

Research Article

Existence of Concave Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation with p -Laplacian Operator

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We consider the existence and multiplicity of concave positive solutions for boundary value problem of nonlinear fractional differential equation with p -Laplacian operator $D_{0+}^{\alpha}(\phi_p(D_{0+}^{\alpha}u(t))) + f(t, u(t), D_{0+}^{\rho}u(t)) = 0$, $0 < t < 1$, $u(0) = u'(1) = 0$, $u''(0) = 0$, $D_{0+}^{\alpha}u(t)|_{t=0} = 0$, where $0 < \gamma < 1$, $2 < \alpha < 3$, $0 < \rho \leq 1$, D_{0+}^{α} denotes the Caputo derivative, and $f : [0, 1] \times [0, +\infty) \times \mathbb{R} \rightarrow [0, +\infty)$ is continuous function, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$. By using fixed point theorem, the results for existence and multiplicity of concave positive solutions to the above boundary value problem are obtained. Finally, an example is given to show the effectiveness of our works.

1. Introduction

As we know, boundary value problems of integer-order differential equations have been intensively studied; see [1–5] and therein. Recently, due to the wide development of its theory of fractional calculus itself as well as its applications, fractional differential equations have been constantly attracting attention of many scholars; see, for example, [6–15].

In [7], Jafari and Gejji used the adomian decomposition method for solving the existence of solutions of boundary value problem:

$$\begin{aligned} D^{\alpha}u(t) + \mu f(t, u(t)) &= 0, & 0 < t < 1, & 1 < \alpha \leq 2, \\ u(0) &= 0, & u(1) &= b. \end{aligned} \tag{1.1}$$

In [9], by using fixed point theorems on cones, Dehghani and Ghanbari considered triple positive solutions of nonlinear fractional boundary value problem:

$$\begin{aligned} D^\alpha x(t) + q(t)f(t, x(t), x'(t)) &= 0, \quad 0 < t < 1, \quad 2 \leq \alpha < 3, \\ x(0) = x(1) &= 0, \end{aligned} \quad (1.2)$$

where D^α is the standard Riemann-Liouville derivative. But we think that Green's function in [9] is wrong; if $\alpha > 2$, then, Green's function cannot be decided by $x(0) = x(1) = 0$.

In [11], using fixed point theorems on cones, Zhang investigated the existence and multiplicity of positive solutions of the following problem:

$$\begin{aligned} D_{0+}^\alpha u(t) &= f(t, u(t)), \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ u(0) + u'(0) &= 0, \quad u(1) + u'(1) = 0, \end{aligned} \quad (1.3)$$

where D_{0+}^α is the Caputo fractional derivative.

In [12], by means of Schauder fixed-point theorem, Su and Liu studied the existence of nonlinear fractional boundary value problem involving Caputo's derivative:

$$\begin{aligned} D_{0+}^\alpha u(t) &= f\left(t, u(t), D_{0+}^\beta u(t)\right) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ u(0) = u'(1) &= 0, \quad \text{or} \quad u'(0) = u(1) = 0, \quad \text{or} \quad u(0) = u(1) = 0. \end{aligned} \quad (1.4)$$

To the best of our knowledge, the existence of concave positive solutions of fractional order equation is seldom considered and investigated. Motivated by the above arguments, the main objective of this paper is to investigate the existence and multiplicity of concave positive solutions of boundary value problem of fractional differential equation with p -Laplacian operator as follows:

$$\begin{aligned} D_{0+}^\gamma (\phi_p(D_{0+}^\alpha u(t))) + f\left(t, u(t), D_{0+}^\rho u(t)\right) &= 0, \quad 0 < t < 1, \\ u(0) = u'(1) = 0, \quad u''(0) = 0, \quad D_{0+}^\alpha u(t)|_{t=0} &= 0, \end{aligned} \quad (1.5)$$

where $0 < \gamma < 1$, $2 < \alpha < 3$, $0 < \rho \leq 1$, D_{0+}^α denotes the Caputo derivative, and $f : [0, 1] \times [0, +\infty) \times \mathbb{R} \rightarrow [0, +\infty)$ is continuous function, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $1/p + 1/q = 1$.

By using fixed point theorem, some results for multiplicity of concave positive solutions to the above boundary value problems are obtained. Finally, an example is given to show the effectiveness of our works.

The rest of the paper is organized as follows. In Section 2, we will introduce some lemmas and definitions which will be used later. In Section 3, the multiplicity of concave positive solutions for the boundary value problem (1.5) will be discussed.

2. Basic Definitions and Preliminaries

Firstly we present here some necessary definitions and lemmas.

