herrlich embedded **TOP** in **MERTOP** as its full bicoreflective subcategory. Therefore **TOP** can be embedded as a full subcategory in **AMT** both bicoreflectively and bireflectively.

Next, let us introduce ordering between approach merotopological spaces and discuss their order structure. The exact meet and join of a family of merotopologies on X are constructed.

Definition 3.10. Let ν and ν' be approach merotopologies with respect to the closures cl and cl′ on X, respectively. Define the ordering "$\leq$" as follows: $(X, \nu, \mathrm{cl}) \leq (X, \nu', \mathrm{cl}')$ if $\nu \leq \nu'$ and the topology induced by cl is weaker than the topology induced by cl′ on X.

Theorem 3.11. *The family of all approach merotopologies on X forms a completely distributive complete lattice with respect to the partial order "$\leq$". The zero of this lattice is the indiscrete approach merotopology ν_i with respect to the closure cl_i induced by the indiscrete topology (called indiscrete closure) on X and the unit is the discrete approach merotopology ν_d with respect to the discrete closure cl_d induced by the discrete topology (called discrete closure) on X.*

Proof. Let $\{(X_j, \nu_j, \mathrm{cl}_j) : j \in J\}$ (where J is an arbitrary index set) be a family of approach merotopological spaces. If τ_j denotes the topology induced by cl_j for all $j \in J$, then $\sup_{j\in J}\tau_j = \bigcap\{V_\alpha : V_\alpha$ is a topology on X and $\tau_j \subseteq V_\alpha$, for all $j \in J\}$. Let cl denote the closure operator induced by $\sup_{j\in J}\tau_j$. Then for $A \subseteq X, \mathrm{cl}(A) = \bigcap\{B \subseteq X : A \subseteq B$ and B is $\sup_{j\in J}\tau_j$-closed$\}$. Define $\nu_{\sup} : \mathcal{P}^2(X) \to [0,\infty]$ as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$,

$$\nu_{\sup}(\mathcal{A}) = \sup\left\{\inf_{i=1}^{n} \sup_{j\in J} \nu_j(\mathcal{A}_i) : (\mathcal{A}_i)_{i=1}^{n} \in \mathfrak{C}(\mathcal{A})\right\}, \tag{3.9}$$

where $\mathfrak{C}(\mathcal{A})$ is the collection of all finite families $(\mathcal{A}_i)_{i=1}^n \subseteq \mathcal{P}^2(X)$ such that $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n \prec \mathcal{A}$. Then $\nu_{\sup}$ with respect to the closure operator cl is an approach merotopology on X and is the supremum of the family of merotopologies $\{\nu_j : j \in J\}$ with respect to the closures $\{\mathrm{cl}_j : j \in J\}$ on X, respectively (techniques of the proof are similar to those of Theorem 3.4). Now we construct the infimum of the given family. Define $\nu_{\inf} : \mathcal{P}^2(X) \to [0,\infty]$ as follows: for $\mathcal{A} \in \mathcal{P}^2(X)$,

$$v_{\inf}(A) = \inf_{j\in J} v_j(\mathrm{cl}(A)), \tag{3.10}$$

where $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: for $A \subseteq X, \mathrm{cl}(A) = \bigcap\{B \subseteq X : A \subseteq B$ and $cl_j(B) = B$, for all $j \in J\}$. Then $\nu_{\inf}$ is an approach merotopology with respect to the closure cl on X which is also the infimum of the merotopologies $\{\nu_j : j \in J\}$ with respect to the closures $\{cl_j : j \in J\}$ on X, respectively (proof follows similarly as in Proposition 3.6). $\square$

4. Completion of Approach Merotopological Spaces

In this section, we construct bunch completion of any approach merotopological space. The concept of regularity in an approach merotopic space is introduced to establish a relationship between the bunch completion and the cluster completion (constructed in [22]) of an approach nearness space in the classical theory. For regularity in a merotopic space, we refer to [1]. It is noteworthy to discuss here that clusters and bunches of an approach merotopological space are essentially the clusters and bunches of the associated merotopology $\xi_\nu = \{\mathcal{A} \in \mathcal{P}(L^X) : \nu(\mathcal{A}) = 0\}$. Similar is the case with the definition of bunch completeness of an approach merotopological space. But in this section, using these definitions, we have constructed the completion $(X^+, \nu^+, \mathrm{cl}_{X^+})$ of an approach merotopological space (X, ν, cl). Observe that ν^+ takes several values in $[0, \infty]$ if ν does so, and the completion of ξ_ν, that is, ξ_ν^+, is not, in general, equal to ν^+. A similar type of study can also be seen in [17], where Lowen et al. had constructed the completion of a T_0-approach space (X, δ) using the clusters of δ which are precisely the clusters of the associated merotopic space.

Definition 4.1. Let (X, ν, cl) be an approach merotopological space. A nonempty grill $\mathcal{G}$ on X is said to be a *ν-bunch* if $\nu(\mathcal{G}) = 0$ and $\mathrm{cl}(G) \in \mathcal{G} \Rightarrow G \in \mathcal{G}$.

By a ν-cluster $\mathcal{A}$ in an approach merotopological space (X, ν, cl), we mean that $\mathcal{A}$ is a ν-cluster of the approach merotopic space (X, ν).

