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Research Article

A Class of Weak Hopf Algebras

Dongming Cheng[1,2]

[1] *Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang 310027, China*
[2] *Department of Mathematics, Henan University of Science and Technology, Luoyang, Henan 471003, China*

Correspondence should be addressed to Dongming Cheng, dmcheng8@gmail.com

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We introduce a class of noncommutative and noncocommutative weak Hopf algebras with infinite Ext quivers and study their structure. We decompose them into a direct sum of two algebras. The coalgebra structures of these weak Hopf algebras are described by their Ext quiver. The weak Hopf extension of Hopf algebra H_n has a quotient Hopf algebra and a sub-Hopf algebra which are isomorphic to H_n.

1. Introduction

Weak Hopf algebra was introduced by Li in 1998 as a generalization of Hopf algebras [1]. It had been proved in [1, 2]; for some sorts of finite dimensional weak Hopf algebras H, the quantum quasidouble $D(H)$ of H is quasibraided equipped with some quasi-R-matrix R. Hence R is a solution of the Quantum Yang-Baxter Equation.

First two examples of noncommutative and noncocommutative weak Hopf algebras were given in [3]. Up to now, many examples of weak Hopf algebras have been found [2, 4–7]. So far, all examples of weak Hopf algebras were based on some Hopf algebras and were constructed by weak extension.

In this paper, we first give a Hopf algebra, denoted by H_n. By weak extension, we construct a weak Hopf algebra $W(n_1, n_2, n_3)$ corresponding to H_n and study their structure. $W(n_1, n_2, n_3)$ has a quotient Hopf algebra and a sub-Hopf algebra which are isomorphic to H_n. And as an algebra, $W(n_1, n_2, n_3)$ can be decomposed into a direct sum of two algebras, one of which is H_n. The coalgebra structures of these weak Hopf algebras are described by their Ext quiver [8, 9].

We organize our paper as follows. In Section 2, we introduce the Hopf algebra H_n. In Section 3, we define a class of weak Hopf algebras $W(n_1, n_2, n_3)$. In Section 4, we study the structure of $W(n_1, n_2, n_3)$ and decompose $W(n_1, n_2, n_3)$ into a direct sum of H_n and some algebra of polynomials as an algebra. We give the Ext-quiver of coalgebra of $W(n_1, n_2, n_3)$ and prove that $W(n_1, n_2, n_3)$ has a quotient Hopf algebra and a sub-Hopf algebra which are isomorphic to H_n.

2. A Quiver Hopf Algebra

The Hopf Algebra $F_{(q)}$ is defined in [10]. Let $q \in k \setminus 0$. As a k-algebra $F_{(q)}$ is generated by a, b, and x subject to the relations

$$ab = 1, \qquad ba = 1, \qquad xa = qax, \qquad xb = q^{-1}bx. \tag{2.1}$$

The coalgebra structure of $F_{(q)}$ is determined by

$$\begin{gathered}\Delta(a) = a \otimes a, \qquad \Delta(b) = b \otimes b, \qquad \Delta(x) = x \otimes a + 1 \otimes x. \\ \varepsilon(1) = \varepsilon(a) = \varepsilon(b) = 1, \qquad \varepsilon(x) = 0.\end{gathered} \tag{2.2}$$

We generalize $F_{(q)}$ to H_n, which is defined as follows. Let k be a field, $q \in k \setminus 0$, $i = 1, 2, \ldots, n$. As a k-algebra H_n is generated by K, K^{-1}, and X_i, $i = 1, 2, \ldots, n$ subject to the relations

$$KK^{-1} = 1, \qquad K^{-1}K = 1, \qquad X_iK = qKX_i, \qquad X_iK^{-1} = q^{-1}K^{-1}X_i. \tag{2.3}$$

The coalgebra structure of H_n is determined by

$$\begin{gathered}\Delta(K) = K \otimes K, \qquad \Delta\left(K^{-1}\right) = K^{-1} \otimes K^{-1}, \\ \Delta(X_i) = X_i \otimes K + 1 \otimes X_i, \\ \varepsilon(K) = \varepsilon\left(K^{-1}\right) = 1, \qquad \varepsilon(X_i) = 0.\end{gathered} \tag{2.4}$$

The antipode S is induced by

$$S(K) = K^{-1}, \qquad S\left(K^{-1}\right) = K, \qquad S(X_i) = -K^{-1}X_i. \tag{2.5}$$

3. A Class of Weak Hopf Algebras

In this section, we construct a class of weak Hopf algebra corresponding to H_n.