Definition 2.1 (see [6, 16, 17]). The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \rightarrow R$ is given by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \quad (2.1)$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 (see [6, 16, 17]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \rightarrow R$ is given by

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds, \quad (2.2)$$

where $n = [\alpha] + 1$ provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.3 (see [17]). Caputo's derivative of order $\alpha > 0$ of a function $y : (0, \infty) \rightarrow R$ is defined as

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds, \quad n-1 < \alpha < n, \quad (2.3)$$

provided that the right side is pointwise defined on $(0, \infty)$.

Remark 2.4. The following properties are well known:

- (1) $D_{0+}^\alpha D_{0+}^\beta y(t) = D_{0+}^{\alpha+\beta} y(t)$, $\alpha > 0$, $\beta > 0$;
- (2) $D_{0+}^{-\alpha} y(t) = I_{0+}^\alpha y(t)$, $\alpha > 0$;
- (3) $I_{0+}^\alpha : C[0, 1] \rightarrow C[0, 1]$, $\alpha > 0$.

If α is an integer, the derivative for order α is understood in the sense of usual differentiation.

Definition 2.5. Let E be a real Banach space over R . A nonempty convex closed set $P \subset E$ is said to be a cone provided that

- (a) $au \in P$, for all $u \in P$, $a \geq 0$;
- (b) $u, -u \in P$, implies $u = 0$.

Definition 2.6. Let E be a real Banach space and let $P \subset E$ be a cone. A function $\varphi : P \rightarrow [0, \infty)$ is called a nonnegative continuous concave functional if φ is continuous and

$$\varphi(\lambda x + (1 - \lambda)y) \geq \lambda\varphi(x) + (1 - \lambda)\varphi(y), \quad (2.4)$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$.

Definition 2.7. Let E be a real Banach space and let $P \subset E$ be a cone. A function $\varphi : P \rightarrow [0, \infty)$ is called a nonnegative continuous convex functional if φ is continuous and

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y), \quad (2.5)$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$.

Suppose that $\varphi, \theta : P \rightarrow [0, +\infty)$ are two nonnegative continuous convex functionals satisfying

$$\|u\| \leq M \max\{\varphi(u), \theta(u)\}, \quad \text{for } u \in P, \quad (2.6)$$

where M is a positive constant and

$$\Omega = \{u \in P \mid \varphi(u) < r, \theta(u) < L\} \neq \emptyset, \quad \text{for } r > 0, L > 0. \quad (2.7)$$

Let $r > a > 0, L > 0$ be given constants, let $\varphi, \theta : P \rightarrow [0, +\infty)$ be two nonnegative continuous convex functionals satisfying (2.6) and (2.7), and let ψ be a nonnegative continuous concave functional on the cone P . Define the bounded convex sets:

$$\begin{aligned} P(\varphi, r; \theta, L) &= \{u \in P \mid \varphi(u) < r, \theta(u) < L\}, \\ \bar{P}(\varphi, r; \theta, L) &= \{u \in P \mid \varphi(u) \leq r, \theta(u) \leq L\}, \\ P(\varphi, r; \theta, L; \psi, a) &= \{u \in P \mid \varphi(u) < r, \theta(u) < L, \psi(u) > a\}, \\ \bar{P}(\varphi, r; \theta, L; \psi, a) &= \{u \in P \mid \varphi(u) \leq r, \theta(u) \leq L, \psi(u) \geq a\}. \end{aligned} \quad (2.8)$$

Lemma 2.8 (see [2, 5]). Let E be a Banach space, let $P \subset E$ be a cone, and let $r_2 \geq c > b > r_1 > 0, L_2 \geq L_1 > 0$ be given constants. Assume that φ, θ are two nonnegative continuous convex functionals on P , such that (2.6) and (2.7) are satisfied; let ψ be a nonnegative continuous concave functional on P , such that $\psi(u) \leq \varphi(u)$ for all $u \in \bar{P}(\varphi, r_2; \theta, L_2)$ and let $T : \bar{P}(\varphi, r_2; \theta, L_2) \rightarrow \bar{P}(\varphi, r_2; \theta, L_2)$ be

a completely continuous operator. Suppose the following:

$$(C_1) \quad \{u \in \bar{P}(\varphi, c; \theta, L_2; \varphi, b) \mid \varphi(u) > b\} \neq \emptyset, \varphi(Tu) > b \text{ for all } u \in \bar{P}(\varphi, c; \theta, L_2; \varphi, b);$$

$$(C_2) \quad \varphi(Tu) < r_1, \theta(Tu) < L_1, \text{ for all } u \in \bar{P}(\varphi, r_1; \theta, L_1);$$

$$(C_3) \quad \varphi(Tu) > b, \text{ for all } u \in \bar{P}(\varphi, r_2; \theta, L_2; \varphi, b) \text{ with } \varphi(Tu) > c.$$

Then T has at least three fixed points u_1, u_2 , and u_3 in $\bar{P}(\varphi, r_2; \theta, L_2)$. Further $u_1 \in P(\varphi, r_1; \theta, L_1)$, $u_2 \in \{\bar{P}(\varphi, r_2; \theta, L_2; \varphi, b) \mid \varphi(u) > b\}$, and $u_3 \in \bar{P}(\varphi, r_2; \theta, L_2) \setminus (\bar{P}(\varphi, r_2; \theta, L_2; \varphi, b) \cup \bar{P}(\varphi, r_1; \theta, L_1))$.

