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Research Article

A Hilbert Integral-Type Inequality with Parameters

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A Hilbert-type integral inequality with parameters α and $(\alpha, \lambda > 0)$ can be established by introducing a nonhomogeneous kernel function. And the constant factor is proved to be the best possible. And then some important and especial results are enumerated. As applications, some equivalent forms are studied.

1. Introduction

Let $f(x), g(x) \in L^2(0, +\infty)$. Then

$$\iint_0^\infty \frac{f(x)g(y)}{x+y} dx\, dy \le \pi \left\{ \int_0^\infty f^2(x) dx \right\}^{1/2} \left\{ \int_0^\infty g^2(x) dx \right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible, and the equality in (1.1) holds if and only if $f(x) = 0$, or $g(x) = 0$.This is the famous Hilbert integral inequality (see [1, 2]). Owing to the importance of the Hilbert inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1.1) appear in a great deal of papers (see [3–11], etc.). Specially, Gao and Hsu enumerated the research articles to more than 40 in the paper [6]. It is obvious that the integral kernel function of the left hand side of (1.1) is a homogeneous form of degree −1. In general, the Hilbert type integral inequality with a homogeneous kernel of degree −1 has been studied in the paper [1]. The purpose of the present paper is to establish a Hilbert integral type inequality with a non-homogeneous kernel. And the constant factor is proved to be the best possible. And then some important and especial results are enumerated and some equivalent forms are studied.

For convenience, we need to introduce the Catalan constant and define a real function. The Catalan constant is defined by

$$G = \frac{1}{2}\int_0^1 K dk = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}, \tag{1.2}$$

where K is the complete elliptic integral, namely,

$$K = K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}. \tag{1.3}$$

And the approximation of G is that

$$G = 0,915965594\cdots. \tag{1.4}$$

These results can be found in the paper [12, page 503].

Let $\lambda > 0$. Define a real function by

$$S(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)^{\lambda+1}}. \tag{1.5}$$

In particular, $S(1) = G$, where G is the Catalan constant.

In order to prove our main results, we need the following lemmas.

Lemma 1.1. *Let a be a positive number and $b > -1$. Then*

$$\int_0^{\infty} x^b e^{-ax} dx = \frac{\Gamma(b+1)}{a^{b+1}}. \tag{1.6}$$

Proof. According to the definition of the gamma function, we obtain immediately (1.6). This result can be also found in the paper [12, page 226, formula (1053)]. □

Lemma 1.2. *Let $\alpha, \lambda > 0$. Define a function by*

$$C(\alpha, \lambda) = 2\left(\frac{2}{\alpha}\right)^{\lambda+1} \Gamma(\lambda+1) S(\lambda), \tag{1.7}$$

where $\Gamma(z)$ is the gamma function, and $S(\lambda)$ is a function defined by (1.5)*. Then*

$$\int_0^{\infty} k(1,u) u^{\alpha/2-1} du = C(\alpha, \lambda), \tag{1.8}$$

where $k(1,u) = |\ln(1/u)|^{\lambda} / (1 + u^{\alpha})$.

Proof. It is easy to deduce that

$$\begin{aligned}\int_0^\infty k(1,u)u^{\alpha/2-1}du &= \int_0^1 \frac{(\ln(1/u))^\lambda}{1+u^\alpha}u^{\alpha/2-1}du + \int_1^\infty \frac{(\ln u)^\lambda}{1+u^\alpha}u^{\alpha/2-1}du \\ &= \int_0^1 \frac{(\ln(1/u))^\lambda}{1+u^\alpha}u^{(\alpha/2)-1}du + \int_0^1 \frac{(\ln(1/v))^\lambda}{1+v^\alpha}u^{(\alpha/2)-1}du \\ &= 2\int_0^1 \frac{(\ln(1/u))^\lambda}{1+u^\alpha}u^{(\alpha/2)-1}du = 2\int_0^\infty \frac{t^\lambda e^{-(\alpha/2)t}}{1+e^{-\alpha t}}dt.\end{aligned} \tag{1.9}$$