First recall the definition of weak Hopf algebra [1].

Definition 3.1. A k-bialgebra $H = (H, \mu, \eta, \Delta, \varepsilon)$ is called a weak Hopf algebra if there exists $T \in \mathrm{Hom}_k(H, H)$ such that $id * T * id = id$ and $T * id * T = T$ where T is called a weak antipode of H.

A weak Hopf algebra is called pointed if it is pointed as a coalgebra. If a weak Hopf algebra H is pointed, then the set of all group-like elements $G(H)$ is a regular monoid [6].

Now we construct weak Hopf algebra W corresponding to H_n. The set $G(W)$ of group-like elements of weak Hopf algebra W is a regular monoid which has generators g, $\overline{g}$, 1, subject to $g\overline{g} = g\overline{g}$, $g^2\overline{g} = g$, $\overline{g}^2 g = \overline{g}$.

To construct all possible weak extension we need the following discussion.

Recall, for any coalgebra C, that the group-like elements in C are the set $G(C) = \{a \in C \mid a \neq 0 \text{ and } \Delta(a) = a \otimes a\}$; necessarily $\varepsilon(a) = 1$ for $a \in G(C)$. Note that a simple subcoalgebra D of C is one-dimensional $\Leftrightarrow$ $D = ka$ for some $a \in G(C)$. A coalgebra is pointed if all of its simple subcoalgebras are one-dimensional. For $a, b \in G(C)$, the a, b-primitive elements in C are the set $P_{a,b}(C) = \{c \in C \mid \Delta(c) = c \otimes a + b \otimes c\}$; necessarily $\varepsilon(c) = 0$ for $c \in P_{a,b}(C)$. Note that $k(a-b) = \{l(a-b) \mid l \in k\} \subset P_{a,b}(C)$; an a, b-primitive element c is nontrivial if $c \notin k(a-b) = \{l(a-b) \mid l \in k\}$. If $a = b = 1$, the $1,1$-primitives are simply called primitive; otherwise they are called skew primitive.

The following result is a generalization of [11].

Lemma 3.2. *Let W be the weak Hopf algebra defined above. One has*

$$gP_{a,b}(W) \subseteq P_{ga,gb}(W), \qquad \overline{g}P_{a,b}(W) \subseteq P_{\overline{g}a,\overline{g}b}(W). \tag{3.1}$$

Proof. Let $u \in P_{a,b}(W)$, then $\Delta(u) = u \otimes a + b \otimes u$. Hence,

$$\begin{aligned}\Delta(gu) &= \Delta(g)\Delta(u) \\ &= (g \otimes g)(u \otimes a + b \otimes u) \\ &= gu \otimes ga + gb \otimes gu \in P_{ga,gb}(W).\end{aligned} \tag{3.2}$$

The second inclusion is proved similarly. □

Corollary 3.3. *For W, one has*

$$\begin{aligned}\dim P_{g^{i+1},g^i}(W) &= \dim P_{g^i,g^{i-1}}(W), \quad i \geq 2, \\ \dim P_{\overline{g}^i,\overline{g}^{i+1}}(W) &= \dim P_{\overline{g}^{i-1},\overline{g}^i}(W), \quad i \geq 2, \\ \dim P_{\overline{g},\overline{g}^2}(W) = \dim P_{g\overline{g},\overline{g}}(W) &= \dim P_{g,g\overline{g}}(W) = \dim P_{g^2,g}(W).\end{aligned} \tag{3.3}$$