Lemma 2.9 (see [12]). Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order α that belongs to $C(0, 1) \cap L(0, 1)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_0 + C_1 t + \cdots + C_{n-1} t^{n-1}, \quad (2.9)$$

for some $C_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$, where n is the smallest integer greater than or equal to α .

Lemma 2.10. Let $y \in C[0, 1]$; then the boundary value problem

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \quad (2.10)$$

$$u(0) = u'(1) = u''(0) = 0 \quad (2.11)$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) y(s) ds, \quad (2.12)$$

where

$$G(t, s) = \begin{cases} \frac{(\alpha - 1)t(1 - s)^{\alpha-2} - (t - s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(\alpha - 1)t(1 - s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.13)$$

Proof. We may apply Lemma 2.9 to reduce (2.10) to an equivalent integral equation:

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds + C_0 + C_1 t + C_2 t^2. \quad (2.14)$$

From (2.14), we have

$$\begin{aligned} u'(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (\alpha-1)(t-s)^{\alpha-2} y(s) ds + C_1 + 2C_2 t, \\ u''(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (\alpha-1)(\alpha-2)(t-s)^{\alpha-3} y(s) ds + 2C_2, \end{aligned} \quad (2.15)$$

and by (2.10) and (2.11), there are $C_0 = C_2 = 0$, $C_1 = (1/\Gamma(\alpha)) \int_0^1 (\alpha-1)(1-s)^{\alpha-2} y(s) ds$.
Therefore, the unique solution of problem (2.10) and (2.11) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha-1)t(1-s)^{\alpha-2} y(s) ds \\ &= \int_0^t \frac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_t^1 \frac{(\alpha-1)t(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s) ds \\ &= \int_0^1 G(t,s) y(s) ds. \end{aligned} \quad (2.16)$$

□

Lemma 2.11. *The function $G(t,s)$ defined by (2.13) satisfies the following conditions:*

- (1) $G(t,s) \geq 0$, $G(t,s) \leq G(1,s)$ for $0 \leq t, s \leq 1$,
- (2) $G(t,s) \geq t^{\alpha-1} G(1,s)$, for $0 \leq t, s \leq 1$.

Proof. Since

$$\begin{aligned} g(t,s) &=: (\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1} \\ &\geq (\alpha-1)t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1} \\ &= t^{\alpha-1} \left[(\alpha-1)(1-s)^{\alpha-2} - \left(1 - \frac{s}{t}\right)^{\alpha-1} \right] \\ &\geq t^{\alpha-1} \left[(\alpha-1)(1-s)^{\alpha-2} - (1-s)^{\alpha-1} \right] \\ &\geq 0, \quad \text{for } 2 < \alpha < 3, \quad 0 \leq s \leq t \leq 1, \end{aligned} \quad (2.17)$$

observing (2.13), we have $G(t,s) \geq 0$.

Form (2.13), we obtain

$$G'_t(t, s) = \begin{cases} \frac{(\alpha - 1)(1 - s)^{\alpha - 2} - (\alpha - 1)(t - s)^{\alpha - 2}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(\alpha - 1)(1 - s)^{\alpha - 2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.18)$$

Clearly, $G'_t(t, s) \geq 0$, for $0 \leq t, s \leq 1$, we have that $G(t, s)$ is increasing with respect to $t \in [0, 1]$, and therefore, $G(t, s) \leq G(1, s)$, for $0 \leq t, s \leq 1$. (1) of Lemma 2.11 holds.

On the other hand, if $t \geq s$, then,

$$\begin{aligned} \frac{G(t, s)}{G(1, s)} &= \frac{(\alpha - 1)t(1 - s)^{\alpha - 2} - (t - s)^{\alpha - 1}}{(\alpha - 1)(1 - s)^{\alpha - 2} - (1 - s)^{\alpha - 1}} \\ &\geq \frac{t^{\alpha - 1} [(\alpha - 1)(1 - s)^{\alpha - 2} - (1 - s)^{\alpha - 1}]}{(\alpha - 1)(1 - s)^{\alpha - 2} - (1 - s)^{\alpha - 1}} \\ &= t^{\alpha - 1}. \end{aligned} \quad (2.19)$$

If $t \leq s$, then $G(t, s)/G(1, s) = t^{\alpha - 1}$; therefore, $G(t, s) \geq t^{\alpha - 1}G(1, s)$, for $0 \leq s, t \leq 1$. (2) of Lemma 2.11 holds. \square

3. Existence of Three Concave Positive Solutions

In this section, we study the existence of concave positive solution for problem (1.5).