Expanding $1/(1+e^{-\alpha t})$ into power series of $e^{-\alpha t}$ and then using Lemma 1.1, we have

$$\begin{aligned}\int_0^\infty \frac{t^\lambda e^{-(\alpha/2)t}}{1+e^{-\alpha t}}dt &= \int_0^\infty t^\lambda e^{-(\alpha/2)t}\left(1-e^{-\alpha t}+e^{-2\alpha t}+e^{-3\alpha t}-e^{-4\alpha t}+\cdots\right)dt \\ &= \int_0^\infty \left(t^\lambda e^{-(\alpha/2)t} - t^\lambda e^{-(3/2)\alpha t} + t^\lambda e^{-(5/2)\alpha t} - t^\lambda e^{-(7/2)\alpha t}+\cdots\right)dt \\ &= \left(\frac{2}{\alpha}\right)^{\lambda+1}\Gamma(\lambda+1)\sum_{n=0}^\infty (-1)^n \frac{1}{(2n+1)^{\lambda+1}} \\ &= \left(\frac{2}{\alpha}\right)^{\lambda+1}\Gamma(\lambda+1)S(\lambda).\end{aligned} \tag{1.10}$$

It follows from (1.9), (1.10), and (1.7) that the equality (1.8) holds. □

2. Main Results

In the section we will formulate our main results.

Theorem 2.1. *Let f and g be two real functions, and let α and λ be arbitrary two positive numbers. If $\int_0^\infty x^{1-\alpha}f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\alpha}g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{|\ln xy|^\lambda f(x)g(y)}{1+(xy)^\alpha}dx\,dy \le C(\alpha,\lambda)\left\{\int_0^\infty x^{1-\alpha}f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty x^{1-\alpha}g^2(x)dx\right\}^{1/2}, \tag{2.1}$$

where $C(\alpha,\lambda)$ is defined by (1.7). And the constant factor $C(\alpha,\lambda)$ in (2.1) is the best possible. And the equality in (2.1) holds if and only if $f(x)=0$, or $g(x)=0$.

Proof. We may apply Hardy's technique and Cauchy-Schwarz's inequality to estimate the left-hand side of (2.1) as follows:

$$\begin{aligned}
&\iint_0^\infty \frac{|\ln xy|^\lambda f(x)g(y)}{1+\ (xy)^\alpha}dx\,dy\\
&=\iint_0^\infty \frac{|\ln xy|^{\lambda/2} f(x)}{\left(1+(xy)^\alpha\right)^{1/2}}\left(\frac{x}{y}\right)^{(2-\alpha)/4}\frac{|\ln xy|^{\lambda/2} f(x)}{\left(1+\ (xy)^\alpha\right)^{1/2}}\left(\frac{y}{x}\right)^{(2-\alpha)/4}g(y)dx\,dy\\
&\le\left\{\iint_0^\infty \frac{|\ln xy|^\lambda}{1+\ (xy)^\alpha}\left(\frac{x}{y}\right)^{(2-\alpha)/2}f^2(x)dx\,dy\right\}^{1/2}\left\{\iint_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha}\left(\frac{y}{x}\right)^{(2-\alpha)/2}g^2(y)dx\,dy\right\}^{1/2}\\
&=\left\{\int_0^\infty \omega(\alpha,\lambda,x)f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty \omega(\alpha,\lambda,x)g^2(x)dx\right\}^{1/2},
\end{aligned}\tag{2.2}$$

where $\omega(\alpha,\lambda,x)=\int_0^\infty(|\ln xy|^\lambda/(1+(xy)^\alpha))(x/y)^{(2-\alpha)/2}dy$.

