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Research Article

Estimation of the Parameters of the Reversed Generalized Logistic Distribution with Progressive Censoring Data

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The reversed generalized logistic (RGL) distributions are very useful classes of densities as they posses a wide range of indices of skewness and kurtosis. This paper considers the estimation problem for the parameters of the RGL distribution based on progressive Type II censoring. The maximum likelihood method for RGL distribution yields equations that have to be solved numerically, even when the complete sample is available. By approximating the likelihood equations, we obtain explicit estimators which are in approximation to the MLEs. Using these approximate estimators as starting values, we obtain the MLEs using iterative method. We examine numerically MLEs estimators and the approximate estimators and show that the approximation provides estimators that are almost as efficient as MLEs. Also we show that the value of the MLEs decreases as the value of the shape parameter increases. An exact confidence interval and an exact joint confidence region for the parameters are constructed. Numerical example is presented in the methods proposed in this paper.

1. Introduction

There are many scenarios in life-testing and reliability experiments whose units are lost or removed from experimentation before failure. The loss may occur un intentionally, or it may have been designed so in the study. In many situations, however, the removal of units prior to failure is preplanned in order to provide saving in terms of time and cost associated with testing. There are types of censored test. Type I and Type II have been investigated extensively by many authors (see, e.g., [1–3]). A generalization of type II censoring is progressive Type II censoring. Under this scheme, n units are placed in test at time zero. Immediately following the first failure, R_1 surviving units are removed from the test at random.

Then, Immediately following the second failure, R_2 surviving units are removed from the test at random. This process continues until, at the time of the mth observed failure, the remanding $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ are all removed from the experiment. In this censoring scheme, $R_1, R_2, \ldots, R_m$ are all prefixed. However, in some particle situations, the size R_i may occur at random (see [4]). Note that if $R_1 = R_2 = \cdots - R_{m-1} = 0$, then $R_m = n - m$ which corresponds to the type II censoring. If $R_1 = R_2 = \cdots = R_m = 0$, then $m = n$ which corresponds to the complete sample.

The statistical inference on the parameters of lifetime distribution under progressive Type II censoring has been studied by many authors such as Cohen [5], Mann [1], Viveros and Balakrishnan [6], and Balakrishnan and Sandhu [7]. Viveros and Balakrishnan has discussed inference for the Weiblull and exponential distributions under progressive Type II censoring and derived explicit expression for the best linear unbiased estimator (BLUE) of the parameters of the one- and two-parameter exponential distribution. Balakrishnan et al. [8] have discussed the inference from the extreme value distribution under progressive Type II censored sample. Wu [9, 10] obtained estimation results concerning a progressive Type II censored sample from two-parameter Weibuul distribution and Pareto distribution, respectively. Balakrishnan et al. [11] have examined numerically the bias and the mean square error of the MLEs based on a progressive Type II censored sample from a Gaussian distribution. Point and interval estimation for the parameters of the logistic distribution based on progressive Type II censored samples are obtained by Balakrishnan and Kannon [12]. Arturo [13] have investigated the estimation problem of exponential parameters, on the basis of general progressive Type II censored sample. Balakrishnan et al. [14] have discussed the inference from the extreme value distribution under progressive Type II censored sample. Nigm and Aboeleneen [15] have obtained the maximum likelihood estimators of the location and scale parameters of inverse Weibull distribution and observed Fisher information matrix based on progressive Type II censored samples and other inference. Patel [16] has obtained MLE for exponential model with changing failure rates based on two-stage progressively multiply type II censored samples. Recently Gajjar and Patel [17] studied the estimation for a mixture of exponential distributions based on progressively type II censored sample. The reversed generalized Logistic (RGL) distribution are very useful classes of densities as they posses a wide range of indices of skewness and kurtosis. Therefore, an important application of RGL distribution is its use in studying robustness of estimators and tests. Balakrishnan and Leung [18] present two real data examples for the usefulness of RGL distribution, one about oxygen concentration and another about resistance of automobile paint. In both examples the authors choose $\phi = 2$ by eye and verify the validity of this assumption by Q-Q plots. Possible applications of this distribution in bioassays as a dose-response curve are discussed and illustrated with some examples by El-Said et al. [19]. This paper considers RGL model and discuss inference based on progressive Type II censored samples. In Section 2, we discuss the maximum likelihood estimators (MLEs) of the location and scale parameters of RGL distribution when the shape parameter is known and we provide an approximation of the maximum likelihood function that leads to explicit estimators. In Section 3, the observed Fisher information matrix is obtained. In Section 4, we obtain MLEs for shape and scale parameters of RGL distribution. We also obtained an exact confidence interval for the scale parameter and an exact joint region for the scale and shape parameters are constructed. A numerical example to show the usefulness of our results is provided in Section 5. Results of simulation study conducted to evaluate the performance of these approximate estimators to the MLEs, in terms of both bias and mean squared error, also to show how the change in the value of shape parameter is after the estimators, are provided in Section 6.

2. Location-Scale Parameters Estimation When the Shape Parameter Is Known

Assume the failure time distribution to be the RGL distribution with probability density function (pdf)

$$f(x,\mu,\sigma) = \frac{\phi}{\sigma}\exp\left(\frac{x-\mu}{\sigma}\right)\left[1+\exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-(\phi+1)}, \quad x>\mu,\ \sigma>0,\ \phi>0, \tag{2.1}$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(x,\mu,\sigma) = 1-\left[1+\exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-\phi}, \quad x>\mu,\ \sigma>0,\ \phi>0. \tag{2.2}$$

With $\mu = 0$ and $\sigma = 1$, we have a standard RGL distribution with pdf and cdf

$$f(x,0,1) = \phi\exp(x)\left[1+\exp(x)\right]^{-(\phi+1)}, \quad x>0,\ \phi>0, \tag{2.3}$$

and the corresponding survival distribution function is given by

$$1-F(x,0,1) = \left[1+\exp(x)\right]^{-\phi}, \quad \phi>0, \tag{2.4}$$

respectively.

With $\phi = 1$, we get the Logistic distribution with pdf and cdf, respectively,

$$g(x,\mu,\sigma) = \frac{1}{\sigma}\exp\left(\frac{x-\mu}{\sigma}\right)\left[1+\exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-2}, \quad x>\mu,\ \sigma>0, \tag{2.5}$$

and the corresponding cumulative distribution function (cdf) is given by

$$1-G(x,\mu,\sigma) = \left[1+\exp\left(\frac{x-\mu}{\sigma}\right)\right]^{-1}, \quad x>\mu,\ \sigma>0. \tag{2.6}$$

Let $(X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ be a progressively type II censored sample from (2.1) with censoring scheme $(R_1, R_2, \ldots R_m)$. The likelihood function based on the progressive Type II censored sample is given by

$$L(\mu,\sigma) = k\prod_{i=1}^{m} f(x_{i:m:n},\mu,\sigma)\left[1-F(x_{i:m:n},\mu,\sigma)\right]^{R_i}, \tag{2.7}$$

where

$$k = n(n-R_1-1)(n-R_1-R_2-2)\cdots(n-R_1-R_2-R_3\cdots-R_{m-1}-m+1). \tag{2.8}$$

The likelihood function may be rewritten as

$$L(\mu,\sigma) = k\prod_{i=1}^{m} f\left(y_{i:m:n}\right)\left[1 - F\left(y_{i:m:n}\right)\right]^{R_i}, \tag{2.9}$$

where $y_{i:m:n} = (x_{i:m:n} - \mu)/\sigma$.