Proposition 4.2. *Let (X, ν, cl) be an approach merotopological space. Then every ν-cluster is a maximal near ν-bunch.*

Proof. Let $\mathcal{C}$ be a ν-cluster. Then clearly $\mathcal{C}$ is a near element in ν. Let $\mathrm{cl}(G) \in \mathcal{C}$. Then $\nu(\{\mathrm{cl}(C), \mathrm{cl}(G)\}) = 0$, for all $C \in \mathcal{C}$ which by (AM5) yields that $\nu(\{C, G\}) = 0$, for all $C \in \mathcal{C}$. Consequently $G \in \mathcal{C}$ as $\mathcal{C}$ is ν-closed. □

For any approach merotopological space (X, ν, cl), let X^+ denote the family of all ν-bunches on X. It must be clarified here that a ν-bunch need not be a ν-cluster. For example, let $\mathcal{G}(\mathfrak{m}) = \{A \subseteq X : |\mathrm{cl}(A)| \geq \mathfrak{m}\}$ where $\mathfrak{m}$ is an infinite cardinal number. Consider the approach merotopological space (X, ν, cl) of the last case of Example 3.2 (ii) (f). Then for $\mathfrak{m} > c$ (where $\mathfrak{c}$ is the cardinality of the set $\mathbb{R}$ of all real numbers), $\mathcal{G}(\mathfrak{m})$ is a ν-bunch which is not a ν-cluster because $\mathcal{G}(\mathfrak{m})$ is a proper subset of $\mathcal{G}(\aleph_0)$ and $\nu(\mathcal{G}(\aleph_0)) = 0$.

Theorem 4.3. *Let (X, ν, cl) be an approach merotopological space and let cl_{X^+} denote the operator defined by $\mathrm{cl}_{X^+}(\omega) = \{\mathcal{B} \in X^+ : \bigcap \omega \subseteq \mathcal{B}\}$, for all $\omega \subseteq X^+$. Then cl_{X^+} is a Kuratowski closure operator on X^+.*

Proof. Clearly $\mathrm{cl}_{X^+}(\emptyset) = \emptyset$ and $\omega \subseteq \mathrm{cl}_{X^+}(\omega)$, where $\omega \subseteq X^+$. Let $\mathcal{B} \in \mathrm{cl}_{X^+}(\omega_1 \cup \omega_2)$ but $\mathcal{B} \notin \mathrm{cl}_{X^+}(\omega_1) \cup \mathrm{cl}_{X^+}(\omega_2)$. Then $\bigcap \omega_1 \cap \bigcap \omega_2 \subseteq \mathcal{B}$ but $\bigcap \omega_1 \nsubseteq \mathcal{B}$ and $\bigcap \omega_2 \nsubseteq \mathcal{B}$. Therefore there exist $A \in \bigcap \omega_1$ and $B \in \bigcap \omega_2$ such that $A \notin \mathcal{B}$ and $B \notin \mathcal{B}$, which gives $A \cup B \notin \mathcal{B}$. Also $A \in \mathcal{A}$ and $B \in \mathcal{B}$, for every $\mathcal{A} \in \omega_1$ and $\mathcal{B} \in \omega_2$, and consequently $A \cup B \in \bigcap(\omega_1 \cup \omega_2)$, which contradicts that $\bigcap(\omega_1 \cup \omega_2) \subseteq \mathcal{B}$. Thus $\mathcal{B} \in \mathrm{cl}_{X^+}(\omega_1) \cup \mathrm{cl}_{X^+}(\omega_2)$. Hence $\mathrm{cl}_{X^+}(\omega_1 \cup \omega_2) \subseteq \mathrm{cl}_{X^+}(\omega_1) \cup \mathrm{cl}_{X^+}(\omega_2)$. For the reverse inclusion, let $\mathcal{B} \notin \mathrm{cl}_{X^+}(\omega_1 \cup \omega_2)$. Then $\bigcap \omega_1 \nsubseteq \mathcal{B}$ and $\bigcap \omega_2 \nsubseteq \mathcal{B}$, giving that $\mathcal{B} \notin \mathrm{cl}_{X^+}(\omega_1) \cup \mathrm{cl}_{X^+}(\omega_2)$. Finally let $\mathcal{B} \in \mathrm{cl}_{X^+}(\mathrm{cl}_{X^+}(\omega))$ but $\mathcal{B} \notin \mathrm{cl}_{X^+}(\omega)$. Then $\bigcap \mathrm{cl}_{X^+}(\omega) \subseteq \mathcal{B}$ but $\bigcap \omega \nsubseteq \mathcal{B}$. Therefore there exists $A \in \bigcap \omega$ such that $A \notin \mathcal{B}$ which implies that $A \notin \bigcap \mathrm{cl}_{X^+}(\omega)$

and thus $\bigcap \omega \nsubseteq \mathcal{N}$, for some $\mathcal{N} \in \omega$. Consequently $\mathcal{N} \notin \omega$, which is not possible. Hence $\mathrm{cl}_{X^+}(\mathrm{cl}_{X^+}(\omega)) = \mathrm{cl}_{X^+}(\omega)$. □

It can be easily verified that, for any approach merotopological space (X, ν, cl), the family $\{A \subseteq X : x \in \mathrm{cl}(A)\}$ is a ν-bunch. Therefore we can define the map $\mathfrak{e}_X : X \to X^+$ as follows: for $x \in X$, $\mathfrak{e}_X(x) = \{A \subseteq X : x \in \mathrm{cl}(A)\}$.