By substituting u for xy, it is easy to deduce that

$$\begin{aligned}
\omega(\alpha,\lambda,x)&=\int_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha}\left(\frac{x}{y}\right)^{(2-\alpha)/2}dy\\
&=x^{1-\alpha}\int_0^\infty \frac{|\ln u|^\lambda}{1+u^\alpha}\left(\frac{1}{u}\right)^{(2-\alpha)/2}du\\
&=x^{1-\alpha}\int_0^\infty \frac{|\ln(1/u)|^\lambda}{1+u^\alpha}u^{(\alpha/2)-1}du.
\end{aligned}\tag{2.3}$$

Based on (1.8), we obtain

$$\omega(\alpha,\lambda,x)=x^{1-\alpha}C(\alpha,\lambda).\tag{2.4}$$

It is known from (2.2) and (2.4) that the inequality (2.1) is valid.

If $f(x)=0$, or $g(x)=0$, then the equality in (2.1) obviously holds. If $f(x)\neq 0$ and $g(x)\neq 0$, then

$$0<\int_0^\infty x^{1-\alpha}f^2(x)dx<+\infty,\qquad 0<\int_0^\infty x^{1-\alpha}g^2(x)dx<+\infty.\tag{2.5}$$

If (2.2) takes the form of the equality, then there exists a pair of non-zero constants c_1 and c_2 such that

$$c_1 \frac{|\ln xy|^{\lambda}}{1+(xy)^{\alpha}} f^2(x)\left(\frac{x}{y}\right)^{1-\alpha/2} = c_2 \frac{|\ln xy|^{\lambda}}{1+(xy)^{\alpha}} g^2(y)\left(\frac{y}{x}\right)^{1-\alpha/2} \quad \text{a.e. on } (0,+\infty)\times(0,+\infty). \tag{2.6}$$

Then we have

$$c_1 x^{2-\alpha} f^2(x) = c_2 y^{2-\alpha} g^2(y) = C_0. \text{ (constant)} \quad \text{a.e. on } (0,+\infty)\times(0,+\infty). \tag{2.7}$$

Without losing the generality, we suppose that $c_1 \neq 0$; then

$$\int_0^{\infty} x^{1-\alpha} f^2(x)dx = \frac{C_0}{c_1}\int_0^{\infty} x^{-1}dx. \tag{2.8}$$

This contradicts that $0 < \int_0^{\infty} x^{1-\alpha} f^2(x)dx < +\infty$. Hence it is impossible to take the equality in (2.2). It shows that it is also impossible to take the equality in (2.1).

It remains to need only to show that $C(\alpha,\lambda)$ in (2.1) is the best possible. Below we will apply Yang's technique (see [13]) to verify our assertion.

For all $n \in N$, define two functions by

$$f_n(x) = \begin{cases} x^{(\alpha/2)-1+1/2n} & x \in (0,1), \\ 0 & x \in [1,\infty), \end{cases}$$
$$g_n(x) = \begin{cases} 0 & x \in (0,1], \\ x^{(\alpha/2)-1-1/2n} & x \in (1,\infty). \end{cases} \tag{2.9}$$

Then we have

$$\left(\int_0^1 x^{1-\alpha} f_n^2(x)dx\right)^{1/2} = \left(\int_1^{\infty} x^{1-\alpha} g_n^2(x)dx\right)^{1/2} = \sqrt{n}. \tag{2.10}$$

Let $0 \le k \le C(\alpha,\lambda)$ such that the inequality (2.1) is still valid when $C(\alpha,\lambda)$ is replaced by k. Specially, we have

$$\frac{1}{n}\iint_0^{\infty} \frac{|\ln xy|^{\lambda} f_n(x) g_n(y)}{1+(xy)^{\alpha}} dx\,dy \le k\left(\frac{1}{n}\right)\left\{\int_0^{\infty} x^{1-\alpha} f_n^2(x)dx\right\}^{1/2}\left\{\int_0^{\infty} x^{1-\alpha} g_n^2(x)dx\right\}^{1/2} = k. \tag{2.11}$$