Using the relation $f(y) = (\phi/\sigma)(\exp(y)/(1+\exp(y)))[1 - F(y)]$, the log-likelihood function is then given by

$$\begin{aligned} L^* = {} & \log k + m\log\phi - m\log\sigma + \sum_{i=1}^{m} y_{i:m:n} - \sum_{i=1}^{m}\log\left(1+\exp\left(y_{i:m:n}\right)\right) \\ & + \sum_{i=1}^{m}(R_i+1)\log\left[1 - F\left(y_{i:m:n}\right)\right], \end{aligned} \tag{2.10}$$

form (2.10), we derive the likelihood equation for μ and σ as

$$\begin{aligned} \frac{\partial L^*}{\partial\mu} &= \frac{-m}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{m}\frac{\exp\left(y_{i:m:n}\right)}{1+\exp\left(y_{i:m:n}\right)} + \frac{\phi}{\sigma}\sum_{i=1}^{m}(R_i+1)\frac{\exp\left(y_{i:m:n}\right)}{1+\exp\left(y_{i:m:n}\right)}, \\ \frac{\partial L^*}{\partial\sigma} &= -\frac{m}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{m} y_{i:m:n} + \frac{1}{\sigma}\sum_{i=1}^{m}\frac{y_{i:m:n}\exp\left(y_{i:m:n}\right)}{1+\exp\left(y_{i:m:n}\right)} \\ &\quad + \frac{\phi}{\sigma}\sum_{i=1}^{m}(R_i+1)\frac{y_{i:m:n}\exp\left(y_{i:m:n}\right)}{1+\exp\left(y_{i:m:n}\right)}. \end{aligned} \tag{2.11}$$

Equation (2.11) may be written, respectively, as

$$\begin{aligned} \frac{\partial L^*}{\partial\mu} &= \frac{-m}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{m}\left\{(R_i+1)\phi + 1\right\}G\left(y_{i:m:n}\right), \\ \frac{\partial L^*}{\partial\sigma} &= \frac{-m}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{m} y_{i:m:n} + \frac{1}{\sigma}\sum_{i=1}^{m}\left\{(R_i+1)\phi + 1\right\}y_{i:m:n}G\left(y_{i:m:n}\right), \end{aligned} \tag{2.12}$$

where $G(y)$ is the cdf of the Logistic distribution defined in (2.6). Equation (2.12) do not provide explicit solution for the parameters and have to be solved numerically to obtain the MLEs of the two-parameter. Hence it may be needed to develop approximation solutions to MLEs equations which yields explicit solutions to the parameters. These explicit solution may be a good starting value for the iterative solution for the MLEs equations. Several Approximate solutions for MLEs have been discussed in the book by Tiku et al. [20]. In this text we approximate the Logistic cdf $G(y_{i:m:n})$ by expanding it in Taylor series around $E(y_{i:m:n}) = \gamma_{i:m:n}$. From the book of Balakrishnan and Aggarwala [21], it is known that

$$G\left(y_{i:m:n}\right) = U_{i:m:n}, \tag{2.13}$$

where $U_{i:m:n}$ is the ith order statistics from a progressively Type-II censored sample from the uniform $U(0,1)$ distribution.

So, we have

$$y_{i:m:n} = G^{-1}(U_{i:m:n}), \tag{2.14}$$

and hence

$$\nu_{i:m:n} = E(y_{i:m:n}) = G^{-1}(\alpha_{i:m:n}), \tag{2.15}$$

where $\alpha_{i:m:n} = E(U_{i:m:n})$ and is given by the book of Balakrishnan and Aggarwala [21],

$$\alpha_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} \frac{j + R_{m-j+1} + \cdots + R_m}{j + 1 + R_{m-j+1} + \cdots + R_m}, \quad i = 1, \ldots, m. \tag{2.16}$$

For the logistic distribution, $G^{-1}(\cdot)$ is easily seen as

$$G^{(-1)}(u) = \ln\left(\frac{1-u}{u}\right). \tag{2.17}$$

By expanding $G(y_{i:m:n})$ around $\nu_{i:m:n}$ and keeping only the first two terms, we have

$$\begin{aligned} G(y_{i:m:n}) &= G(\nu_{i:m:n}) + (y_{i:m:n} - \nu_{i:m:n})\grave{G}(y_{i:m:n})\big|_{y_{i:m:n}=\nu_{i:m:n}} \\ &= \gamma_{i:m:n} + \delta_{i:m:n} y_{i:m:n}, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} \gamma_{i:m:n} &= G(\nu_{i:m:n}) - (\nu_{i:m:n}) g(\nu_{i:m:n}), \\ \delta_{i:m:n} &= g(\nu_{i:m:n}) \geq 0, \quad i = 1, \ldots, m. \end{aligned} \tag{2.19}$$

Using the above expression, we obtain the approximate MLEs equations as

$$\frac{\partial L^*}{\partial \mu} = -m + \sum_{i=1}^{m} \{(R_i + 1)\phi + 1\}\{\gamma_{i:m:n} + \delta_{i:m:n} y_{i:m:n}\} = 0, \tag{2.20}$$

$$\frac{\partial L^*}{\partial \sigma} = -m - \sum_{i=1}^{m} y_{i:m:n} + \sum_{i=1}^{m} \{(R_i + 1)\phi + 1\} y_{i:m:n}\{\gamma_{i:m:n} + \delta_{i:m:n} y_{i:m:n}\} = 0. \tag{2.21}$$

We may write (2.20) as

$$-m + \sum_{i=1}^{m} \{(R_i + 1)\phi + 1\}\gamma_{i:m:n} + \sum_{i=1}^{m} \{(R_i + 1)\phi + 1\}\delta_{i:m:n}\left(\frac{x_{i:m:n} - \mu}{\sigma}\right) = 0, \tag{2.22}$$

which yields the estimator of μ as

$$\widehat{\mu} = \frac{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}x_{i:m:n}}{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}} + \frac{\sigma\{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\gamma_{i:m:n}-m\}}{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}} = K + L\sigma, \tag{2.23}$$

where

$$K = \frac{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}x_{i:m:n}}{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}},$$
$$L = \frac{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\gamma_{i:m:n}-m}{\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}}. \tag{2.24}$$

Also (2.21) may be written as

$$-m - \sum_{i=1}^{m}\left(\frac{x_{i:m:n}-\mu}{\sigma}\right) + \sum_{i=1}^{m}\{(R_i+1)\phi+1\}\left(\frac{x_{i:m:n}-\mu}{\sigma}\right)\gamma_{i:m:n}$$
$$+\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}\left(\frac{x_{i:m:n}-\mu}{\sigma}\right)^2 = 0. \tag{2.25}$$

Now replacing μ in (2.25) by $k + L\sigma$, we have

$$-m + \frac{1}{\sigma}\sum_{i=1}^{m}(\{(R_i+1)\phi+1\}\gamma_{i:m:n}-1)(x_{i:m:n}-k)$$
$$+\frac{1}{\sigma^2}\sum_{i=1}^{m}(\{(R_i+1)\phi+1\}\delta_i)(x_{i:m:n}-k)^2$$
$$+mL - L\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\gamma_{i:m:n} + L^2\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}$$
$$-\frac{2L}{\sigma}\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n}(x_{i:m:n}-k) = 0. \tag{2.26}$$

It is easy to see that the last four terms vanish, so we got the quadratic equation of σ as

$$m\sigma^2 - \sum_{i=1}^{m}(\{(R_i+1)\phi+1\}\gamma_i-1)(x_{i:m:n}-k) - \sum_{i=1}^{m}(\{(R_i+1)\phi+1\}\delta_i)(x_{i:m:n}-k)^2 = 0, \tag{2.27}$$

or

$$m\sigma^2 - A\sigma - B = 0, \tag{2.28}$$

where

$$A = \sum_{i=1}^{m} \left(\{ (R_i + 1)\phi + 1 \} \gamma_i - 1 \right) (x_{i:m:n} - k),$$
$$B = \sum_{i=1}^{m} \left(\{ (R_i + 1)\phi + 1 \} \delta_i \right) (x_{i:m:n} - k)^2 > 0. \tag{2.29}$$

Hence, we have

$$\widehat{\sigma} = \frac{A \pm \sqrt{A^2 + 4mB}}{2m}. \tag{2.30}$$

Since $B \geq 0$, only one root is admissible, and hence the approximate MLE of σ is given by

$$\widehat{\sigma} = \frac{A + \sqrt{A^2 + 4mB}}{2m}, \tag{2.31}$$

the approximate MLEs are thus given explicitly by (2.23) and (2.31).