Theorem 4.4. *The map* $F : \mathbf{AMT} \to \mathbf{AMT}$ *defined by* $F((X, \nu, \mathrm{cl})) = (X^+, \nu^+, \mathrm{cl}_{X^+})$ *and* $F(f) = f^+$, *where* $\nu^+ : \mathcal{P}^2(X^+) \to [0, \infty]$ *is defined as follows: for* $\Omega \in \mathcal{P}^2(X^+)$, $\nu^+(\Omega) = \nu(\bigcup\{\bigcap \omega : \omega \in \Omega\})$, *and* $f^+ : X^+ \to Y^+$ *(when* $f : X \to Y$*) is defined as follows: for* $\mathcal{B} \in X^+$, $f^+(\mathcal{B}) = \{A \subseteq Y : f(B) \subseteq \mathrm{cl}(A)$, *for some* $B \in \mathcal{B}\}$, *is a functor. Moreover,* $e : 1 \to F$ *is a natural transformation from the identity functor.*

Proof. First we will show that $(X^+, \nu^+, \mathrm{cl}_{X^+})$ is an approach merotopological space. Let $\Omega, \Im \in \mathcal{P}^2(X^+)$ and $\Omega \prec \Im$. Then $\bigcup\{\bigcap \omega : \omega \in \Omega\} \prec \bigcup\{\bigcap \lambda : \lambda \in \Im\}$. Thus $\nu^+(\Omega) \le \nu^+(\Im)$. If $\bigcap \Omega \neq \emptyset$, then there exists $\mathcal{A} \in \bigcap \Omega$ and therefore $\bigcap \omega \subseteq \mathcal{A}$, for every $\omega \in \Omega$. Consequently $\bigcup\{\bigcap \omega : \omega \in \Omega\} \subseteq \mathcal{A}$ which yields $\nu^+(\Omega) = 0$. Condition (AM3) follows by the convention $\emptyset \in \bigcap \emptyset$. Now

$$\begin{aligned} \nu^+(\Omega \vee \Im) &= \nu\Big(\bigcup\Big\{\bigcap \omega : \omega \in \Omega \vee \Im\Big\}\Big) \\ &\ge \nu\Big(\bigcup\Big\{\bigcap \omega : \omega \in \Omega\Big\} \vee \bigcup\Big\{\bigcap \omega : \omega \in \Im\Big\}\Big) \\ &\ge \nu\Big(\bigcup\Big\{\bigcap \omega : \omega \in \Omega\Big\}\Big) \wedge \nu\Big(\bigcup\Big\{\bigcap \omega : \omega \in \Im\Big\}\Big) \\ &= \nu^+(\Omega) \wedge \nu^+(\Im). \end{aligned} \tag{4.1}$$

For (AM5) we have to show that $\bigcup\{\bigcap \omega : \omega \in \Omega\} \subseteq \bigcup\{\bigcap \mathrm{cl}_{X^+}(\omega) : \mathrm{cl}_{X^+}(\omega) \in \mathrm{cl}_{X^+}(\Omega)\}$, where $\mathrm{cl}_{X^+}(\Omega) = \{\mathrm{cl}_{X^+}(\omega) : \omega \in \Omega\}$. Let $A \in \bigcup\{\bigcap \omega : \omega \in \Omega\}$. Then $A \in \bigcap \omega$, for some $\omega \in \Omega$. So, $A \in \mathcal{A}$ for every $\mathcal{A} \in \mathrm{cl}_{X^+}(\omega)$ as $\bigcap \omega \subseteq \mathcal{A}$, for each $\mathcal{A} \in \mathrm{cl}_{X^+}(\omega)$. Thus $A \in \bigcup\{\bigcap \mathrm{cl}_{X^+}(\omega) : \mathrm{cl}_{X^+}(\omega) \in \mathrm{cl}_{X^+}(\Omega)\}$. Hence $(X^+, \nu^+, \mathrm{cl}_{X^+})$ is an approach merotopological space.

Next let $f : X \to Y$ be an **AMT**-morphism. Then we will show that $f^+ : X^+ \to Y^+$ is an **AMT**-morphism. That f^+ is a map follows from the fact that $\mathrm{cl}(f^+(\mathcal{B})) \prec f(\mathcal{B})$, for all $\mathcal{B} \in \mathcal{P}^2(X)$. For the continuity of $f^+ : X^+ \to Y^+$, let $\omega \subseteq X^+$ and $\mathcal{B} \in \mathrm{cl}_{X^+}(\omega)$ and $A \in \bigcap f^+(\omega)$. Then $A \in f^+(\mathcal{C})$, for every $\mathcal{C} \in \omega$. Consequently $f(B) \subseteq \mathrm{cl}(A)$, for some $B \in \mathcal{B}$ and hence $\bigcap f^+(\omega) \subseteq f^+(\mathcal{B})$ which in turn gives that $f^+ : X^+ \to Y^+$ is continuous. Now to show that $f^+ : X^+ \to Y^+$ is a contraction, let $\Omega \in \mathcal{P}^2(X^+)$ and $A \in \bigcup\{\bigcap f^+(\omega) : f^+(\omega) \in f^+(\Omega)\}$. Then there is an $A \in \bigcap f^+(\omega_0)$, for some $\omega_0 \in \Omega$. Thus for every $\mathcal{B} \in \omega_0$, there is a $B \in \mathcal{B}$ such that $f(B) \subseteq \mathrm{cl}(A)$. Since $\bigcap f(\mathcal{B}) \subseteq f(B)$, therefore $\mathrm{cl}(\bigcup\{\bigcap f^+(\omega) : f^+(\omega) \in f^+(\Omega)\}) \prec f(\bigcup\{\bigcap \omega : \omega \in \Omega\})$. Thus $\nu^+_Y(f^+(\Omega)) \le \nu^+_X(\Omega)$ and $f^+ : (X^+, \nu^+_X, \mathrm{cl}_{X^+}) \to (Y^+, \nu^+_Y, \mathrm{cl}_{Y^+})$ is an **AMT**-morphism. Hence F is well defined. Clearly F preserves identity. To show that F is a functor, we finally show that, for **AMT**-morphisms $f : X \to Y$ and $g : Y \to Z$, $(g \circ f)^+ = g^+ \circ f^+$. Let