Let $k(1,xy) = |\ln xy|^{\lambda}/(1+(xy)^{\alpha})$. By Fubini's theorem, we have

$$\begin{aligned}
k &\geq \frac{1}{n}\iint_0^{\infty} k(1,xy) f_n(x) g_n(y)\,dx\,dy \\
&= \frac{1}{n}\int_1^{\infty} y^{(\alpha/2)-1-1/2n}\left(\int_0^1 k(1,xy)x^{(\alpha/2)-1+1/2n}dx\right)dy \\
&= \frac{1}{n}\int_1^{\infty} y^{-1-1/n}\left(\int_0^y k(1,u)u^{(\alpha/2)-1+1/2n}du\right)dy \\
&= \frac{1}{n}\left\{\int_1^{\infty} y^{-1-1/n}\left(\int_0^1 k(1,u)u^{(\alpha/2)-1+1/2n}du\right)dy \right.\\
&\qquad \left. + \int_1^{\infty} y^{-1-1/n}\left(\int_1^y k(1,u)u^{(\alpha/2)-1+1/2n}du\right)dy\right\} \\
&= \frac{1}{n}\left\{\int_1^{\infty} n\left(\int_0^1 k(1,u)u^{(\alpha/2)-1+1/2n}du\right) \right.\\
&\qquad \left. + \int_1^{\infty} k(1,u)u^{(\alpha/2)-1+1/2n}\left(\int_u^{\infty} y^{-1-1/n}dy\right)du\right\} \\
&= \int_0^1 k(1,u)u^{(\alpha/2)-1+1/2n}du + \int_1^{\infty} k(1,u)u^{(\alpha/2)-1-1/2n}du.
\end{aligned} \tag{2.12}$$

By Fatou's lemma and (1.8), we have

$$\begin{aligned}
k &\geq \varliminf_{n\to\infty}\int_0^1 k(1,u)u^{\alpha/2-1+1/2n}du + \varliminf_{n\to\infty}\int_1^{\infty} k(1,u)u^{\alpha/2-1-1/2n}du \\
&\geq \int_0^1 \varliminf_{n\to\infty} k(1,u)u^{\alpha/2-1+1/2n}du + \int_1^{\infty} \varliminf_{n\to\infty} k(1,u)u^{\alpha/2-1-1/2n}du \\
&= \int_0^1 k(1,u)u^{\alpha/2-1}du + \int_1^{\infty} k(1,u)u^{\alpha/2-1}du = \int_0^{\infty} k(1,u)u^{\alpha/2-1}du = C(\alpha,\lambda).
\end{aligned} \tag{2.13}$$

It follows that $k = C(\alpha,\lambda)$ in (2.1) is the best possible. Thus the proof of theorem is completed. □

Based on Theorem 2.1, we may build some important and interesting inequalities.

Theorem 2.2. *Let n be a nonnegative integer and $\alpha > 0$. If $\int_0^\infty x^{1-\alpha} f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\alpha} g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{(\ln xy)^{2n} f(x)g(y)}{1+(xy)^\alpha} dx\,dy \leq C(\alpha, 2n)\left\{\int_0^\infty x^{1-\alpha} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty x^{1-\alpha} g^2(x)dx\right\}^{1/2}, \tag{2.14}$$

where $C(\alpha, 2n) = (\pi/\alpha)^{2n+1} E_n$,the $E_{n's}$ are the Euler numbers, namely, $E_0 = 1, E_1 = 1, E_2 = 5$, $E_3 = 61, E_4 = 1385$, and so forth, and the constant factor $C(\alpha, 2n)$ in (2.14) is the best possible. And the equality in (2.14) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We need only to verify the constant factor $C(\alpha, 2n)$ in (2.14). When $\lambda = 2n$, it is known from (1.10) that

$$C(\alpha, 2n) = 2\left(\frac{2}{\alpha}\right)^{2n+1} \Gamma(2n+1)\sum_{k=0}^\infty (-1)^k \frac{1}{(2k+1)^{2n+1}}. \tag{2.15}$$

It is known from the paper [14] that

$$\sum_{k=0}^\infty (-1)^k \frac{1}{(2k+1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_n, \tag{2.16}$$

where the $E_{n's}$ are the Euler numbers, namely, $E_0 = 1, E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385$, and so forth.