Let $E = C^1[0, 1]$. From Definitions 2.1 and 2.3, we obtain $D_{0+}^\rho u(t) = I_{0+}^{1-\rho} u'(t)$, $0 < \rho < 1$, and $D_{0+}^\rho u(t) = u'(t)$, $\rho = 1$. So, by (3) of Remark 2.4, we know that $D_{0+}^\rho u(t)$ is continuous for all $u(t) \in E$. Hence, for all $u(t) \in E$, we can define

$$\|u\| = \begin{cases} \max_{0 \leq t \leq 1} |u(t)| + \max_{0 \leq t \leq 1} |D_{0+}^\rho u(t)| + \max_{0 \leq t \leq 1} |u'(t)|, & 0 < \rho < 1, \\ \max_{0 \leq t \leq 1} |u(t)| + \max_{0 \leq t \leq 1} |u'(t)|, & \rho = 1. \end{cases} \quad (3.1)$$

Lemma 3.1 (see [12]). $(E, \|\cdot\|)$ is a Banach space.

Define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \geq 0, u(t) \text{ is concave on } [0, 1]\}$.

Let the nonnegative continuous concave functional φ and the nonnegative continuous convex functionals θ, θ be defined on the cone P by

$$\begin{aligned} \varphi(u) &= \min_{1/k \leq t \leq (k-1)/k} |u(t)|, & \varphi(u) &= \max_{0 \leq t \leq 1} |u(t)|, \\ \theta(u) &= \begin{cases} \max_{0 \leq t \leq 1} |D_{0+}^{\rho} u(t)| + \max_{0 \leq t \leq 1} |u'(t)|, & 0 < \rho < 1, \\ \max_{0 \leq t \leq 1} |u'(t)|, & \rho = 1. \end{cases} \end{aligned} \quad (3.2)$$

Lemma 3.2 (see [1]). Let $u \in P$, $k \geq 3$; then

$$\min_{1/k \leq t \leq (k-1)/k} |u(t)| \geq \frac{1}{k} \max_{0 \leq t \leq 1} |u(t)|. \quad (3.3)$$

Lemma 3.3. BVP (1.5) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^{\rho} u(\tau)) d\tau \right) ds. \quad (3.4)$$

Proof. From BVP (1.5) and Lemma 2.9, we have

$$\begin{aligned} \phi_p(D_{0+}^{\alpha} u(t)) &= -I_{0+}^{\gamma} f(t, u(t), D_{0+}^{\rho} u(t)) + C, \\ &= -\frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^{\rho} u(\tau)) d\tau + C. \end{aligned} \quad (3.5)$$

By $D_{0+}^{\alpha} u(t)|_{t=0} = 0$, there is $C = 0$, and then,

$$D_{0+}^{\alpha} u(t) = -\phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^{\rho} u(\tau)) d\tau \right). \quad (3.6)$$

Therefore, BVP(1.5) is equivalent to following problem:

$$\begin{aligned} D_{0+}^{\alpha} u(t) + \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^{\rho} u(\tau)) d\tau \right) &= 0, \quad 0 < t < 1, \quad 2 < \alpha < 3, \\ u(0) = u'(1) = u''(0) &= 0. \end{aligned} \quad (3.7)$$

By Lemma 2.10, BVP (1.5) is equivalent to the integral equation (3.4).

Let $T : P \rightarrow E$ be the operator defined by

$$Tu(t) = \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds =: F(t). \quad (3.8)$$

□

Lemma 3.4. $T : P \rightarrow P$ is completely continuous.

Proof. Let $u \in P$; in view of nonnegativeness and continuity of $G(t,s)$ and $f(t,u,v)$, we have $Tu \geq 0$, and $t \in [0,1]$ is continuous:

$$\begin{aligned} (Tu)'(t) &= \int_0^1 G_t'(t,s) \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \\ &= \int_0^t \frac{(\alpha-1)(1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \\ &\quad + \int_t^1 \frac{(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds. \end{aligned} \quad (3.9)$$

Clearly, $(Tu)'(t)$ is continuous for $\alpha < 2$.

By Remark 2.4 and noting (3.4) and (3.6), we have

$$\begin{aligned} D_{0+}^{2-\alpha}(Tu)(t) &= D_{0+}^{2-\alpha} D_{0+}^\alpha (Tu)(t) \\ &= D_{0+}^{2-\alpha} (D_{0+}^\alpha F(t)) \\ &= -D_{0+}^{2-\alpha} \left(\phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) \right) \\ &= -I_{0+}^{\alpha-2} \left(\phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) \right) \\ &= -\frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} \left(\phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right) \\ &\leq 0. \end{aligned} \quad (3.10)$$

So, Tu is concave on $[0,1]$ and $Tu \in C^1[0,1]$; we obtain $T(P) \subset P$.