$\mathcal{B} \in X^+$. Then

$$\begin{aligned}(g \circ f)^+(\mathcal{B}) &= \{G \subseteq Z : \text{there is a } B \in \mathcal{B} \text{ such that } (g \circ f)(\mathcal{B}) \subseteq \mathrm{cl}_Z(G)\} \\ &= \left\{G \subseteq Z : (g \circ f)^{-1}(\mathrm{cl}_Z(G)) \in \mathcal{B}\right\} \\ &= f^+\left(\left\{G \subseteq Z : g^{-1}(\mathrm{cl}_Z(G)) \in \mathcal{B}\right\}\right) = (g^+ \circ f^+)(\mathcal{B}).\end{aligned} \tag{4.2}$$

Now to prove that $e : 1 \to F$ is a natural transformation, we will first show that $\mathfrak{e}_X : X \to X^+$ is an **AMT**-morphism. For this, let $A \subseteq X$ and $\mathcal{A} \in \mathfrak{e}_X(\mathrm{cl}(A))$. Then $\bigcap \mathfrak{e}_X(\mathrm{cl}(A)) \subseteq \mathcal{A}$ and thus $A \in \mathcal{A}$ because $\mathcal{A} \in X^+$. So, $\mathcal{A} \in \mathrm{cl}_{X^+}\mathfrak{e}_X(A)$. Therefore $\mathfrak{e}_X(\mathrm{cl}(A)) \subseteq \mathrm{cl}_{X^+}(\mathfrak{e}_X(A))$ and hence $\mathfrak{e}_X : X \to X^+$ is continuous. Also since $\bigcap \mathfrak{e}_X(A) \subseteq \mathcal{A}$ for all $A \in \mathcal{A} \in X^+$, therefore $\bigcup\{\bigcap \mathfrak{e}_X(A) : A \in \mathcal{A}\} \subseteq \mathcal{A}$ which in turn yields that $\nu^+(\mathfrak{e}_X(\mathcal{A})) \le \nu(\mathcal{A})$. Hence $\mathfrak{e}_X : X \to X^+$ is an **AMT**-morphism. Finally let $x \in X$ and $B \in \mathfrak{e}_Y(f(x))$. Then $f(x) \in \mathrm{cl}_Y(B)$ and therefore $f^{-1}(\mathrm{cl}_Y(B)) \in \mathfrak{e}_X(x)$ which gives $B \in (f^+ \circ \mathfrak{e}_X)(x)$. On the other hand if $B \in f^+(\mathfrak{e}_X(x))$, then there exists $C \in \mathfrak{e}_X(x)$ such that $f(C) \subseteq cl_Y(B)$. Consequently $f(x) \in cl_Y(B)$ and $B \in \mathfrak{e}_Y(f(x))$. Hence $f^+ \circ \mathfrak{e}_X = \mathfrak{e}_Y \circ f$. □

Definition 4.5. (i) Let $(X, \nu_X, \mathrm{cl}_X)$ and $(Y, \nu_Y, \mathrm{cl}_Y)$ be approach merotopological spaces. A mapping $f : (X, \nu_X, \mathrm{cl}_X) \to (Y, \nu_Y, \mathrm{cl}_Y)$ is said to be *strict* in **AMT** if $\{\mathrm{cl}_Y(f(A)) : A \subseteq X\}$ is a base for closed subsets of Y (i.e., *strict* in **TOP** (see [25])), and $\nu_Y(\mathcal{B}) \le \sup\{\nu_X(\mathcal{A}) : \mathrm{cl}(f(\mathcal{A})) \prec \mathcal{B}\}$, for all $\mathcal{B} \in \mathcal{P}^2(X)$ (i.e., *strict* in **AMER**).

(ii) Further, f said to be *initial* in **AMT** if $f^{-1}(\mathrm{cl}_Y(f(A))) \subseteq \mathrm{cl}_X(A)$ for all $A \subseteq X$ (called *initial* in **TOP**), and $\nu_X(\mathcal{A}) \le \nu_Y(f(\mathcal{A}))$ for all $\mathcal{A} \in \mathcal{P}^2(X)$ (called *initial* in **AMER**).

Observe that an initial map $f : X \to Y$ is strict in **AMER** if and only if the approach merotopy ν_Y on Y is defined as: $\nu_Y(\mathcal{B}) = \sup\{\nu_X(\mathcal{A}) : \mathrm{cl}_Y(f(\mathcal{A})) \prec \mathcal{B}\}$, for all $B \in \mathcal{P}^2(Y)$. Also for any initial map $f : (X, \nu_X, \mathrm{cl}_X) \to (Y, \nu_Y, \mathrm{cl}_Y)$ in **AMER**, there is a strict approach merotopology μ_Y with respect to the closure operator cl_Y on Y defined as: $\mu_Y(\mathcal{B}) = \sup\{\nu_X(\mathcal{A}) : \mathrm{cl}_Y(f(\mathcal{A})) \prec \mathcal{B}\}$, for all $\mathcal{B} \in \mathcal{P}^2(Y)$.

Proposition 4.6. *Let* (X, ν, cl) *be an approach merotopological space. Then*

(i) *the map* $\mathfrak{e}_X : X \to X^+$ *is initial and strict;*

(ii) $\mathrm{cl}_{X^+}(\mathfrak{e}_X(X)) = X^+$;

(iii) $\mathfrak{e}_X^+ : X^+ \to X^{++}$ *is an injective map.*

Thus X^+ *can be regarded as a dense subspace of* X^{++}.