It follows that the constant factor $C(\alpha, 2n)$ in (2.14) is correct. □

In particular, based on Theorem 2.2, the following results are gotten.

Corollary 2.3. *If $\int_0^\infty f^2(x)dx < +\infty$ and $\int_0^\infty g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{f(x)g(y)}{1+xy} dx\,dy \leq \pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{2.17}$$

where the constant factor π in (2.17) is the best possible. And the equality in (2.17) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Corollary 2.4. *If $\int_0^\infty f^2(x)dx < +\infty$ and $\int_0^\infty g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{(\ln xy)^2 f(x)g(y)}{1+xy} dx\,dy \leq \pi^3\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{2.18}$$

where the constant factor π^3 in (2.18) is the best possible. And the equality in (2.18) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Theorem 2.5. *Let $f$and g be two real functions, and α a positive number. If $\int_0^\infty x^{1-\alpha} f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\alpha} g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{|\ln xy| f(x)g(y)}{1+(xy)^\alpha} dx\, dy \leq \frac{8G}{\alpha^2} \left\{ \int_0^\infty x^{1-\alpha} f^2(x)dx \right\}^{1/2} \left\{ \int_0^\infty x^{1-\alpha} g^2(x)dx \right\}^{1/2}, \tag{2.19}$$

where G is the Catalan constant, that is, $G = 0,915965594\cdots$. And the constant factor $8G/\alpha^2$ in (2.19) *is the best possible. And the equality in* (2.19) *holds if and only if $f(x) = 0$, or $g(x) = 0$.*

Proof. We need only to verify the constant factor in (2.19). For case $\lambda = 1$, based on (1.7), (1.5) and (1.2), it is easy to deduce that the constant factor is that $C(\alpha, 1) = 8G/\alpha^2$. □

In particular, when $\alpha = \lambda = 1$, the following result is obtained

Corollary 2.6. *Let f and g be two real functions. If $\int_0^\infty f^2(x)dx < +\infty$ and $\int_0^\infty g^2(x)dx < +\infty$, then*

$$\iint_0^\infty \frac{|\ln xy| f(x)g(y)}{1+xy} dx\, dy \leq 8G \left\{ \int_0^\infty f^2(x)dx \right\}^{1/2} \left\{ \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{2.20}$$

where G is the Catalan constant, that is, $G = 0,915965594\cdots$. And the constant factor $8G$ in (2.20) *is the best possible. And the equality in* (2.20) *holds if and only iff $f(x) = 0$, or $g(x) = 0$.*

A great deal of the Hilbert type inequalities can be established provided that the parameters are properly selected

3. Some Equivalent Forms

As applications, we will build some new inequalities.

Theorem 3.1. *Let f be a real function, and let α and λ be arbitrary two positive numbers. If $\int_0^\infty x^{1-\alpha} f^2(x)dx < +\infty$, then*

$$\int_0^\infty y^{\alpha-1} \left\{ \int_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)dx \right\}^2 dy \leq (C(\alpha,\lambda))^2 \int_0^\infty x^{1-\alpha} f^2(x)dx, \tag{3.1}$$

where $C(\alpha, \lambda)$ is defined by (1.7) *and the constant factor $(C(\alpha, \lambda))^2$ in* (3.1) *is the best possible. And the equality in* (3.1) *holds if and only if $f(x) = 0$. And the inequality* (3.1) *is equivalent to* (2.1).