Proof. We only prove the first equation. In fact, the map $\varphi : P_{g^i,g^{i-1}}(W) \to P_{g^{i+1},g^i}(W)$, $u \mapsto gu$ is a linear map with inverse $\varphi : P_{g^{i+1},g^i}(W) \mapsto P_{g^i,g^{i-1}}(W)$, $v \mapsto \overline{g}v$. Hence, $P_{g^i,g^{i-1}}(W)$ and $P_{g^{i+1},g^i}(W)$ are isomorphic as vector spaces. □

Since all the dimensions in Corollary 3.3 are same, we have the following corollary.

Corollary 3.4. *One has*

$$\dim P_{1,\overline{g}}(W) \leq \dim P_{g,g\overline{g}}(W), \qquad \dim P_{g,1}(W) \leq \dim P_{g,g\overline{g}}(W). \tag{3.4}$$

Proof. The map $\varphi : P_{g,1}(W) \to P_{g,g\overline{g}}(W)$, $u \mapsto g\overline{g}u$ is a linear map. If $\varphi(u) = g\overline{g}u = l(g - g\overline{g})$, for some $l \in k$, then $u \in kG(W)$, the vector space spanned by all group-like elements, because W is graded. Hence, $u = l(g-1)$. Therefore, the linear map φ is an injection. Consequently,

$$\dim P_{1,\overline{g}}(W) \leq \dim P_{g,g\overline{g}}(W). \tag{3.5}$$

The proof of the second inequality is similar. □

By the above discussion we know that weak Hopf algebra W is determined by $P_{1,\overline{g}}(W)$, $P_{g,1}(W)$, and $P_{g,g\overline{g}}(W)$. Take $x_1, \dots, x_{n_1}$ to be linearly independent nontrivial elements in $P_{1,\overline{g}}(W)$, and $y_1, \dots, y_{n_2}$ linearly independent nontrivial elements in $P_{g,1}(W)$. Let

$$P_{g,g\overline{g}}(W) = \left(g\overline{g}P_{1,\overline{g}}(W) + gP_{g,1}(W)\right) \oplus V, \tag{3.6}$$

and $z_1, \dots, z_{n_3}$ a basis of V. Then W is determined by $x_1, \dots, x_{n_1}$, $y_1, \dots, y_{n_2}$, $z_1, \dots, z_{n_3}$.

To summarize, we define weak Hopf algebra $W(n_1, n_2, n_3)$ corresponding to H_n as follows.

Definition 3.5. Let k be a field. For any positive integers n_1, n_2, n_3, and nonzero element $q \in k$, we define $W(n_1, n_2, n_3)$ to be associative algebra over field k generated by $1, g, \overline{g}, x_i, y_j, z_k$, $i = 1, 2, \dots, n_1$, $j = 1, 2, \dots, n_2$, $k = 1, 2, \dots, n_3$, subject to

$$g\overline{g} = g\overline{g}, \qquad g\overline{g}^2 = g, \qquad \overline{g}^2 g = \overline{g}, \tag{3.7}$$

$$gx_i = qx_ig, \qquad \overline{g}x_i = q^{-1}x_i\overline{g}, \quad i = 1, 2, \dots, n_1, \tag{3.8}$$

$$gy_j = qy_jg, \qquad \overline{g}y_j = q^{-1}y_j\overline{g}, \quad j = 1, 2, \dots, n_2, \tag{3.9}$$

$$gz_k\overline{g} = qz_k, \quad k = 1, 2, \dots, n_3. \tag{3.10}$$

$W(n_1, n_2, n_3)$ can be endowed with coalgebra structure by

$$\Delta(g) = g \otimes g, \tag{3.11}$$

$$\Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \tag{3.12}$$

$$\Delta(y_j) = y_j \otimes 1 + \overline{g} \otimes y_j, \tag{3.13}$$

$$\Delta(z_k) = z_k \otimes g + g\overline{g} \otimes z_k, \tag{3.14}$$

$$\varepsilon(1) = \varepsilon(g) = \varepsilon(\overline{g}) = 1, \qquad \varepsilon(x_i) = 0, \qquad \varepsilon(y_j) = 0, \qquad \varepsilon(z_k) = 0, \tag{3.15}$$