Proof. (i) Let $A \subseteq X$ and $x \in \mathfrak{e}_X^{-1}(\mathrm{cl}_{X^+}(\mathfrak{e}_X(A)))$. Then we have $\mathfrak{e}_X(x) \in \mathrm{cl}_{X^+}(\mathfrak{e}_X(A))$, that is $\bigcap \mathfrak{e}_X(A) \subseteq \mathfrak{e}_X(x)$ which yields $x \in \mathrm{cl}(A)$. Therefore $\mathfrak{e}_X : X \to X^+$ is initial in **TOP**. That $\mathfrak{e}_X$ is initial in **AMER**, follows immediately by the relation $\mathcal{A} \subseteq \bigcup\{\bigcap \omega : \omega \in \mathfrak{e}_X(\mathcal{A})\}$, for all $\mathcal{A} \in \mathcal{P}^2(X)$. Thus $\mathfrak{e}_X : X \to X^+$ is initial in **AMT**. For strictness in **TOP**, we refer to [20]. For strictness in **AMER**, let $\Omega \subseteq \mathcal{P}(X^+)$ and $\mathcal{A} = \bigcup\{\bigcap \omega : \omega \in \Omega\}$. Then $\mathrm{cl}_{X^+}(\mathfrak{e}_X(\mathcal{A})) \prec \Omega$. Thus $\nu^+(\Omega) = \sup\{\nu(\mathcal{A}) : \mathrm{cl}_{X^+}(\mathfrak{e}_X(\mathcal{A})) \prec \Omega\}$ as $\mathfrak{e}_X$ is initial in **AMER**.

(ii) Clearly $\mathrm{cl}_{X^+}(\mathfrak{e}_X(X)) \subseteq X^+$. Let $A \in \bigcap \mathfrak{e}_X(X)$. Then $X = \mathrm{cl}(A)$ implying that $A \in \mathcal{B}$, for all $\mathcal{B} \in X^+$. Hence $\mathrm{cl}_{X^+}(\mathfrak{e}_X(X)) = X^+$.

(iii) See [20, Proposition 22]. □

Definition 4.7. An approach merotopological space (X, ν, cl) is said to be *bunch complete* if $\bigcap \mathrm{cl}(\mathcal{B}) \neq \emptyset$, for all $\mathcal{B} \in X^+$.

Proposition 4.8. *An approach merotopological space* (X, ν, cl) *is bunch complete if and only if for every grill* $\mathcal{G}$ *on one X such that one* $\nu(\mathcal{G}) = 0$, *has* $\bigcap \mathrm{cl}(\mathcal{G}) \neq \emptyset$.

Proof. Let (X, ν, cl) be bunch complete and let $\mathcal{G}$ be a grill on X such that $\nu(\mathcal{G}) = 0$. Then $\mathcal{G} \subseteq \{G \subseteq X : \mathrm{cl}(G) \in \mathcal{G}\}$. Thus $\bigcap \mathrm{cl}(\mathcal{G}) \neq \emptyset$. Converse is obvious. □

Proposition 4.9. *Let* (X, ν, cl) *be an approach merotopological space. Then* $(X^+, \nu^+, \mathrm{cl}_{X^+})$ *is a bunch complete approach merotopological space and* (X^+, cl_{X^+}) *is a* T_0*-space.*

Proof. That $(X^+, \nu^+, \mathrm{cl}_{X^+})$ is an approach merotopological space follows from Theorem 4.4. Let Ω be a ν^+-bunch and $\mathcal{B} = \bigcup\{\bigcap \omega : \omega \in \Omega\}$. Then $\mathcal{B}$ is a ν-bunch: let $A \cup B \in \mathcal{B}$. Then there exists $\omega_0 \in \Omega$ such that $A \cup B \in \bigcap \omega_0$. Consequently $\omega_0 \subseteq \mathrm{cl}_{X^+}(\mathfrak{e}_X(A \cup B)) = \mathrm{cl}_{X^+}(\mathfrak{e}_X(A)) \cup \mathrm{cl}_{X^+}(\mathfrak{e}_X(B))$. Since Ω is a grill on X^+, therefore $\mathrm{cl}_{X^+}(\mathfrak{e}_X(A)) \cup \mathrm{cl}_{X^+}(\mathfrak{e}_X(B)) \in \Omega$ which in turn gives that $\mathrm{cl}_{X^+}(\mathfrak{e}_X(A)) \in \Omega$ or $\mathrm{cl}_{X^+}(\mathfrak{e}_X(B)) \in \Omega$. Also $A \in \bigcap \mathrm{cl}_{X^+}(\mathfrak{e}_X(A)) \in \Omega$ and $B \in \bigcap \mathrm{cl}_{X^+}(\mathfrak{e}_X(B)) \in \Omega$ yield that $A \in \mathcal{B}$ or $B \in \mathcal{B}$. Now we will show that $\bigcap \mathrm{cl}_{X^+}(\Omega) \neq \emptyset$, for which it is sufficient to note that $\bigcap \omega \subseteq \mathcal{B}$, for all $\omega \in \Omega$. Thus ν^+ is bunch complete. Now let $\mathcal{B}_1, \mathcal{B}_2 \in X^+$ and $\mathcal{B}_1 \in \mathrm{cl}_{X^+}(\{\mathcal{B}_2\})$ and $\mathcal{B}_2 \in \mathrm{cl}_{X^+}(\{\mathcal{B}_1\})$. Then $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$. Hence cl_{X^+} is a T_0-closure operator on X^+. □

Remark 4.10. Since any topological space (X, cl) can be viewed as an approach merotopological space (X, ν^r, cl), following the convention established previously, therefore each topological space (X, cl) is bunch complete.

The following theorem shows that the bunch completion X^+ of X possesses a universal mapping property that shows it to be very large.