Let $\Omega \in P$ be bounded; that is, there exists a positive constant $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$.

Let $N = \max_{(t,u,v) \in [0,1] \times [0,M] \times [-M,M]} |f(t, u(t), D_{0+}^\rho u(t))|$; then, for all $u \in \Omega$, we have

$$\begin{aligned}
 |Tu(t)| &= \left| \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right| \\
 &\leq \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \int_0^1 G(1,s) ds \\
 &= \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \left[\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha+1)} \right] \\
 &\leq \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)}, \\
 |Tu'(t)| &= \left| \int_0^1 G'_t(t,s) \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right| \\
 &\leq \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \int_0^1 G'_t(t,s) ds \\
 &\leq \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)}, \\
 |D_{0+}^\rho(Tu)(t)| &= \left| \frac{1}{\Gamma(1-\rho)} \int_0^t (t-s)^{-\rho} (Tu)'(s) ds \right| \\
 &\leq \frac{1}{\Gamma(1-\rho)} \int_0^t (t-s)^{-\rho} \\
 &\quad \times \left| \int_0^1 G'_s(s,v) \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^v (v-\tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right| \\
 &\leq \frac{1}{\Gamma(1-\rho)} \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\rho} ds \\
 &= \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \frac{t^{1-\rho}}{(1-\rho)\Gamma(1-\rho)\Gamma(\alpha)} \\
 &\leq \left(\frac{N}{\Gamma(\gamma+1)} \right)^{q-1} \frac{1}{\Gamma(2-\rho)\Gamma(\alpha)}.
 \end{aligned} \tag{3.11}$$

So, for all $u \in \Omega$, $\|Tu(t)\| \leq (N/\Gamma(\gamma+1))^{q-1} [2/\Gamma(\alpha)+1/\Gamma(2-\rho)\Gamma(\alpha)]$. Hence, $T(\Omega)$ is uniformly bounded.

Since $G(t, s)$ is continuous on $[0, 1] \times [0, 1]$, it is uniformly continuous on $[0, 1] \times [0, 1]$. Thus for fixed $s \in [0, 1]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that any $t_1, t_2 \in [0, 1]$ and $|t_1 - t_2| < \delta$,

$$|G(t_1, s) - G(t_2, s)| < \varepsilon \left(\frac{\Gamma(\gamma + 1)}{N} \right)^{q-1}. \tag{3.12}$$

Therefore,

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\leq \int_0^1 |G(t_2, s) - G(t_1, s)| \phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s - \tau)^{\gamma-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \\ &\leq \left(\frac{N}{\Gamma(\gamma + 1)} \right)^{q-1} \int_0^1 |G(t_2, s) - G(t_1, s)| ds < \varepsilon. \end{aligned} \tag{3.13}$$

That is to say, $T(\Omega)$ is equicontinuous. By the means of the Arzela-Ascoli Theorem, we have that $T : P \rightarrow P$ is completely continuous. The proof is completed. \square

Let

$$\begin{aligned} A &= \left(\frac{1}{\Gamma(\gamma + 1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)}, \\ B &= \left(\frac{1}{k} \right)^{\gamma(q-1)} \left(\frac{1}{\Gamma(\gamma + 1)} \right)^{q-1} \int_{1/k}^1 G(1, s) ds, \\ M &= \left(\frac{1}{\Gamma(\gamma + 1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)\Gamma(2 - \rho)}. \end{aligned} \tag{3.14}$$

Theorem 3.5. *Suppose that there exist constants $0 < r_1 < b < kb \leq r_2$, $L_2 \geq L_1 > 0$, such that $kb/B \leq \min(r_2/A, L_2/2A, L_2/2M)$, and the following conditions hold:*

- (H₁) $f(t, u, v) < \min\{\phi_p(r_1/A), \phi_p(L_1/2A), \phi_p(L_1/2M)\}$, for $(t, u, v) \in [0, 1] \times [0, r_1] \times [-L_1, L_1]$;
- (H₂) $f(t, u, v) > \phi_p(kb/B)$, for $(t, u, v) \in [1/k, 1] \times [b, kb] \times [-L_2, L_2]$;
- (H₃) $f(t, u, v) \leq \min\{\phi_p(r_2/A), \phi_p(L_2/2A), \phi_p(L_2/2M)\}$, for $(t, u, v) \in [0, 1] \times [0, r_2] \times [-L_2, L_2]$.