while the weak antipode T is induced by

$$T(1) = 1, \qquad T(g) = \overline{g}, \qquad T(\overline{g}) = g, \tag{3.16}$$

$$T(x_i) = -x_i\overline{g}, \qquad T(y_j) = -gy_j, \qquad T(z_k) = -z_k\overline{g}, \tag{3.17}$$

Theorem 3.6. *For any positive integers n_1, n_2, n_3, $W(n_1, n_2, n_3)$ is a weak Hopf algebra.*

Proof. First we must check that the coproduct Δ is an algebra map. It suffices to prove that Δ preserves the relations (3.7)–(3.10). It is easy to see that Δ preserves the relations (3.7). And

$$\begin{aligned}
\Delta(gx_i) &= (g\otimes g)(x_i\otimes g + 1\otimes x_i)\\
&= gx_i\otimes g^2 + g\otimes gx_i\\
&= (qx_ig)\otimes g^2 + g\otimes (qx_ig)\\
&= q(x_i\otimes g + 1\otimes x_i)(g\otimes g)\\
&= \Delta(qx_ig),\\
\Delta(gy_j) &= (g\otimes g)(y_j\otimes 1 + \overline{g}\otimes y_j)\\
&= gy_j\otimes g + g\overline{g}\otimes gy_j\\
&= (qy_jg)\otimes g + g\overline{g}\otimes (qy_jg)\\
&= q(y_j\otimes 1 + \overline{g}\otimes y_j)(g\otimes g)\\
&= \Delta(qy_jg),\\
\Delta(gz_k\overline{g}) &= (g\otimes g)(z_k\otimes g + g\overline{g}\otimes z_k)(\overline{g}\otimes \overline{g})\\
&= gz_k\overline{g}\otimes gg\overline{g} + gg\overline{g}\overline{g}\otimes gz_k\overline{g}\\
&= (qz_k)\otimes g + g\overline{g}\otimes (qz_k)\\
&= \Delta(qz_k).
\end{aligned}\tag{3.18}$$

Next we prove that T is the weak antipode. It suffices to prove that for each generator g, $\overline{g}, x_i, y_j, z_k$, the action of $T * id * T$ is the same as that of T, and the action of $id * T * id$ is the same as that of id.

Since

$$\begin{aligned}
(\Delta\otimes id)\Delta(x_i) &= (\Delta\otimes id)(x_i\otimes g + 1\otimes x_i)\\
&= (x_i\otimes g + 1\otimes x_i)\otimes g + 1\otimes 1\otimes x_i\\
&= x_i\otimes g\otimes g + 1\otimes x_i\otimes g + 1\otimes 1\otimes x_i,
\end{aligned}\tag{3.19}$$

we get

$$\begin{aligned}
(id * T * id)(x_i) &= x_i\overline{g}g + (-x_i\overline{g})g + x_i = x_i = id(x_i),\\
(T * id * T)(x_i) &= (-x_i\overline{g})g\overline{g} + x_i\overline{g} + (-x_i\overline{g})\\
&= -x_i(\overline{g}g\overline{g}) = -x_i\overline{g} = T(x_i).
\end{aligned}\tag{3.20}$$

Since

$$
\begin{aligned}
(\Delta \otimes id)\Delta(y_j) &= (\Delta \otimes id)(y_j \otimes 1 + \overline{g} \otimes y_j) \\
&= (y_j \otimes 1 + \overline{g} \otimes y_j) \otimes 1 + \overline{g} \otimes \overline{g} \otimes y_j \\
&= y_j \otimes 1 \otimes 1 + \overline{g} \otimes y_j \otimes 1 + \overline{g} \otimes \overline{g} \otimes y_j,
\end{aligned}
\tag{3.21}
$$

it follows that

$$
\begin{aligned}
(id * T * id)(y_j) &= y_j + \overline{g}(-gy_j) + \overline{g}gy_j = y_j = id(y_j), \\
(T * id * T)(y_j) &= (-gy_j) + gy_j + g\overline{g}(-gy_j) = -gy_j = T(y_j).
\end{aligned}
\tag{3.22}
$$