Theorem 4.11. *Let* $(X, \nu_X, \mathrm{cl}_X)$ *be an approach merotopological space and let* $\mathfrak{e}_X : X \to X^+$ *be the canonical mapping into the bunch completion. If* $f : X \to W$ *is an initial **AMT**-morphism into the approach merotopological space* $(W, \nu_W, \mathrm{cl}_W)$ *such that* $\mathrm{cl}_W(f(X)) = W$, *then the map* $v : W \to X^+$ *that is defined as follows: for* $w \in W, v(w) = \{A \subseteq X : w \in \mathrm{cl}_W(f(A))\}$, *is such that* $v \circ f = \mathfrak{e}_X$. *Moreover,*

(i) $v : W \to X^+$ *is an **AMT**-morphism*

(ii) *if* $f : X \to W$ *is strict, then* $v : W \to X^+$ *is initial, strict and* $\mathrm{cl}_{X^+}(v(w)) = X^+$.

Proof. That $v : W \to X^+$ is a map follows from the assumption that $f : X \to W$ is initial. Let $A \in \mathfrak{e}_X(x)$. Then $x \in \mathrm{cl}_X(A)$ and therefore $f(x) \in \mathrm{cl}_W(f(A))$, that is, $A \in v(f(x))$. Hence $v \circ f = \mathfrak{e}_X$.

(i) For the continuity of v, let $B \subseteq W$ and $w \in \mathrm{cl}_W(B)$ but $v(w) \notin \mathrm{cl}_{X^+}(v(B))$. Then $\bigcap v(B) \nsubseteq v(w)$, that is, there exists $A \in \bigcap v(B)$ such that $A \notin v(w)$. Consequently $B \nsubseteq \mathrm{cl}_W(f(A))$, and thus there is $b \in B$ such that $A \notin v(b)$, a contradiction. Therefore $v(w) \in \mathrm{cl}_{X^+}(v(B))$ and $v(\mathrm{cl}_W(B)) \subseteq \mathrm{cl}_{X^+}(v(B))$, for all $B \subseteq W$. Hence $v : W \to X^+$ is continuous. Now we will show that v is a contraction. Let $\mathcal{C} \in \mathcal{P}^2(W)$ and $\mathcal{O} = \bigcup\{\bigcap v(C) : C \in \mathcal{C}\}$. Then $\mathrm{cl}_W(f(\mathcal{O})) \prec \mathcal{C}$. By the initiality of $f, \nu^+(v(\mathcal{C})) \leq \nu_W(\mathcal{C})$. Hence v is a contraction.

(ii) It can be verified that if $\mathrm{cl}_W(f(\mathcal{A})) \prec \mathcal{B}$, then $\mathrm{cl}_{X^+}(\mathfrak{e}_X(\mathcal{A})) \prec v(\mathcal{B})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$ and $\mathcal{B} \in \mathcal{P}^2(W)$. So, $\nu_X(\mathcal{A}) \leq \nu^+(v(\mathcal{B}))$, for all $\mathcal{A} \in \mathcal{P}^2(X)$ such that $\mathrm{cl}_W(f(\mathcal{A})) \prec \mathcal{B}$, since $\mathfrak{e}_X : X \to X^+$ is initial. Thus $\sup\{\nu_X(\mathcal{A}) : \mathrm{cl}_W(f(\mathcal{A})) \prec \mathcal{B}\} \leq \nu^+(v(\mathcal{B}))$ implying that $\nu_W(\mathcal{B}) \leq \nu^+(v(\mathcal{B}))$, since f is strict in **AMER**. Hence $v : M(W) \to M(X^+)$ is initial in **AMER**. To show that $v : T(X) \to T(X^+)$ is initial in **TOP**, we refer to [20].

Now to show that v is strict, first we will show that $\mathrm{cl}_{X^+}(\mathfrak{e}_X(\mathcal{O})) \prec \Omega$, where $\mathcal{O} = \bigcup\{\bigcap\omega : \omega \in \Omega\}$. On the contrary, suppose that $\mathrm{cl}_{X^+}(\mathfrak{e}_X(\mathcal{O})) \not\prec \Omega$. Then there exists $B \in \mathcal{O}$ such that $\omega \nsubseteq \mathrm{cl}_{X^+}(\mathfrak{e}_X(B))$, for all $\omega \in \Omega$, that is, for every $\omega \in \Omega$, there is $\mathcal{E} \in \omega$ such that $\mathcal{E} \notin \mathrm{cl}_{X^+}(\mathfrak{e}_X(B))$, that is, for every $\omega \in \Omega$, there is $\mathcal{E} \in \omega$ such that $B \notin \mathcal{E}$. Consequently, $B \notin \bigcap\omega$ for all $\omega \in \Omega$, and thus $B \notin \mathcal{O}$, a contradiction. Hence $\mathrm{cl}_{X^+}(\mathfrak{e}_X(\mathcal{O})) \prec \Omega$. Since $v \circ f = \mathfrak{e}_X$, therefore $\nu_W(f(\mathcal{O})) \leq \sup\{\nu_W(\mathcal{C}) : \mathrm{cl}_{X^+}(v(\mathcal{C})) \prec \Omega\}$. But by the initiality of f, $\nu^+(\Omega) \leq \sup\{\nu_W(\mathcal{C}) : \mathrm{cl}_{X^+}(v(\mathcal{C})) \prec \Omega\}$ giving that v is initial.