Remark 2.1. When the shape parameter $\phi = 1$ the approximate MlEs estimators of μ and σ are identical to those derived in Balakrishnan and Kannan [12] for the case of Logistic Distribution under progressive Type II censoring.

3. Observed Fisher Information Matrix

We need to compute the asymptotic variances-covariance matrix. In this section, we derive the observed Fisher information matrix for the full and approximate equations.

Now, we derive the observed Fisher information matrix for the Likelihood equation (2.11). We have

$$\frac{\partial^2 L^*}{\partial \mu^2} = -\frac{1}{\sigma^2} \sum_{i=1}^{m} \{ (R_i + 1)\phi + 1 \} g(y_{i:m:n}),$$
$$\frac{\partial^2 L^*}{\partial \sigma^2} = \frac{m}{\sigma^2} + \frac{1}{\sigma^2} \sum_{i=1}^{m} y_{i:m:n} - \frac{1}{\sigma^2} \sum_{i=1}^{m} \{ (R_i + 1)\phi + 1 \} y_{i:m:n} G(y_{i:m:n})$$
$$- \sum_{i=1}^{m} \{ (R_i + 1)\phi + 1 \} (y_{i:m:n})^2 g(y_{i:m:n}), \tag{3.1}$$
$$\frac{\partial^2 L^*}{\partial \sigma \partial \mu} = \frac{m}{\sigma^2} - \frac{1}{\sigma^2} \sum_{i=1}^{m} \{ (R_i + 1)\phi + 1 \} G(y_{i:m:n})$$
$$- \frac{1}{\sigma^2} \sum_{i=1}^{m} \{ (R_i + 1)\phi + 1 \} y_{i:m:n} g(y_{i:m:n}).$$

Similarly, from the approximate likelihood equations, we obtain from (2.20) and (2.21)

$$\frac{\partial^2 L^*}{\partial \mu^2} = -\frac{1}{\sigma^2}\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\delta_{i:m:n},$$

$$\frac{\partial^2 L^*}{\partial \sigma^2} = \frac{m}{\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{m} y_{i:m:n} - \frac{1}{\sigma^2}\sum_{i=1}^{m}\{(R_i+1)\phi+1\} y_{i:m:n}\left(\gamma_{i:m:n} + \delta_{i:m:n} y_{i:m:n}\right)$$
$$- \sum_{i=1}^{m}\{(R_i+1)\phi+1\}\left(y_{i:m:n}\right)^2 \delta_{i:m:n}, \tag{3.2}$$

$$\frac{\partial^2 L^*}{\partial \sigma \partial \mu} = \frac{m}{\sigma^2} - \frac{1}{\sigma^2}\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\left(\gamma_{i:m:n} + \delta_{i:m:n} y_{i:m:n}\right)$$
$$- \frac{1}{\sigma^2}\sum_{i=1}^{m}\{(R_i+1)\phi+1\}\left(y_{i:m:n}\right)^2 \delta_{i:m:n}.$$

Now, let

$$-\frac{\partial^2 L^*}{\partial \mu^2} = \frac{V_1}{\sigma^2}, \qquad -\frac{\partial^2 L^*}{\partial \sigma^2} = \frac{V_2}{\sigma^2}, \qquad -\frac{\partial^2 L^*}{\partial \sigma \partial \mu} = \frac{V_3}{\sigma^2}. \tag{3.3}$$

The observed information matrix can be inverted to obtain the asymptotic variance-covariance matrix of the estimators as

$$\frac{1}{\sigma^2}\begin{pmatrix} V_1 & V_2 \\ V_2 & V_3 \end{pmatrix}^{-1} = \sigma^2 \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix}, \tag{3.4}$$

where

$$V_{11} = \frac{V_3}{V_1 V_3 - V_2^2}, \qquad V_{12} = \frac{V_2}{V_1 V_3 - V_2^2}, \qquad V_{22} = \frac{V_1}{V_1 V_3 - V_1^2}. \tag{3.5}$$

4. Scale and Shape Parameters Estimation When the Location Parameter Is Known

The maximum likelihood estimators (MLEs) for the scale and shape parameters of the RGL distribution based on progressive Type II censoring are derived. An exact confidence interval for the scale parameter and an exact joint confidence region for the scale and shape parameters are investigated also. The pdf in (2.1) for the shape and scale parameters may be written as

$$f(x,\sigma) = \frac{\phi}{\sigma}\exp\left(\frac{x}{\sigma}\right)\left[1+\exp\left(\frac{x}{\sigma}\right)\right]^{-(\phi+1)}, \quad \sigma > 0,\ \phi > 0, \tag{4.1}$$

and the associated cumulative distribution function (cdf) is given by

$$F(x,\sigma) = 1 - \left[1 + \exp\left(\frac{x}{\sigma}\right)\right]^{-\phi}, \quad \sigma > 0,\ \alpha > 0. \tag{4.2}$$

The likelihood function is given by

$$L_1(c,\sigma) = k\prod_{i=1}^{m} f(x_{i:m:n},\sigma)[1 - F(x_{i:m:n},\sigma)]^{R_i}, \tag{4.3}$$

where

$$k = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\cdots(n - R_1 - R_2 - R_3 \cdots - R_{m-1} - m + 1), \tag{4.4}$$

and $y_{i:m:n} = x_{i:m:n}/\sigma$.

The log-likelihood function is then given by

$$\begin{aligned} L_1^* = {} & \log k + m\log\phi - m\log\sigma + \sum_{i=1}^{m} y_{i:m:n} - \sum_{i=1}^{m}\log(1 + \exp(y_{i:m:n})) \\ & + \sum_{i=1}^{m}(R_i + 1)\log\left[1 - F(y_{i:m:n})\right]. \end{aligned} \tag{4.5}$$

Hence, we have the likelihood equations for ϕ and σ as

$$\frac{\partial L_1^*}{\partial\sigma} = -\frac{m}{\sigma} - \sum_{i=1}^{m} y_{i:m:n} - \frac{1}{\sigma}\sum_{i=1}^{m}\frac{y_{i:m:n}\exp(y_{i:m:n})}{1 + \exp(y_{i:m:n})} + \frac{\phi}{\sigma}\sum_{i=1}^{m}(R_i + 1)\frac{y_{i:m:n}\exp(y_{i:m:n})}{1 + \exp(y_{i:m:n})}, \tag{4.6}$$

$$\frac{\partial L_1^*}{\partial\phi} = -\frac{m}{\phi} + \sum_{i=1}^{m}(R_i + 1)\log\left[1 + \exp(y_{i:m:n})\right]. \tag{4.7}$$

The MLEs $\widehat{\phi}$ and $\widehat{\sigma}$ can be obtained by solving the likelihood equations. Equation (4.7) yields the MLe of ϕ to be

$$\widehat{\phi} = \frac{m}{\sum_{i=1}^{m}(R_i + 1)\log\left[1 + \exp(x_{i:m:n}/\widehat{\sigma})\right]}. \tag{4.8}$$

Equation (4.6), in conjunction with the MLE of ϕ in (4.8), reduces to

$$m - \sum_{i=1}^{m} x_{i:m:n} - \sum_{i=1}^{m} x_{i:m:n} G\left(\frac{x_{i:m:n}}{\widehat{\sigma}}\right) + \frac{m\sum_{i=1}^{m}(R_i + 1)x_{i:m:n}G(x_{i:m:n}/\widehat{\sigma})}{\sum_{i=1}^{m}(R_i + 1)\ln(1 + \exp(x_{i:m:n}/\widehat{\sigma}))} = 0. \tag{4.9}$$

Since (4.9) can not be solved analytically for $\widehat{\sigma}$, some numerical methods such as Newton's method must be employed.