Since

$$
\begin{aligned}
(\Delta \otimes id)\Delta(z_k) &= (\Delta \otimes id)(z_k \otimes g + g\overline{g} \otimes z_k) \\
&= (z_k \otimes g + g\overline{g} \otimes z_k) \otimes g + g\overline{g} \otimes g\overline{g} \otimes z_k \\
&= z_k \otimes g \otimes g + g\overline{g} \otimes z_k \otimes g + g\overline{g} \otimes g\overline{g} \otimes z_k, \\
z_k g\overline{g} &= q^{-1} g z_k \overline{g} g \overline{g} = q^{-1} g z_k \overline{g} = z_k, \\
g\overline{g} z_k &= q^{-1} g \overline{g} g z_k \overline{g} = q^{-1} g z_k \overline{g} = z_k,
\end{aligned}
\tag{3.23}
$$

we get

$$
\begin{aligned}
(id * T * id)(z_k) &= z_k g\overline{g} + g\overline{g}(-z_k\overline{g})g + gg\overline{g}\overline{g} z_k \\
&= z_k - q\overline{g} z_k g + g\overline{g} z_k \\
&= z_k - z_k + z_k = z_k = id(z_k), \\
(T * id * T)(z_k) &= (-z_k\overline{g})g\overline{g} + g\overline{g} z_k \overline{g} + gg\overline{g}\overline{g}(-z_k\overline{g}) \\
&= (-z_k\overline{g}) + z_k\overline{g} + (-z_k\overline{g}) = -z_k\overline{g} = T(z_k).
\end{aligned}
\tag{3.24}
$$

□

4. The Structure of $W(n_1, n_2, n_3)$

In this section we study the algebra and coalgebra structure of $W(n_1, n_2, n_3)$.

It is easy to prove that the elements $g\overline{g}$ and $1 - g\overline{g}$ are a pair of orthogonal central idempotents. Set $W_1 = W(n_1, n_2, n_3)g\overline{g}$, $W_2 = W(n_1, n_2, n_3)(1 - g\overline{g})$. We have the following.

Theorem 4.1. *$W(n_1, n_2, n_3)$ can be written as a direct sum of two-sided ideals $W(n_1, n_2, n_3) = W_1 \bigoplus W_2$. And one has the following.*

(1) *As an algebra, W_1 is isomorphic to H_n, where $n = n_1 + n_2 + n_3$.*

(2) *As an algebra, W_2 is isomorphic to the free associative algebra $k\langle Y_1, \ldots, Y_t\rangle$ of t generators, where $t = n_1 + n_2$.*

Proof. (1) Since $g\overline{g}$ and $1-g\overline{g}$ are a pair of orthogonal central idempotents,

$$W(n_1,n_2,n_3) = W(n_1,n_2,n_3)g\overline{g} \oplus W(n_1,n_2,n_3)(1-g\overline{g}) = W_1 \oplus W_2. \tag{4.1}$$

The isomorphism $W_1 \to H_n$ is induced by $x_i g\overline{g} \mapsto X_i$, $y_j g\overline{g} \mapsto X_{n_1+j}$, $z_k g\overline{g} \mapsto X_{n_1+n_2+k}$, $g\overline{g} \mapsto 1$, $g^2\overline{g} \mapsto K$.