Finally for denseness of v, we know that $\mathrm{cl}_W(f(X)) = W$, which gives $v(\mathrm{cl}_W(f(X))) = v(W)$. But $v(\mathrm{cl}_W(f(X))) = \mathrm{cl}_{X^+}((v \circ f)(X)) = \mathrm{cl}_{X^+}(\mathfrak{e}_X(X)) = X^+$. Hence $v(W) = X^+$ and therefore $\mathrm{cl}_{X^+}(v(W)) = X^+$. □

We now concentrate on approach nearness spaces. In [21], we have constructed the cluster completion (X^*, ν^*) of an approach nearness space (X, ν) (where X^* is the family of all ν-clusters) by defining ν^* in the same way as ν^+ and the map $\mathfrak{e}_X : X \to X^*$ similarly as in the case of bunch completion. Since each approach nearness space can be viewed as an approach merotopological space, therefore, by Proposition 4.9, we can obtain a bunch completion of an approach nearness space. Also $X^* \subseteq X^+$ by Proposition 4.2. We introduce the property of regularity in approach merotopic spaces and prove that X^* is a retract of X^+ for a regular approach nearness space (X, ν). For this, we first consider the following.

Lowen and Lee [19] also gave an equivalent description of an approach merotopy on X (by generalizing the concept of micromeric collections of a nonempty set) as a function $\gamma : \mathcal{P}^2(X) \to [0, \infty]$ such that the following conditions are satisfied for any $\mathcal{A}, \mathcal{B} \in \mathcal{P}^2(X)$:

(AM1′) $\mathcal{A} \prec \mathcal{B} \Rightarrow \gamma(\mathcal{A}) \geq \gamma(\mathcal{B})$,

(AM2′) there exists $A \in \mathcal{A}$ such that $|A| \leq 1 \Rightarrow \gamma(\mathcal{A}) = 0$,

(AM3′) $\gamma(\emptyset) = \infty$,

(AM4′) $\gamma(\mathcal{A} \cup \mathcal{B}) \geq \gamma(\mathcal{A}) \wedge \gamma(\mathcal{B})$.

The function γ is called an *approach merotopy* on X and (X, γ) is an *approach merotopic space*. The relation between an approach merotopy γ (generalizing micromeric collections) and an approach merotopy ν (generalizing near collections) on X is given by $\gamma \mapsto \nu_\gamma$ and $\nu \mapsto \gamma_\nu$, where, for any $\mathcal{A} \in \mathcal{P}^2(X)$, $\nu_\gamma(\mathcal{A}) = \gamma(\sec \mathcal{A})$ and $\gamma_\nu(\mathcal{A}) = \nu(\sec \mathcal{A})$.

Definition 4.12. An approach merotopic space (X, ν) is called *regular* if one of the following equivalent conditions holds:

(i) $\nu(\mathcal{A}) = \nu(\{B \subseteq X : \text{there is } A \in \mathcal{A} \text{ satisfying } \nu(\{A, X - B\}) > 0\})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$;

(ii) $\gamma(\mathcal{A}) = \nu(\{B \subseteq X : \text{for all } A \in \mathcal{A}, \nu(\{A, B\}) = 0\})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$;

(iii) $\nu(\mathcal{A}) = \gamma(\{B \subseteq X : \text{for all } A \in \mathcal{A}, \nu(\{A, B\}) = 0\})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$;

(iv) $\gamma(\mathcal{A}) = \gamma(\{B \subseteq X : \text{there is } A \in \mathcal{A} \text{ satisfying } \nu(\{A, X - B\}) > 0\})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$.

The above equivalences hold by noting that sec $\{B \subseteq X$: there is $A \in \mathcal{A}$ satisfying $\nu(\{A, X-B\}) > 0\} = \{B \subseteq X$: for all $A \in \mathcal{A}, \nu(\{A, B\}) = 0\}$ and by the following transitions: $\nu(\mathcal{A}) = \gamma(\sec \mathcal{A})$ and $\gamma(\mathcal{A}) = \nu(\sec \mathcal{A})$, for all $\mathcal{A} \in \mathcal{P}^2(X)$.

Definition 4.13. An approach merotopic space (X, ν) is called *separated* if, for all $\mathcal{A} \in \mathcal{P}^2(X)$,

$$\nu(\{A \subseteq X : \nu(\{A\} \cup \mathcal{A}) = 0\}) \leq \nu(\mathcal{A}) \vee \gamma(\mathcal{A}). \tag{4.3}$$

Proposition 4.14. *Every regular approach merotopic space is separated.*

Proof. Let (X, ν) be a regular approach merotopic space. Then since $\{A \subseteq X : \nu(\{A\} \cup \mathcal{A}) = 0\} \subseteq \{B \subseteq X$: for all $A \in \mathcal{A}, \nu(\{A, B\}) = 0\}$, therefore

$$\begin{aligned}\nu(\{A \subseteq X : \nu(\{A\} \cup \mathcal{A}) = 0\}) &\leq \nu(\{B \subseteq X : \ \forall A \in \mathcal{A}, \nu(\{A, B\}) = 0\}) \\ &= \gamma(\mathcal{A}) \leq \nu(\mathcal{A}) \vee \gamma(\mathcal{A}).\end{aligned} \tag{4.4}$$

□

A morphism f in **TOP** is a retraction if and only if there exists a topological retraction r and a homeomorphism h such that $f = h \circ r$. In other words, the retractions in **TOP** are (up to homeomorphism) exactly the topological retractions. Therefore a subspace Y of a space X is called a *retract* in **TOP** if there exists a continuous map $f : X \to Y$ with $f(y) = y$ for all $y \in Y$ so that if $e : Y \to X$ is an inclusion, then $f \circ e = \mathrm{id}_Y$ (see [22]).