One can obtained an approximate MLEs estimators $\tilde{\phi}$ and $\tilde{\sigma}$ by approximating the logistic cdf $G(y_{i:m:n})$ as we did in Section 2.

In the rest of this section, an exact confidence interval for the scale parameter and an exact joint confidence region for the scale and shape parameters are investigated. Let $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ denote a progressive type II censored sample from a two-parameter RLG distribution, with censoring scheme $R_1, R_2, \ldots, R_m$. Further, Let $Y_i = \log(1 + \exp(X_{i:m:n}/\sigma))^{\phi}$, $i = 1, 2, \ldots, m$. It can be seen that $Y_1 < Y_2 < \cdots < Y_m$ is a progressive Type II censored order statistic from an exponential distribution with mean 1.

Let us consider the following transformation:

$$\begin{gathered} Z_1 = nY_1 \\ Z_2 = (n - R_1 - 1)(Y_1 - Y_2) \\ \vdots \\ Z_m = (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)(Y_m - Y_{m-1}). \end{gathered} \tag{4.10}$$

Thomas and Wilson [22] established that the generalized spacing $Z_1, Z_2, \ldots, Z_m$ as defined above are independent and identically distributed as an exponential distribution with mean 1. Hence,

$$W_1 = 2Z_1 = 2nY_1 \tag{4.11}$$

has a Chi-square distribution with 2 degree of freedom and

$$W_2 = 2\sum_{i=1}^{m} Z_i = 2\left\{ \sum_{i=1}^{m} (R_i + 1)Y_i - nY_1 \right\} \tag{4.12}$$

has a Chi-square distribution with $2m - 2$ degree of freedom. Now, let

$$\begin{gathered} U = \frac{W_2}{(m-1)W_1} = \frac{\sum_{i=1}^{m}(R_i + 1)Y_i - nY_1}{n(m-1)Y_1}, \\ V = W_1 + W_2 = (m-1)W_1 = 2\sum_{i=1}^{m}(R_i + 1)Y_i. \end{gathered} \tag{4.13}$$

It is easy to see that U has an F distribution with $2m - 2$ and 2 degree for freedom and V has a Chi-square distribution with 2 degree of freedom. Also U and V are independent.

Lemma 4.1. *Suppose that* $0 < Y_1 < Y_2 < \cdots < Y_m$. *Let*

$$U(\sigma) = \frac{\sum_{i=1}^{m}(R_i + 1)Y_i - nY_1}{n(m-1)Y_1}, \tag{4.14}$$

where $Y_i = \log(1+\exp(X_{i:m:n}/\sigma))^{\phi}$, $i = 1, 2, \ldots, m$, *then,* $U(\sigma)$ *is strictly decreasing function in* σ *for any* $\sigma > 0$*. Furthermore, if* $t > 0$*, the equation* $U(\sigma) = t$ *has a unique solution for any* $\sigma > 0$*.*

Proof. The proof is similar to the proof of Lemma 1 in Wu [9]. □

The following theorem gives an exact confidence interval for the parameter σ.

Theorem 4.2. *Suppose that* $X_{i:m:n}$, $i = 1, 2, \ldots, m$, *are the order statistics of a progressively type II right censored sample of size* n*, from a two-parameter RGL distribution, with censoring scheme* $(R_1, R_2, \ldots R_m)$*, then* $100(1-\alpha)\%$ *confidence interval for the scale parameter* σ *is given by*

$$(h(X_1, \ldots, X_n), F_{\alpha/2}(2m-2, 2), h(X_1, \ldots, X_n), F_{1-\alpha/2}(2m-2, 2)), \tag{4.15}$$

where $0 < \alpha < 1$ *and* $h(X_1, \ldots, X_n, t)$ *is the solution of* σ *for the equation*

$$\frac{\sum_{i=1}^{m}(R_i+1)\log\left(1+\exp(Y_{i:m:n})\right) - n\log\left(1+\exp(Y_{1:m:n})\right)}{n(m-1)\log\left(1+\exp(Y_{1:m:n})\right)} = t. \tag{4.16}$$

Proof. The proof is similar to the proof of Theorem 1 in Wu [9]. □

Let $\chi^2_{\alpha(\delta)}$ denote the percentile of Chi-square distribution with the right-tail probability α and δ degree of freedom. An exact joint confidence region for the parameters ϕ and σ is given by the following theorem.

Theorem 4.3. *Suppose that* $X_{i:m:n}$, $i = 1, 2, \ldots, m$, *are the order statistics of a progressively type II right censored sample of size* n*, from a two-parameter RGL distribution, with censoring scheme* $(R_1, R_2, \ldots R_m)$*, then* $100(1-\alpha)\%$ *confidence region for the shape and scale parameters* ϕ *and* σ *is given by the following inequalities:*

$$h\Big(X_1, \ldots, X_n, F_{(1-\sqrt{1-\alpha})/2}(2m-2, 2)\Big) < \sigma < h\Big(X_1, \ldots, X_n, F_{(1+\sqrt{1-\alpha})/2}(2m-2, 2)\Big),$$
$$\frac{\chi^2_{(1+\sqrt{1-\alpha})/2(2m)}}{2\sum_{i=1}^{m}(R_i+1)\log\left(1+\exp(X_{i:m:n}/\sigma)\right)} < \phi < \frac{\chi^2_{(1-\sqrt{1-\alpha})/2(2m)}}{2\sum_{i=1}^{m}(R_i+1)\log\left(1+\exp(X_{i:m:n}/\sigma)\right)}, \tag{4.17}$$

where $0 < \alpha < 1$ *and* $h(X_1, \ldots, X_n, t)$ *is the solution of* σ *for the equation*

$$\frac{\sum_{i=1}^{m}(R_i+1)\log\left(1+\exp(X_{i:m:n}/\sigma)\right) - n\log\left(1+\exp(y_{1:m:n})\right)}{n(m-1)\log\left(1+\exp(X_{i:m:n}/\sigma)\right)} = t. \tag{4.18}$$

Proof. The proof is similar to the proof of Theorem 2 in Wu [9]. □

5. Illustrative Example

To illustrate the methods of inference proposed in this paper, the following example is discussed.

Table 1: MLEs of estimators and the approximated MLEs of estimators for $\phi = 1, 1.5, 2$, and 3.

	MLES		Approximate MLEs	
ϕ	$\hat{\mu}$	$\hat{\sigma}$	$\check{\mu}$	$\check{\sigma}$
1	1.875	0.903	1.685	0.920
1.5	2.364	0.941	2.321	0.960
2	2.705	0.961	2.718	0..989
3	3.176	0.982	3.202	1.032

Table 2: Average, variances, and covariance of the MLEs when the shape parameter $\phi = 1$.