Proof. First, we assume that the inequality (2.1) is valid. Define a function $g(y)$ by

$$g(y) = y^{\alpha-1}\int_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)dx, \quad y \in (0,+\infty). \tag{3.2}$$

By using (2.1), we have

$$\begin{aligned}
&\int_0^\infty y^{\alpha-1}\left\{\int_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)dx\right\}^2 dy \\
&= \iint_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)g(y)dx\,dy \\
&\le C(\alpha,\lambda)\left\{\int_0^\infty x^{1-\alpha} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{1-\alpha} g^2(y)dy\right\}^{1/2} \\
&= C(\alpha,\lambda)\left\{\int_0^\infty x^{1-\alpha} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{\alpha-1}\left(\int_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)dx\right)^2 dy\right\}^{1/2}.
\end{aligned} \tag{3.3}$$

It follows from (3.3) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid; by applying in turn Cauchy's inequality and (3.1), we have

$$\begin{aligned}
&\iint_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)g(y)dx\,dy \\
&= \int_0^\infty y^{(\alpha-1)/2}\left\{\int_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)dx\right\} y^{(\alpha-1)/2} g(y)dy \\
&\le \left\{\int_0^\infty y^{\alpha-1}\left(\int_0^\infty \frac{|\ln xy|^\lambda}{1+(xy)^\alpha} f(x)dx\right)^2 dy\right\}^{1/2}\left\{\int_0^\infty y^{1-\alpha} g^2(y)dy\right\}^{1/2} \\
&\le \left\{(C(\alpha,\lambda))^2\int_0^\infty x^{1-\alpha} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{1-\alpha} g^2(y)dy\right\}^{1/2} \\
&= C(\alpha,\lambda)\left\{\int_0^\infty x^{1-\alpha} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty y^{1-\alpha} g^2(y)dy\right\}^{1/2}.
\end{aligned} \tag{3.4}$$

Therefore, the inequality (3.1) is equivalent to (2.1).

If the constant factor $(C(\alpha,\lambda))^2$ in (3.1) is not the best possible, then it is known from (3.4) that the constant factor $C(\alpha,\lambda)$ in (2.1) is also not the best possible. This is a contradiction. The theorem is proved. □

Theorem 3.2. *Let n be a nonnegative integer and $\alpha > 0$. If $\int_0^\infty x^{1-\alpha} f^2(x)dx < +\infty$, then*

$$\int_0^\infty y^{\alpha-1}\left\{\int_0^\infty \frac{(\ln xy)^{2n}}{1+(xy)^\alpha} f(x)dx\right\}^2 dy \le (C(\alpha,2n))^2 \int_0^\infty x^{1-\alpha} f^2(x)dx, \tag{3.5}$$

where $C(\alpha,2n) = (\pi/\alpha)^{2n+1}E_n$, the $E_{n's}$ are the Euler numbers,namely, $E_0 = 1, E_1 = 1, E_2 = 5, E_3 = 61$, $E_4 = 1385$, and so forth, and the constant factor $C(\alpha,2n)$ is the best possible. And the equality in (3.5) *holds if and only if $f(x) = 0$. And the inequality* (3.5) *is equivalent to* (2.14).

Theorem 3.3. *Let f and g be two real functions, and let α be a positive numbers. If $\int_0^\infty x^{1-\alpha} f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\alpha} g^2(x)dx < +\infty$, then*

$$\int_0^\infty y^{\alpha-1}\left\{\int_0^\infty \frac{|\ln xy|}{1+(xy)^\alpha} f(x)dx\right\}^2 dy \le \frac{8G}{\alpha^2}\left\{\int_0^\infty x^{1-\alpha} f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty x^{1-\alpha} g^2(x)dx\right\}^{1/2}, \tag{3.6}$$

where G is the Catalan constant, that is, $G = 0,915965594\cdots$. And the constant factor $8G/\alpha^2$ in (3.6) *is the best possible. And the equality in* (3.6) *holds if and only if $f(x) = 0$. And the inequality* (3.6) *is equivalent to* (2.19).

The proofs of Theorems 3.2 and 3.3 are similar to one of Theorem 3.1; they are omitted here.

Similarly, we can establish also some new inequalities which they are, respectively, equivalent to the inequalities (2.17), (2.18) and (2.20). These are omitted here.

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