Then, the BVP (1.5) has at least three concave positive solutions u_1 , u_2 , and u_3 , such that

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| &< r_1, & \max_{0 \leq t \leq 1} \left| D_{0+}^\rho u_1(t) \right| + \max_{0 \leq t \leq 1} |u_1'(t)| &< L_1, \\ b &< \min_{1/k \leq t \leq (k-1)/k} |u_2(t)| &< \max_{0 \leq t \leq 1} |u_2(t)| &\leq r_2 \\ \max_{0 \leq t \leq 1} \left| D_{0+}^\rho u_2(t) \right| + \max_{0 \leq t \leq 1} |u_2'(t)| &< L_2, & & (3.15) \\ r_1 &< \max_{0 \leq t \leq 1} |u_3(t)| \leq kb, & \min_{1/k \leq t \leq (k-1)/k} |u_3(t)| &< b, \\ L_1 &< \max_{0 \leq t \leq 1} \left| D_{0+}^\rho u_3(t) \right| + \max_{0 \leq t \leq 1} |u_3'(t)| &< L_2. \end{aligned}$$

Proof. By Lemmas 3.3 and 3.4, we have that $T : P \rightarrow P$ is completely continuous and problem (1.5) has a solution $u = u(t)$ if and only if $u(t)$ satisfies the operator equation $u = Tu$.

Now, we show that all the conditions of Lemma 2.8 hold.

Step 1. We will show that $T : \bar{P}(\varphi, r_2; \theta, L_2) \rightarrow \bar{P}(\varphi, r_2; \theta, L_2)$.

If $u \in \bar{P}(\varphi, r_2; \theta, L_2)$, then $\varphi(u) \leq r_2$, $\theta(u) \leq L_2$.

From (H_3) , we have

$$\begin{aligned} \varphi(Tu(t)) &= \max_{0 \leq t \leq 1} |(Tu)(t)| \\ &= \max_{0 \leq t \leq 1} \left| \int_0^1 G(t, s) \phi_q \left(\frac{1}{\Gamma(r)} \int_0^s (s-\tau)^{r-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right| \\ &\leq \frac{r_2}{A} \int_0^1 G(1, s) \phi_q \left(\frac{1}{\Gamma(r)} \int_0^s (s-\tau)^{r-1} d\tau \right) ds \\ &\leq \frac{r_2}{A} \left[\frac{1}{r\Gamma(r)} \right]^{q-1} \left(\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha+1)} \right) \\ &\leq \frac{r_2}{A} \left(\frac{1}{\Gamma(r+1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)} \\ &= r_2, \end{aligned}$$

$$\begin{aligned}
 \max_{0 \leq t \leq 1} |(Tu)'(t)| &= \max_{0 \leq t \leq 1} \left| \int_0^1 G'_t(t,s) \phi_q \left(\frac{1}{\Gamma(r)} \int_0^s (s-\tau)^{r-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right| \\
 &\leq \frac{L_2}{2A} \left[\frac{1}{\Gamma(r+1)} \right]^{q-1} \int_0^1 G'_t(t,s) ds \\
 &= \frac{L_2}{2A} \left(\frac{1}{\Gamma(r+1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)} \\
 &\leq \frac{L_2}{2}, \\
 \max_{0 \leq t \leq 1} |D_{0+}^\rho (Tu)(t)| &= \max_{0 \leq t \leq 1} \left| \frac{1}{\Gamma(1-\rho)} \int_0^t (t-s)^{-\rho} (Tu)'(s) ds \right| \\
 &\leq \frac{1}{\Gamma(1-\rho)} \int_0^t (t-s)^{-\rho} \\
 &\quad \times \left| \int_0^1 G'_s(s,v) \phi_q \left(\frac{1}{\Gamma(r)} \int_0^v (v-\tau)^{r-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right| \\
 &\leq \frac{L_2}{2M} \frac{1}{\Gamma(1-\rho)} \left(\frac{1}{\Gamma(r+1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\rho} ds \\
 &= \frac{L_2}{2M} \left(\left(\frac{1}{\Gamma(r+1)} \right)^{q-1} \frac{1}{\Gamma(\alpha)\Gamma(2-\rho)} \right) \\
 &\leq \frac{L_2}{2}.
 \end{aligned}
 \tag{3.16}$$

Then, $\theta((Tu)(t)) \leq L_2$.
 So, $T : \bar{P}(\varphi, r_2; \theta, L_2) \rightarrow \bar{P}(\varphi, r_2; \theta, L_2)$.

Step 2. Let $u(t) = kb/2, 0 \leq t \leq 1$. It is easy to see that $u(t) = kb/2 \in \bar{P}(\varphi, kb; \theta, L_2, \varphi, b)$, and $\varphi(u) = \varphi(kb/2) > b$. Consequently, $\{u \in \bar{P}(\varphi, kb; \theta, L_2, \varphi, b) \mid \varphi(u) > b\} \neq \emptyset$.