n	m	Scheme	$\hat{\mu}$	$\hat{\sigma}$	$\text{Var}(\hat{\mu})$	$\text{Var}(\hat{\sigma})$	$\text{Cov}(\hat{\mu}, \hat{\sigma})$
10	2	$(8, 0)$	−0.5470	0.7741	0.9621	0.1799	0.2466
10	2	$(6, 2)$	−0.7365	0.6491	0.7874	0.1772	0.2531
10	2	$(5, 3)$	−0.7951	0.6155	0.7839	0.1824	0.2678
10	2	$(4, 4)$	−0.8434	0.5892	0.7908	0.1883	0.2827
10	2	$(0, 8)$	−0.9828	0.5190	0.8507	0.2173	0.34011
10	4	$(6, 0, 0, 0)$	−0.1515	0.9053	0.5497	0.1243	0.0854
10	4	$(5, 1, 0, 0)$	−0.1626	0.8938	0.5243	0.1721	0.0914
10	4	$(4, 2, 0, 0)$	−0.1721	0.8856	0.5106	0.0971	0.1296
10	4	$(0, 0, 2, 4)$	−0.2533	0.7925	0.4089	0.11380	0.1668
10	4	$(0, 0, 0, 6)$	−0.2583	0.7876	0.4103	0.1725	0.1434
10	5	$(5, 0, 0, 0, 0)$	−0.0996	0.9116	0.4473	0.1070	0.05302
10	5	$(4, 1, 0, 0, 0)$	−0.1070	0.9050	0.4315	0.1088	0.0576
10	5	$(3, 2, 0, 0, 0)$	−0.1135	0.9003	0.4218	0.1104	0.0616
10	5	$(2, 2, 1, 0, 0)$	−0.1220	0.8906	0.4008	0.1142	0.0661
10	5	$(1, 2, 2, 0, 0)$	−0.1303	0.8830	0.3384	0.1179	0.0713
10	5	$(0, 0, 0, 0, 5)$	−0.1533	0.8325	0.3173	0.1393	0.08164
20	5	$(0, 0, 0, 0, 15)$	−0.3329	0.8126	0.4683	0.1519	0.1998
20	5	$(15, 0, 0, 0, 0)$	−0.1442	0.9424	0.4784	0.0844	0.0777
20	5	$(10, 5, 0, 0, 0)$	−0.1693	0.9249	0.4503	0.0873	0.0928
20	5	$(5, 10, 0, 0, 0)$	−0.1827	0.9190	0.4536	0.0897	0.1023
20	5	$(3, 3, 3, 3, 3)$	−0.2600	0.8538	0.3904	0.1120	0.1320
20	5	$(5, 5, 5, 0, 0)$	−0.2031	0.9015	0.4316	0.0964	0.1130
20	10	$(10, 0, \ldots, 0)$	−0.0416	0.9623	0.2571	0.0576	0.05671
20	10	$(4, 3, 3, \ldots, 0)$	−0.0541	0.9574	0.2422	0.0578	0.0297
20	10	$(0, 1, 2, 3, 4, \ldots, 0)$	−0.0663	0.9471	0.2219	0.0631	0.0397
20	10	$(0, \ldots, 0, 10)$	−0.0726	0.9222	0.1781	0.0776	0.0429

Example 5.1. A sample of size $m = 8$ of a progressively Type II censored data giving the log-time to breakdown of an insulating fluid tested at 34 kilovolts from RGL distribution was randomly selected from the $n = 19$ observation in Nelson's [23, Table 5.1], refer to Table 10 in this paper, as utilized earlier by Viveros and Balakrishnan [6] and studied by Balakrishnan and Kannan [12] as a data came from logistic distribution.

Table 3: Average, variances, and covariance of the approximate MLEs when the shape parameter $\phi = 1$.

n	m	Scheme	$\tilde{\mu}$	$\tilde{\sigma}$	$\mathrm{Var}(\tilde{\mu})$	$\mathrm{Var}(\tilde{\sigma})$	$\mathrm{Cov}(\tilde{\mu},\tilde{\sigma})$
10	2	$(8,0)$	−0.5455	0.8071	1.0456	0.1955	0.2680
10	2	$(6,2)$	−0.7365	0.6491	0.7874	0.1772	0.2531
10	2	$(5,3)$	−0.8205	0.6554	0.8886	0.2068	0.3036
10	2	$(4,4)$	−0.8536	0.6321	0.9101	0.2168	0.3253
10	2	$(0,8)$	−0.9617	0.5487	0.9507	0.2399	0.3801
10	4	$(6,0,0,0)$	−0.1526	0.9680	0.6292	0.1424	0.0977
10	4	$(5,1,0,0)$	−0.1588	0.9597	0.6054	0.1468	0.1055
10	4	$(4,2,0,0)$	−0.1602	0.9526	0.5916	0.1501	0.1125
10	4	$(0,0,2,4)$	−0.2630	0.8231	0.4412	0.1801	0.1490
10	4	$(0,0,0,6)$	−0.2638	0.8184	0.4434	0.1864	0.1550
10	5	$(5,0,0,0,0)$	−0.1076	0.9753	0.5131	0.1123	0.0608
10	5	$(4,1,0,0,0)$	−0.1077	0.9697	0.4965	0.1251	0.0662
10	5	$(3,2,0,0,0)$	−0.1069	0.9641	0.4847	0.1268	0.0708
10	5	$(2,2,1,0,0)$	−0.1074	0.9533	0.4602	0.1311	0.0759
10	5	$(1,2,2,0,0)$	−0.1073	0.9430	0.4439	0.1348	0.0815
10	5	$(0,0,0,0,5)$	−0.1685	0.88613	0.3402	0.1493	0.0875
20	5	$(15,0,0,0,0)$	−0.1459	1.0027	0.5425	0.0957	0.0875
20	5	$(10,5,0,0,0)$	−0.1537	0.9907	0.5200	0.1004	0.1068
20	5	$(5,10,0,0,0)$	−0.1509	0.9773	0.5141	0.1016	0.1160
20	5	$(3,3,3,3,3)$	−0.2656	0.8897	0.4251	0.1220	0.1437
20	5	$(5,5,5,0,0)$	−0.1638	0.9649	0.4956	0.1106	0.1297
20	5	$(0,0,0,0,15)$	−0.3210	0.8342	0.4940	0.1602	0.2108
20	10	$(10,0,\ldots,0)$	−0.0569	1.0164	0.2873	0.0634	0.0234
20	10	$(4,3,3,\ldots,0)$	−0.0518	1.0044	0.2668	0.0637	0.0326
20	10	$(0,1,2,3,4,\ldots,0)$	−0.0452	0.9855	0.2404	0.0684	0.0431
20	10	$(10,0,\ldots,0)$	−0.0790	0.9392	0.1848	0.0806	0.0445

For this example,

(1) We obtain the MLEs of μ and σ and the approximated MLEs of μ and σ, for different value of the shape parameters ϕ as Table 1 shows.

Table 1 provides the values of MLEs of μ and σ and the approximated values of μ and σ, for $\phi = 1$, 1.5, 2, and 3. The entries were computed using (2.12) for the MLEs values and (2.23) and (2.31) for the approximated values and MATHEMATICA.

Note that for $\phi = 1$, the RGL distribution encompasses to the logistic distribution as a further special case moreover, these values agree with the reported MLEs and the approximated MLEs in Balakrishnan and Kannan [12] for the logistic distribution.

(2) We can determine numerically the values of the MLEs $\widehat{\phi}$ and $\widehat{\sigma}$ by solving (4.8) and (4.9) numerically to be [0.2248] and [0.6445], respectively. By Theorem 4.2 and using Mathematica Package we can get a 95% confidence interval for the scale parameter σ as $(0.6200, 2.3043)$, where the percentile

$$F_{0.025(14,2)} = 39.4265, \qquad F_{0.975(14,2)} = 0.2059. \tag{5.1}$$

Table 4: Average, variances, and covariance of the MLEs when the shape parameter $\phi = 1.5$.