(2) Note that $z_k(1-g\overline{g}) = 0$ and $x_i(1-g\overline{g})y_j(1-g\overline{g}) = y_j(1-g\overline{g})x_i(1-g\overline{g})$. Since $x_i(1-g\overline{g})$, $y_j(1-g\overline{g})$ are generators of W_2, the isomorphism $W_2 \to k\langle Y_1,\ldots,Y_t\rangle$ is defined by $(1-g^2) \mapsto 1$, $x_i(1-g^2) \mapsto Y_i$, $y_j(1-g^2) \mapsto Y_{n_1+j}$. □

A weak Hopf ideal J of a weak Hopf algebra H is a bi-ideal such that $T(J) \subset J$, where T is the weak antipode of H. It is easy to see that H/J has a natural structure of a weak Hopf algebra.

Theorem 4.2. *The ideal J in $W(n_1,n_2,n_3)$ generated by $1-g\overline{g}$ is a weak Hopf ideal. And the quotient weak Hopf algebra $W(n_1,n_2,n_3)/J$ is a Hopf algebra, which is isomorphic to H_n, where $n = n_1+n_2+n_3$.*

Proof. Since

$$\begin{aligned}
\Delta(1-g\overline{g}) &= 1\otimes 1 - g\overline{g}\otimes g\overline{g} \\
&= 1\otimes 1 - g\overline{g}\otimes 1 + g\overline{g}\otimes 1 - g\overline{g}\otimes g\overline{g} \\
&= (1-g\overline{g})\otimes 1 + g\overline{g}\otimes(1-g\overline{g}), \\
T(1-g\overline{g}) &= T(1) - T(\overline{g})T(g) = 1-g\overline{g},
\end{aligned} \tag{4.2}$$

J is a weak Hopf ideal in $W(n_1,n_2,n_3)$.

The isomorphism $W(n_1,n_2,n_3)/J \to H_n$ is defined by $g+J \mapsto K$, $\overline{g}+J \mapsto K^{-1}$, $x_i+J \mapsto X_i$, $gy_j+J \mapsto X_{n_1+j}$, $z_k+J \mapsto X_{n_1+n_2+k}$. □

Now we give the Ext quiver of $W(n_1,n_2,n_3)$. For the definition and calculation of Ext quiver, we refer to [5, 8, 9, 12].

The Ext quiver of $W(n_1,n_2,n_3)$ is shown in Figure 1. The multiplicity of arrow $g\cdot \to \cdot 1$ is n_1. The multiplicity of arrow $1\cdot \to \cdot\overline{g}$ is n_2. The multiplicity of other arrows is all n.

Theorem 4.3. *The sub-coalgebra H related to the subquiver in Figure 2 is isomorphic to H_n as coalgebra.*

Proof. The isomorphism $H \to H_n$ is induced by $g\overline{g} \mapsto 1$, $g \mapsto K$, $\overline{g} \mapsto K^{-1}$, $x_i \mapsto X_i$, $gy_j \mapsto X_{n_1+j}$, $z_k \mapsto X_{n_1+n_2+k}$. □

Remark 4.4. The isomorphisms described in Theorem 4.1 are not isomorphisms of bialgebras.

Remark 4.5. The weak Hopf algebras discussed in [4, 5] also have quotient Hopf algebras and sub-Hopf algebras which are isomorphic to the related Hopf algebras.

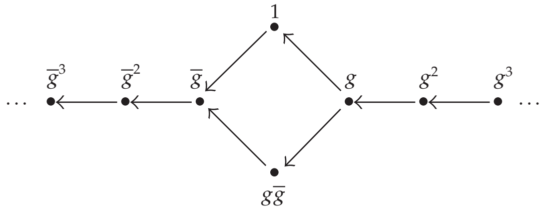

Figure 1: Ext quiver of $W(n_1, n_2, n_3)$.

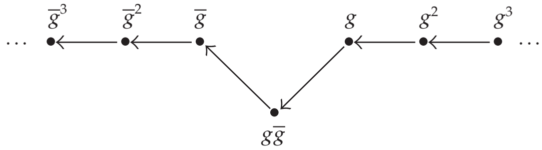

Figure 2: A subquiver of Ext quiver of $W(n_1, n_2, n_3)$.

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