Proposition 4.15. *Let* (X, ν) *be a regular approach nearness space. Then the map* $g : X^+ \to X^*$ *that is defined as follows: for* $\mathcal{B} \in X^+, g(\mathcal{B}) = \{A \subseteq X : \nu(\{A\} \cup \mathcal{B}) = 0\}$ *is a retraction.*

Proof. Let $\mathcal{B} \in X^+$. Then $\sec \mathcal{B} \subseteq \mathcal{B}$ as $\mathcal{B}$ is a grill on X yielding $\gamma(\mathcal{B}) = 0$ which in turn gives that $\nu(g(\mathcal{B})) = 0$ applying the separability of ν. Thus $g(\mathcal{B})$ is a ν-cluster. Since $\mathcal{B} \subseteq g(\mathcal{B})$, therefore $\mathcal{B} = g(\mathcal{B})$, whenever $\mathcal{B} \in X^*$. To prove that g is continuous, let $\omega \subseteq X^+$ and $\mathcal{B} \in \mathrm{cl}_{X^+}(\omega)$. Then we only need to show that $\bigcap g(\omega) \subseteq g(\mathcal{B})$. On the contrary if there is an $A \in \bigcap g(\omega)$ such that $A \notin g(\mathcal{B})$, then $\nu(\{A\} \cup \mathcal{B}) \neq 0$. But $\nu(\mathcal{B} \cup \{A\}) = \nu(\{D \subseteq X : \nu(\{A, X-D\}) > 0$ or there is $B \in \mathcal{B}$ satisfying $\nu(\{B, X-D\}) > 0\})$. Taking $\mathfrak{D} = \{D \subseteq X : \nu(\{A, X-D\}) > 0$ or there is $B \in \mathcal{B}$ satisfying $\nu(\{B, X-D\}) > 0\}$, we get, by regularity of $\nu, 0 \neq \nu(\mathfrak{D}) \nleq \nu(\mathcal{B}) = 0$ and so $\mathfrak{D} \not\prec \mathcal{B}$. Since $\bigcap \omega \subseteq \mathcal{B}$, therefore $D \notin \bigcap \omega$. Consequently there exists $\mathcal{H} \in \omega$ such that $D \notin \mathcal{H}$. Also $A \in g(\mathcal{H})$ and $\nu(\{A, X-D\}) > 0$ imply that $\nu(\{A\} \cup \mathcal{H}) = 0$. But $\mathcal{H}$ is a grill implying that $X - D \in \mathcal{H}$ and therefore $\nu(\{A, X-D\}) = 0$, a contradiction. So, $A \in g(\mathcal{B})$ and hence $\bigcap g(\omega) \subseteq g(\mathcal{B})$. To show that g is a contraction, let $\Omega \subseteq \mathcal{P}(X^+), \mathcal{A} = \bigcup\{\bigcap \tau : \tau \in g(\Omega)\}$, and $\mathcal{E} = \{E \subseteq X$: there is $A \in \mathcal{A}$ satisfying $\nu(\{A, X-E\}) > 0\}$. Then $\mathcal{E} \subseteq \bigcup\{\bigcap \omega : \omega \in \Omega\}$: for if $E \in \mathcal{E}$, then there exists $A \in \mathcal{A}$ such that $\nu(\{A, X-E\}) > 0$, for some $E \subseteq X$. Suppose that $E \notin \bigcap \omega$, for all $\omega \in \Omega$. Then for $\omega \in \Omega$, there exists $\mathcal{B} \in \omega$ such that $E \notin \mathcal{B}$ which yields that $X - E \in \mathcal{B}$ because $\mathcal{B}$ is a grill. Thus $\{A, X-E\} \prec \{A\} \cup \mathcal{B}$, where $A \in \mathcal{A}$. So $\nu(\{A, X-E\}) \leq \nu(\{A\} \cup \mathcal{B}) = 0$ which gives that $\nu(\{A, X-E\}) = 0$ for all $A \in \mathcal{A}$, a contradiction. Thus $\nu^*(g(\Omega)) = \nu(\mathcal{A}) = \nu(\mathcal{E}) \leq \nu^+(\Omega)$. Hence g is a contraction. □

5. Concluding Remark and Future Applications

The present paper is a unified study of the categories **MET**, **TOP**, **ANEAR**, **AMER**, **AP**, and **AMT**. Such type of unified study is relevant as can be seen in [16]. Various examples that are provided support the existence of approach merotopological spaces. Given an approach merotopological space, we can obtain a new approach merotopological space in general, employing the method given in Example 3.2(i) and (iv). Since **ANEAR** $\subseteq$ **AMT**, therefore in this paper we have obtained a bunch completion (X^+, ν^+) of an approach nearness space (X, ν). In [21], we have obtained a cluster completion (X^*, ν^*) of the same structure. In fact, X^* is a retract of X^+, for a regular approach nearness space.

In 1988, Smyth [26] suggested that nearness-like concepts provide a suitable vehicle for the study of problems in Theoretical Computer Science. For a comprehensive account with references, see Section 11 (Applications to Theoretical Science) in [27]. Using tools from nearness-like structures, Vakarelov [28] established a topological representation theorem for a connection-based class of systems. More clearly, all digital images used in computer vision and computer graphics can be viewed as nontrivial semi-proximity spaces, which can be seen in [29]. In 2002, work on a perceptual basis for near sets begun, which was motivated by image analysis. This work was inspired by a study of perception of the nearness of familiar physical objects in a philosophical manner in a poem entitled "*How Near*" written in 2002 and published in 2007 [30]. Since an approach merotopology measures how near a collection of sets is and **AMT** is a supercategory of almost all of the nearness-like structures, therefore it is our prediction that applying tools of approach merotopological spaces would help in obtaining better outputs in such studies of computer science.

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