n	m	Scheme	$\hat{\mu}$	$\hat{\sigma}$	$\mathrm{Var}(\hat{\mu})$	$\mathrm{Var}(\hat{\sigma})$	$\mathrm{Cov}(\hat{\mu},\hat{\sigma})$
10	2	(8, 0)	−0.6731	0.7603	0.5688	0.11173	0.1267
10	2	(6, 2)	−0.9100	0.6416	0.4326	0.1170	0.1257
10	2	(5, 3)	−0.9809	0.6092	0.4210	0.1215	0.1343
10	2	(4, 4)	−1.0388	0.5836	0.4178	0.1267	0.1434
10	2	(0, 8)	−1.2050	0.5150	0.4355	0.1500	0.1813
10	4	(6, 0, 0, 0)	−0.1961	0.9014	0.3763	0.0850	0.0271
10	4	(5, 1, 0, 0)	−0.2115	0.8902	0.3510	0.0876	0.0304
10	4	(4, 2, 0, 0)	−0.2242	0.8822	0.3358	0.0900	0.0338
10	4	(0, 0, 2, 4)	−0.2444	0.8713	0.3190	0.0943	0.0401
10	4	(0, 0, 0, 6)	−0.3712	0.7816	0.2304	0.1375	0.0539
10	5	(5, 0, 0, 0, 0)	−0.1424	0.9089	0.3216	0.0741	0.0065
10	5	(4, 1, 0, 0, 0)	−0.1517	0.9028	0.3049	0.0759	0.0094
10	5	(3, 2, 0, 0, 0)	−0.1594	0.8984	0.2937	0.0776	0.0378
10	5	(2, 2, 1, 0, 0)	−0.1726	0.8888	0.2735	0.0817	0.0137
10	5	(1, 2, 2, 0, 0)	−0.1841	0.8813	0.2599	0.0856	0.0165
10	5	(0, 0, 0, 0, 5)	−0.2487	0.8263	0.1988	0.1124	0.0130
20	5	(15, 0, 0, 0, 0)	−0.1690	0.9405	0.3246	0.0572	0.0330
20	5	(10, 5, 0, 0, 0)	−0.1985	0.9238	0.2911	0.0602	0.0438
20	5	(5, 10, 0, 0, 0)	−0.2136	0.9182	0.2864	0.0625	0.0507
20	5	(3, 3, 3, 3, 3)	−0.3356	0.8493	0.2171	0.0837	0.0653
20	5	(5, 5, 5, 0, 0)	−0.2429	0.9005	0.2616	0.0686	0.0566
20	5	(0, 0, 0, 0, 15)	−0.4241	0.8097	0.2505	0.1287	0.1173
20	10	(10, . . . , 0)	−0.0611	0.9610	0.1943	0.0398	−0.0036
20	10	(4, 3, 3, . . . , 0)	−0.0732	0.9567	0.1736	0.0411	0.0032
20	10	(0, 1, 2, 3, 4, . . . , 0)	−0.0889	0.9471	0.1490	0.0095	0.0468
20	10	(0, . . . , 0, 10)	−0.1167	0.9190	0.1138	z‘0.0672	0.0041

Furthermore, to obtain a 95% joint confidence interval for the scale parameter σ and the shape parameter ϕ, we need the following percentiles:

$$F_{0.025(14,2)} = 39.4265, \qquad F_{0.975(14,2)} = 0.2059,$$
$$\chi^2_{0.0127(16)} = 31.2069, \qquad \chi^2_{0.9873(16)} = 6.0684. \tag{5.2}$$

By Theorem 4.3, a 95% joint confidence interval for parameter σ and ϕ determined by the following inequalities:

$$0.6200 < \sigma < 2.3043,$$
$$\frac{6.0684}{2\sum_{i=1}^{m}(R_i+1)\log\left(1+\exp\left(X_{i:m:n}/\sigma\right)\right)} < \phi < \frac{31.2069}{2\sum_{i=1}^{m}(R_i+1)\log\left(1+\exp\left(X_{i:m:n}/\sigma\right)\right)}, \tag{5.3}$$

Table 5: Average, variances, and covariance of the approximate MLEs when the shape parameter $\phi = 1.5$.

n	m	Scheme	$\tilde{\mu}$	$\tilde{\sigma}$	$\mathrm{Var}(\tilde{\mu})$	$\mathrm{Var}(\tilde{\sigma})$	$\mathrm{Cov}(\tilde{\mu}, \tilde{\sigma})$
10	2	$(8, 0)$	−0.7506	0.8203	0.6625	0.11365	0.1475
10	2	$(6, 2)$	−0.8987	0.7331	0.5649	0.1528	0.1641
10	2	$(5, 3)$	−0.9386	0.7065	0.5663	0.1635	0.1806
10	2	$(4, 4)$	−0.9479	0.6820	0.5705	0.1729	0.1958
10	2	$(0, 8)$	−1.1518	0.5701	0.5337	0.1838	0.2222
10	4	$(6, 0, 0, 0)$	−0.2725	0.9826	0.4489	0.1013	0.0323
10	4	$(5, 1, 0, 0)$	−0.2757	0.9753	0.4230	0.1056	0.03665
10	4	$(4, 2, 0, 0)$	−0.2780	0.9674	0.4055	0.1087	0.0408
10	4	$(2, 4, 0, 0)$	−0.2851	0.9487	0.3794	0.1121	0.0477
10	4	$(0, 0, 0, 6)$	−0.3640	0.8438	0.2688	0.1604	0.0629
10	5	$(5, 0, 0, 0, 0)$	−0.1076	0.9753	0.5131	0.1123	0.0608
10	5	$(4, 1, 0, 0, 0)$	−0.2058	0.9862	0.3660	0.0912	0.0112
10	5	$(3, 2, 0, 0, 0)$	−0.2071	0.9791	0.3508	0.0927	0.0142
10	5	$(2, 2, 1, 0, 0)$	−0.2097	0.9664	0.3250	0.0970	0.0163
10	5	$(1, 2, 2, 0, 0)$	−0.2146	0.9528	0.3049	0.1004	0.0193
10	5	$(0, 0, 0, 0, 5)$	−0.2045	0.9924	0.3856	0.0889	0.0078
20	5	$(15, 0, 0, 0, 0)$	−0.2252	1.0221	0.3854	0.0680	0.0390
20	5	$(10, 5, 0, 0, 0)$	−0.2314	1.0108	0.3507	0.0725	0.0528
20	5	$(5, 10, 0, 0, 0)$	−0.2394	0.9924	0.3359	0.0732	0.0594
20	5	$(3, 3, 3, 3, 3)$	−0.3161	0.9283	0.2603	0.1004	0.0783
20	5	$(5, 5, 5, 0, 0)$	−0.2490	0.9816	0.3123	0.0819	0.0675
20	5	$(0, 0, 0, 0, 15)$	−0.4124	0.8524	0.2780	0.1428	0.1301
20	10	$(10, 0, \ldots, 0)$	−0.0795	1.0419	0.2300	0.0470	−0.0043
20	10	$(4, 3, 3, \ldots, 0)$	−0.0850	1.0263	0.2002	0.0474	0.0037
20	10	$(0, 1, 2, 3, 4, \ldots, 0)$	−0.1003	0.9972	0.1652	0.0519	0.0106
20	10	$(0, \ldots, 0, 10)$	−0.1291	0.9578	0.1236	0.0730	0.0045

In connecting to the previous work, Lawless [2, pages 147–154, 533–539] used the conditional method to handle data from complete and Type II right censored samples and suggested it for progressively Type II censored samples. Viveros and Balakrishnan [6] extend the conditional method to obtain conditional inference under progressively Type II censored samples. In compassion between the conditional method and the unconditional method used in this paper, it is important to mention here that the intervals obtained by Viveros and Balakrishnan [6] are conditional in that they are based on the specific progressive censored sample, while the interval and region we have obtained here are unconditional in that the above percentiles given by Theorems 1 and 2 can be used to obtain confidence interval and confidence region for any other progressively type censored sample with same scheme, that is, same $(m, n, R_1, R_2, \ldots, R_m)$. Moreover the conditional approach of Viveros and Balakrishnan [6] requires intensive numerical integration computationals for determining percentage points while the method used here does not require any numerical integration.