If $u \in \bar{P}(\varphi, kb; \theta, L_2, \varphi, b)$, then for all $1/k \leq t \leq 1, b \leq u(t) \leq kb, \theta(u(t)) \leq L_2$. By (H_2) , we obtain $f(t, u(t), D_{0+}^\rho u(t)) > \phi_p(kb/B)$, for $1/k \leq t \leq 1$.

From Lemma 3.2, we have

$$\begin{aligned}
 \varphi(Tu) &= \min_{1/k \leq t \leq (k-1)/k} |Tu(t)| \\
 &\geq \frac{1}{k} \max_{0 \leq t \leq 1} |(Tu)(t)| \\
 &= \frac{1}{k} \max_{0 \leq t \leq 1} \left| \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(r)} \int_0^s (s-\tau)^{r-1} f(\tau, u(\tau), D_{0+}^\rho u(\tau)) d\tau \right) ds \right| \\
 &> \frac{kb}{B} \frac{1}{k} \max_{0 \leq t \leq 1} t^{\alpha-1} \int_0^1 G(1,s) \phi_q \left(\frac{s^\gamma}{\gamma \Gamma(\gamma)} \right) ds \\
 &= \frac{b}{B} \left(\frac{1}{\Gamma(\gamma+1)} \right)^{q-1} \int_0^1 G(1,s) \phi_q(s^\gamma) ds \\
 &> \frac{b}{B} \left(\frac{1}{k} \right)^{\gamma(q-1)} \left(\frac{1}{\Gamma(\gamma+1)} \right)^{q-1} \int_{1/k}^1 G(1,s) ds = b;
 \end{aligned} \tag{3.17}$$

that is, $\varphi(Tu) > b$, for all $u \in \bar{P}(\varphi, kb; \theta, L_2, \varphi, b)$. This shows that condition (C_1) of Lemma 2.8 holds.

Step 3. Let $u \in \bar{P}(\varphi, r_1; \theta, L_1)$; by (H_1) , we have

$$f(t, u(t), D_{0+}^\rho u(t)) < \min \left\{ \phi_p \left(\frac{r_1}{A} \right), \phi_p \left(\frac{L_1}{2A} \right), \phi_p \left(\frac{L_1}{2M} \right) \right\}. \tag{3.18}$$

Similarly, we can prove that $T : \bar{P}(\varphi, r_1; \theta, L_1) \rightarrow \bar{P}(\varphi, r_1; \theta, L_1)$. (C_2) of Lemma 2.8 holds.

Step 4. Let $u \in \bar{P}(\varphi, kb; \theta, L_2; \varphi, b)$, and $\varphi(Tu) > kb$; we have

$$\varphi(Tu) = \min_{1/k \leq t \leq (k-1)/k} |Tu(t)| \geq \frac{1}{k} \max_{0 \leq t \leq 1} |Tu(t)| = \frac{1}{k} \varphi(Tu) > b. \tag{3.19}$$

(C_3) of Lemma 2.8 holds. Therefore, the BVP (1.5) has at least three positive solutions u_1, u_2 , and u_3 satisfying

$$\begin{aligned}
 \max_{0 \leq t \leq 1} |u_1(t)| &\leq r_1, & \max_{0 \leq t \leq 1} |D_{0+}^\rho u_1(t)| + \max_{0 \leq t \leq 1} |u_1'(t)| &\leq L_1, \\
 b &< \min_{1/k \leq t \leq (k-1)/k} |u_2(t)| &< \max_{0 \leq t \leq 1} |u_2(t)| &\leq r_2, \\
 \max_{0 \leq t \leq 1} |D_{0+}^\rho u_2(t)| + \max_{0 \leq t \leq 1} |u_2'(t)| &\leq L_2, & r_1 &< \max_{0 \leq t \leq 1} |u_3(t)| \leq kb, \\
 \min_{1/k \leq t \leq (k-1)/k} |u_3(t)| &< b, & L_1 &< \max_{0 \leq t \leq 1} |D_{0+}^\rho u_3(t)| + \max_{0 \leq t \leq 1} |u_3'(t)| \leq L_2.
 \end{aligned} \tag{3.20}$$

The proof is completed. \square

Corollary 3.6. *If there exist constants $0 < r_1 < b_1 < kb_1 \leq r_2 < b_2 < kb_2 \leq \dots \leq r_n$, and $0 < L_1 \leq L_2 \leq \dots \leq L_{n-1}$, $n \in \mathbb{N}$, such that $kb_i/B \leq \min\{r_{i+1}/A, L_{i+1}/2A, L_{i+1}/2M\}$, for $1 \leq i \leq n-1$ and the following conditions are satisfied:*

$$(I_1) \quad f(t, u, v) < \min\{\phi_p(r_i/A), \phi_p(L_i/2A), \phi_p(L_i/M)\} \quad \text{for } (t, u, v) \in [0, 1] \times [0, r_i] \times [-L_i, L_i];$$

$$(I_2) \quad f(t, u, v) > \phi_p(kb_i/B), \quad \text{for } (t, u, v) \in [1/k, 1] \times [b_i, kb_i] \times [-L_{i+1}, L_{i+1}];$$

then the problem (1.5) has at least $2n - 1$ concave positive solutions.