6. Simulation Results

In this section, we introduce a simulation study to compare the performance of the approximate MLEs estimators with the MLEs and also to shows the effect of the change of

Table 6: Average, variances, and covariance of the MLEs when the shape parameter $\phi = 2$.

n	m	Scheme	$\hat{\mu}$	$\hat{\sigma}$	$\mathrm{Var}(\hat{\mu})$	$\mathrm{Var}(\hat{\sigma})$	$\mathrm{Cov}(\hat{\mu},\hat{\sigma})$
10	2	$(0,8)$	−1.3586	0.5130	0.2929	0.1255	0.1264
10	2	$(6,2)$	−1.0302	0.6377	0.2992	0.0911	0.0799
10	2	$(5,3)$	−1.1093	0.6059	0.2881	0.0958	0.0867
10	2	$(4,4)$	−1.1740	0.5808	0.2838	0.1010	0.0941
10	2	$(8,0)$	−0.7664	0.7522	0.4048	0.0886	0.0817
10	4	$(6,0,0,0)$	−0.2347	0.8968	0.2933	0.0661	0.0062
10	4	$(5,1,0,0)$	−0.2532	0.8856	0.2706	0.0687	0.0083
10	4	$(4,2,0,0)$	−0.2684	0.8776	0.2564	0.0711	0.0108
10	4	$(2,4,0,0)$	−0.2919	0.8667	0.2400	0.0753	0.0154
10	4	$(0,0,0,6)$	−0.4485	0.7786	0.1661	0.1302	0.0185
10	5	$(5,0,0,0,0)$	−0.1766	0.9057	0.2603	0.0587	−0.0098
10	5	$(4,1,0,0,0)$	−0.1873	0.8996	0.2446	0.0605	−0.0077
10	5	$(3,2,0,0,0)$	−0.1961	0.8952	0.2339	0.0622	−0.0058
10	5	$(2,2,1,0,0)$	−0.2123	0.8857	0.2162	0.0665	−0.0053
10	5	$(1,2,2,0,0)$	−0.2264	0.8781	0.2036	0.0707	−0.0039
10	5	$(0,0,0,0,5)$	−0.3127	0.8232	0.1586	0.1079	−0.0172
20	5	$(15,0,0,0,0)$	−0.1915	0.9378	0.2499	0.0441	0.0163
20	5	$(10,5,0,0,0)$	−0.2254	0.9212	0.2171	0.0469	0.0249
20	5	$(5,10,0,0,0)$	−0.2422	0.9156	0.2109	0.0489	0.0303
20	5	$(3,3,3,3,3)$	−0.3884	0.8470	0.1511	0.0720	0.0402
20	5	$(5,5,5,0,0)$	−0.2777	0.8976	0.1885	0.0549	0.0344
20	5	$(0,0,0,0,15)$	−0.4869	0.8082	0.1726	0.1297	0.0898
20	10	$(10,0,\ldots,0)$	−0.0760	0.9596	0.1634	0.0320	−0.0124
20	10	$(4,3,3,\ldots,0)$	−0.0883	0.9556	0.1412	0.0332	−0.0063
20	10	$(0,1,2,3,4,\ldots,0)$	−0.1064	0.9462	0.1172	0.0392	−0.0019
20	10	$(0,0,\ldots,10)$	−0.1469	0.9174	0.0933	0.0696	−0.0157

the value of the shape parameter ϕ on the values of MLEs estimators and the approximate MLEs estimators. A progressively Type-II censored samples from the standard RGL distribution were generated by using the algorithm presented in Balakrishnan and Sandhu [24] according to the following steps.

(1) Generate m independent $U(0,1)$ random variables $W_1, W_2, \ldots, W_m$.

(2) For given values of the progressive censoring scheme $R_I, R_2, \ldots, R_m$, set $V = W_i^{1/i+R_m+R_{m-1}+\cdots+R_{m-i+1}}$, $i = 1,2,\ldots,m$.

(3) Set $U = 1-(V_m V_{m-1},\ldots,V_{m-i+1})$, $i = 1,2,\ldots,m$; then $U_1, U_2, \ldots, U_m$ is a progressive Type II censored sample of size m from $U(0,1)$.

(4) Finally, set $\mathrm{X}_i = F^{-1}(U_i)$ for $i = 1,2,\ldots,m$, where $F^{-1}(\cdot)$ is the inverse cdf of the standard RGL distribution.

Table 7: Average, variances, and covariance of the approximate MLEs when the shape parameter $\phi = 2$.

n	m	Scheme	$\tilde{\mu}$	$\tilde{\sigma}$	$\mathrm{Var}(\tilde{\mu})$	$\mathrm{Var}(\tilde{\sigma})$	$\mathrm{Cov}(\tilde{\mu},\tilde{\sigma})$
10	2	$(0,8)$	−1.3001	0.5887	0.3858	0.1653	0.1664
10	2	$(6,2)$	−0.9726	0.7784	0.4459	0.1357	0.01191
10	2	$(5,3)$	−01.0143	0.7517	0.4433	0.1474	0.1334
10	2	$(4,4)$	−1.0564	0.7254	0.4426	0.1546	0.1468
10	2	$(8,0)$	−0.8524	0.8525	0.5200	0.1138	0.1041
10	4	$(6,0,0,0)$	−0.2971	1.0274	0.3876	0.0873	0.0082
10	4	$(5,1,0,0)$	−0.3054	1.0195	0.3613	0.0918	0.1119
10	4	$(4,2,0,0)$	−0.3125	1.0107	0.3427	0.0950	0.0144
10	4	$(2,4,0,0)$	−0.3308	0.9884	0.3139	0.0985	0.0201
10	4	$(0,0,0,6)$	−0.4492	0.8683	0.2070	0.1623	0.0231
10	5	$(5,0,0,0,0)$	−0.2106	1.0403	0.3469	0.0782	−0.0131
10	5	$(4,1,0,0,0)$	−0.2176	1.0335	0.3260	0.0807	−0.0103
10	5	$(3,2,0,0,0)$	−0.2241	1.0252	0.3095	0.0823	−0.0077
10	5	$(2,2,1,0,0)$	−0.2382	1.009	0.2829	0.0870	−0.0070
10	5	$(1,2,2,0,0)$	−0.2539	0.9919	0.2614	0.0907	−0.0050
10	5	$(0,0,0,0,5)$	−0.3250	0.9125	0.1953	0.1329	−0.0213
20	5	$(15,0,0,0,0)$	−0.2173	1.0681	0.3268	0.0578	0.0213
20	5	$(10,5,0,0,0)$	−0.2326	1.0572	0.2889	0.0624	0.0331
20	5	$(5,10,0,0,0)$	−0.2417	1.0355	0.2716	0.0630	0.0390
20	5	$(3,3,3,3,3)$	−0.3590	0.9667	0.1976	0.0942	0.0526
20	5	$(5,5,5,0,0)$	−0.2748	1.022	0.2464	0.0718	0.0449
20	5	$(0,0,0,0,15)$	−0.5105	0.8690	0.1998	0.1501	0.1039
20	10	$(10,0,\dots,0)$	−0.0252	1.0969	0.2150	0.0421	−0.0163
20	10	$(4,3,3,\dots,0)$	−0.0471	1.0802	0.1812	0.0426	−0.0081
20	10	$(0,1,2,3,4,\dots,0)$	−0.0997	1.0416	0.1442	00476	−0.0024
20	10	$(0,0,\dots,10)$	−0.1811	0.99793	0.1064	0.0793	−0.0179

We computed MLEs estimators and the approximate MLEs estimators (table entries) according to the following steps.

(1) For given values of the progressive censoring scheme n, m, $R_I, R_2, \dots, R_m$ and ϕ, generate A progressively Type-II censored samples from the standard RGL distribution using the above algorithm.

(2) Calculate the approximate MLEs estimators from (2.23) and (2.31).

(3) Calculate variances and covariances matrix of the approximate MLEs estimators from (3.2).

(4) The MLEs estimators of the parameters were obtained by solving equation (2.12) numerically by using the approximate MLEs computed from (2.11) as starting values for the numerical iterations.

(5) Calculate variances and covariances matrix of MLEs estimators from (3.1).

(6) Steps (1)–(5) are repeated 10,000 times, and the average of the MLEs estimators, approximate MLEs estimators and variances and covariances matrix are caudated.