Proof. If $n = 1$, by Condition (I_1) and Step 1 of the proof of Theorem 3.5, we can obtain that $T : \bar{P}(\varphi, r_1; \theta, L_1) \rightarrow P(\varphi, r_1; \theta, L_1) \subset \bar{P}(\varphi, r_1; \theta, L_1)$. From the Schauder fixed-point theorem, the problem (1.5) has at least one fixed-point $u_1 \in P(\varphi, r_1; \theta, L_1)$.

If $n = 2$, by Theorem 3.5, there exist at least three concave positive solutions u_2, u_3 , and u_4 . By the induction method, we finish the proof. \square

Finally, we present an example to check our results.

Example 3.7. Consider the boundary value problem:

$$\begin{aligned} D_{0+}^{1/2} \left(\phi_{3/2} \left(D_{0+}^{5/2} u(t) \right) \right) + f(t, u(t), D_{0+}^{1/2} u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \quad D_{0+}^\alpha u(t)|_{t=0} &= 0, \end{aligned} \quad (3.21)$$

where

$$f(t, u, v) = \begin{cases} \frac{t}{20} + 6u^2 + \left(\frac{|v|}{4 \times 10^4} \right)^4, & \text{for } u \leq 4, \\ \frac{t}{20} + 96 + \left(\frac{|v|}{4 \times 10^4} \right)^4, & \text{for } u > 4. \end{cases} \quad (3.22)$$

Let $k = 4$; note that $p = 3/2$, $q = 3$, $\alpha = 5/2$, $\gamma = 1/2$, $\rho = 1/2$; we have $A \approx 0.9585$, $B \approx 0.1089$, $M \approx 1.0819$.

Choosing $r_1 = 1/4$, $b = 1$, $r_2 = 10^4$, $L_1 = 2 \times 10^4$, $L_2 = 4 \times 10^4$. It is easy to see that $kb/B \leq \min(r_2/A, L_2/2A, L_2/2M)$, and $f(t, u, v)$ satisfying

$$(1) \quad f(t, u, v) \leq 0.4875 < \min\{\phi_p(r_1/A), \phi_p(L_1/2A), \phi_p(L_1/2M)\} \approx 0.5107, \quad (t, u, v) \in [0, 1] \times [0, 1/4] \times [-2 \times 10^4, 2 \times 10^4],$$

$$(2) \quad f(t, u, v) \geq 7.0125 > \phi_p(kb/B) \approx 6.060, \quad (t, u, v) \in [1/4, 1] \times [1, 4] \times [-4 \times 10^4, 4 \times 10^4],$$

$$(3) \quad f(t, u, v) \leq 97.05 < \min\{\phi_p(r_2/A), \phi_p(L_2/2A), \phi_p(L_2/2M)\} \approx 102.1419, \quad (t, u, v) \in [0, 1] \times [0, 10^4] \times [-4 \times 10^4, 4 \times 10^4].$$

By Theorem 3.5, problem (3.21) has at least three concave positive solutions u_1 , u_2 , and u_3 satisfying

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| &< \frac{1}{4}, & \max_{0 \leq t \leq 1} \left| D_{0+}^{\rho} u_1(t) \right| + \max_{0 \leq t \leq 1} |u_1'(t)| &< 2 \times 10^4, \\ 1 &< \min_{1/k \leq t \leq (k-1)/k} |u_2(t)| &< \max_{0 \leq t \leq 1} |u_2(t)| &\leq 10^4, \\ \max_{0 \leq t \leq 1} \left| D_{0+}^{\rho} u_2(t) \right| + \max_{0 \leq t \leq 1} |u_2'(t)| &< 4 \times 10^4, & & (3.23) \\ \frac{1}{4} &< \max_{0 \leq t \leq 1} |u_3(t)| &\leq 4, & \min_{1/k \leq t \leq (k-1)/k} |u_3(t)| &< 1, \\ 2 \times 10^4 &< \max_{0 \leq t \leq 1} \left| D_{0+}^{\rho} u_3(t) \right| + \max_{0 \leq t \leq 1} |u_3'(t)| &< 4 \times 10^4. \end{aligned}$$

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