Table 8: Average, variances, and covariance of the MLEs when the shape parameter $\phi = 3$.

n	m	Scheme	$\widehat{\mu}$	$\widehat{\sigma}$	$\mathrm{Var}(\widehat{\mu})$	$\mathrm{Var}(\widehat{\sigma})$	$\mathrm{Cov}(\widehat{\mu},\widehat{\sigma})$
10	2	$(8,0)$	−0.8970	0.7437	0.2580	0.0607	0.0444
10	2	$(6,2)$	−1.1961	0.6338	0.1859	0.0654	0.0436
10	2	$(5,3)$	−1.2866	0.6027	0.1770	0.0702	0.0486
10	2	$(4,4)$	−1.3603	0.5780	0.1729	0.0756	0.0543
10	2	$(0,8)$	−1.5710	0.5110	0.1774	0.1023	0.0812
10	4	$(6,0,0,0)$	−0.2943	0.8904	0.2090	0.047	−0.0079
10	4	$(5,1,0,0)$	−0.3174	0.8792	0.1909	0.0495	−0.0068
10	4	$(4,2,0,0)$	−0.3360	0.8711	0.1793	0.0518	−0.0053
10	4	$(2,4,0,0)$	−0.3642	0.8603	0.1653	0.0557	−0.0251
10	4	$(0,0,0,6)$	−0.5540	0.7756	0.1161	0.1389	−0.0183
10	5	$(5,0,0,0,0)$	−0.2288	0.9006	0.1956	0.0430	−0.0202
10	5	$(4,1,0,0,0)$	−0.2419	0.8946	0.1823	0.0447	−0.190
10	5	$(3,2,0,0,0)$	−0.2524	0.8903	0.1731	0.0436	−0.0178
10	5	$(2,2,1,0,0)$	−0.728	0.8807	0.1597	0.0509	−0.0188
10	5	$(1,2,2,0,0)$	−0.2902	0.8731	0.1497	0.0554	−0.0188
10	5	$(0,0,0,0,5)$	−0.3999	0.8201	0.1364	0.1203	−0.0566
20	5	$(15,0,0,0,0)$	−0.2282	0.9336	0.1739	0.0306	0.0040
20	5	$(10,5,0,0,0)$	−0.2690	0.9169	0.1459	0.0331	0.0102
20	5	$(5,10,0,0,0)$	−0.1827	0.9190	0.4536	0.0897	0.1023
20	5	$(3,3,3,3,3)$	−0.4613	0.8445	0.0948	0.0606	0.0194
20	5	$(5,5,5,0,0)$	−0.3323	0.8930	0.1218	0.0404	0.0164
20	5	$(0,0,0,0,15)$	−0.5733	0.8068	0.1099	0.1611	0.0744
20	10	$(10,0,\ldots,0)$	−0.0986	0.9573	0.1294	0.0241	−0.0179
20	10	$(4,3,3,\ldots,0)$	−0.1115	0.9536	0.1075	0.0251	−0.0127
20	10	$(0,1,2,3,4,\ldots,0)$	−0.02436	1.1515	0.1294	0.0469	−0.0160
20	10	$(0,0,\ldots,10)$	−0.1881	0.9157	0.0927	0.0979	−0.0536

(7) The simulation, were carried out for sample size $n = 10$ and 20 and for each $\phi = 1, 1.5, 2$, and 3. Furthermore we provide a different choices of the effective sample m, and different progressive censoring schemes in each case including the two extreme censoring schemes of $(0,0,\ldots,n-m)$ and $(n-m,0,0,\ldots,0)$.

The results of the MLEs estimators and variances and covariances matrix of MLEs estimators are displayed in Tables 1, 3, 5, and 7 and The results of the approximate MLEs estimators and variances and covariances matrix of the approximate MLEs estimators are displayed in Tables 2, 4, 6, and 8.

From Tables 1–8, we observe that the approximate estimators and the MLEs estimators are seems to be identical in terms of bias and variance. For all sample size and censoring schemes, the approximate estimators are almost as efficient as the MLEs. The values in Tables 1 and 2 where $\phi = 1$ are close to the results reported by Balakrishnan and Kannan [12, Table 1]. For all choices, the censoring scheme ($R_1 = n-m$, $R_2 = \cdots = R_m = 0$) provide the smallest bias and variance for the estimates. Both bias and variance of the estimators decreases significantly as the effective sample proportion m/n increases.

Table 9: Average, variances, and covariance of the approximate MLEs when the shape parameter $\phi = 3$.

n	m	Scheme	$\tilde{\mu}$	$\tilde{\sigma}$	$\mathrm{Var}(\tilde{\mu})$	$\mathrm{Var}(\tilde{\sigma})$	$\mathrm{Cov}(\tilde{\mu},\tilde{\sigma})$
10	2	$(8,0)$	−0.9404	0.9231	3975	0.0934	0.0684
10	2	$(6,2)$	−1.0650	0.8532	0.3370	0.1186	0.0790
10	2	$(5,3)$	−1.1139	0.8251	0.3318	0.1317	0.0911
10	2	$(4,4)$	−1.1666	0.7958	0.3279	0.1433	0.1029
10	2	$(0,8)$	−1.5228	0.6208	0.2619	0.1509	0.1199
10	4	$(6,0,0,0)$	−0.2426	101316	0.3413	0.0768	−0.0129
10	4	$(5,1,0,0)$	−0.2690	1.1216	0.3144	0.0816	−0.0112
10	4	$(4,2,0,0)$	−0.2780	0.9674	0.4055	0.1087	0.0408
10	4	$(2,4,0,0)$	−0.2883	1.1110	0.2950	0.0852	−.0088
10	4	$(0,0,0,6)$	−0.5880	0.91129	0.1611	0.1928	−0.025
10	5	$(5,0,0,0,0)$	−0.1227	1.1528	0.3250	0.0714	−0.0336
10	5	$(4,1,0,0,0)$	−0.1445	1.1447	0.3026	0.0734	−0.0315
10	5	$(3,2,0,0,0)$	−0.1627	1.1348	0.2848	0.0763	−0.0293
10	5	$(2,2,1,0,0)$	−0.2088	1.1106	0.2569	0.0819	−0.00302
10	5	$(1,2,2,0,0)$	−0.2489	1.0688	0.2341	0.0865	−0.0293
10	5	$(0,0,0,0,5)$	−0.4348	0.9616	0.1882	0.1660	−0.0781
20	5	$(15,0,0,0,0)$	−0.1188	1.1694	0.2761	0.0487	0.0064
20	5	$(10,5,0,0,0)$	−0.1579	1.1594	0.2365	0.054	0.0165
20	5	$(5,10,0,0,0)$	−0.1509	0.9773	0.5141	0.1016	0.1160
20	5	$(3,3,3,3,3)$	−0.4289	1.4289	0.1424	0.0910	0.0292
20	5	$(5,5,5,0,0)$	−0.2599	101125	0.1913	0.0634	0.0259
20	5	$(0,0,0,0,15)$	−0.6836	0.8978	0.1363	0.1999	0.0923
20	10	$(10,0,\ldots,0)$	0160	1.221	0.2127	0.0396	−0.0294
20	10	$(4,3,3,\ldots,0)$	0.1091	1.2060	0.1730	0.0404	−0.0205
20	10	$(0,1,2,3,4,\ldots,0)$	−0.0243	1.1515	0.1294	0.0469	−0.0160
20	10	$(0,0,\ldots,10)$	−0.2800	1.020	0.1150	0.1215	−0.0665

Table 10: Progressively censored sample generated by Nelson [23].

i	1	2	3	4	5	6	7	8
$y_{i:8:19}$	−1.6608	−0.2485	−0.0409	0.2700	1.0224	1.5789	1.8718	1.9947
R_i	0	0	3	0	3	0	0	5

Finally the tables shows that for all sample size and censoring schemes, both the bias and variance of the MLEs estimators decreases significantly by increasing the value of shape parameter ϕ.

7. Conclusion

In this paper, we considers the estimation problem for the parameters of the RGL distribution based on progressive Type II censoring. We have obtained explicit estimators which are approximation to the MLEs of the location and scale parameters. Also obtained An exact confidence interval and an exact joint confidence region for the scale and shape parameters beside the MLEs.

In addition, simulation study shows that the approximation estimators are almost as efficient as the MLEs. Also shows that the value of the MLEs decreases as the value of the shape parameter increases.

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