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Research Article

A Rademacher Type Formula for Partitions and Overpartitions

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A Rademacher-type convergent series formula which generalizes the Hardy-Ramanujan-Rademacher formula for the number of partitions of n and the Zuckerman formula for the Fourier coefficients of $\vartheta_4(0 \mid \tau)^{-1}$ is presented.

1. Background

1.1. Partitions

A *partition* of an integer n is a representation of n as a sum of positive integers, where the order of the summands (called *parts*) is considered irrelevant. It is customary to write the parts in nonincreasing order. For example, there are three partitions of the integer 3, namely, 3, $2+1$, and $1+1+1$. Let $p(n)$ denote the number of partitions of n, with the convention that $p(0)=1$, and let $f(x)$ denote the generating function of $p(n)$, that is, let

$$f(x) := \sum_{n=0}^{\infty} p(n) x^n. \tag{1.1}$$

Euler [1] was the first to systematically study partitions. He showed that

$$f(x) = \prod_{m=1}^{\infty} \frac{1}{1-x^m}. \tag{1.2}$$

Euler also showed that

$$\frac{1}{f(x)} = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2}, \tag{1.3}$$

and since the exponents appearing on the right side of (1.3) are the pentagonal numbers, (1.3) is often called "Euler's pentagonal number theorem."

Although Euler's results can all be treated from the point of view of formal power series, the series and infinite products above (and indeed all the series and infinite products mentioned in this paper) converge absolutely when $|x| < 1$, which is important for analytic study of these series and products.

Hardy and Ramanujan were the first to study $p(n)$ analytically and produced an incredibly accurate asymptotic formula [2, page 85, equation (1.74)], namely,

$$\begin{aligned} p(n) = {}& \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\lfloor \alpha\sqrt{n} \rfloor} \sqrt{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} \omega(h,k) e^{-2\pi i h n/k} \frac{d}{dn} \left(\frac{\exp\left((\pi/k)\sqrt{(2/3)(n-1/24)}\right)}{\sqrt{n-1/24}} \right) \\ & + O\left(n^{-1/4}\right), \end{aligned} \tag{1.4}$$

where

$$\omega(h,k) = \exp\left(\pi i \sum_{r=1}^{k-1} \frac{r}{k} \left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2} \right) \right). \tag{1.5}$$

α is an arbitrary constant, and here and throughout (h,k) is an abbreviation for $\gcd(h,k)$.

Later Rademacher [3] improved upon (1.4) by finding the following convergent series representation for $p(n)$:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} \omega(h,k) e^{-2\pi i n h/k} \frac{d}{dn} \left(\frac{\sinh\left((\pi/k)\sqrt{(2/3)(n-1/24)}\right)}{\sqrt{n-1/24}} \right). \tag{1.6}$$

Rademacher's method was used extensively by many practitioners, including Grosswald [4, 5], Haberzetle [6], Hagis [7–15], Hua [16], Iseki [17–19], Lehner [20], Livingood [21], Niven [22], and Subramanyasastri [23] to study various restricted partitions functions.

Recently, Bringmann and Ono [24] have given exact formulas for the coefficients of all harmonic Maass forms of weight $\leqq 1/2$. The generating functions considered herein are weakly holomorphic modular forms of weight $-1/2$, and thus they are harmonic Maass forms of weight $\leqq 1/2$. Accordingly, the results of this present paper could be derived from the general theorem in [24]. However, here we opt to derive the results via classical method of Rademacher.

1.2. Overpartitions

Overpartitions were introduced by Corteel and Lovejoy in [25] and have been studied extensively by them and others including Bringmann, Chen, Fu, Goh, Hirschhorn, Hitczenko, Lascoux, Mahlburg, Robbins, Rødseth, Sellers, Yee, and Zhao [25–43].

An *overpartition* of n is a representation of n as a sum of positive integers with summands in nonincreasing order, where the last occurrence of a given summand may or may not be overlined. Thus the eight overpartitions of 3 are 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{1}$, $\overline{2}+\overline{1}$, $1+1+1$, $1+1+\overline{1}$.

Let $\overline{p}(n)$ denote the number of overpartitions of n and let $\overline{f}(x)$ denote the generating function $\sum_{n=0}^{\infty}\overline{p}(n)x^n$ of $\overline{p}(n)$. Elementary techniques are sufficient to show that

$$\overline{f}(x)=\prod_{m=1}^{\infty}\frac{1+x^m}{1-x^m}=\frac{f(x)^2}{f(x^2)}. \tag{1.7}$$

Note that

$$\frac{1}{\overline{f}(x)}=\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}. \tag{1.8}$$

via an identity of Gauss (see equation (2.2.12), page 23 in [44, 45]) , so that the reciprocal of the generating function for overpartitions is a series wherein a coefficient is nonzero if and only if the exponent of x is a perfect square, just as the reciprocal of the generating function for partitions is a series wherein a coefficient is nonzero if and only if the exponent of x is a pentagonal number.

Hardy and Ramanujan, writing more than 80 years before the coining of the term "overpartition," stated [2, page 109-110] that the function which we are calling $\overline{p}(n)$ "has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of a simple ϑ-function, of particular interest." They went on to state that

$$\overline{p}(n)=\frac{1}{4\pi}\frac{d}{dn}\left(\frac{e^{\pi\sqrt{n}}}{\sqrt{n}}\right)+\frac{\sqrt{3}}{2\pi}\cos\left(\frac{2}{3}n\pi-\frac{1}{6}\pi\right)\frac{d}{dn}\left(e^{\pi\sqrt{n}/3}\right)+\cdots+O\left(n^{-1/4}\right). \tag{1.9}$$

In fact, (1.9) was improved to the following Rademacher-type convergent series by Zuckerman [46, page 321, equation (8.53)]:

$$\overline{p}(n)=\frac{1}{2\pi}\sum_{\substack{k\geqq 1\\ 2\nmid k}}\sqrt{k}\sum_{\substack{0\leqq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi inh/k}\frac{d}{dn}\left(\frac{\sinh(\pi\sqrt{n}/k)}{\sqrt{n}}\right). \tag{1.10}$$

A simplified proof of (1.10) was given by Goldberg in his Ph.D. thesis [47].

1.3. Partitions Where No Odd Part Is Repeated

Let $pod(n)$ denote the number of partitions of n where no odd part appears more than once. Let $g(x)$ denote the generating function of $pod(n)$, so we have

$$g(x) = \sum_{n=0}^{\infty} pod(n)x^n = \prod_{m=1}^{\infty} \frac{1 + x^{2m-1}}{1 - x^{2m}} = \frac{f(x)f(x^4)}{f(x^2)}. \tag{1.11}$$

Via another identity of Gauss (see equation (2.2.13), page 23 in [44, 45]) , it turns out that

$$\frac{1}{g(x)} = \sum_{n=0}^{\infty} (-x)^{n(n+1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n x^{2n^2-n}; \tag{1.12}$$

so in this case the reciprocal of the generating function under consideration has nonzero coefficients at the exponents which are triangular (or equivalently, hexagonal) numbers.

The analogous Rademacher-type formula for $pod(n)$ is as follows:

$$\begin{aligned} pod(n) = \frac{2}{\pi} \sum_{k \geqq 1} &\sqrt{k\left(1 - (-1)^k + \left\lfloor \frac{(k,4)}{4} \right\rfloor\right)} \\ &\times \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} \frac{\omega(h,k) \quad \omega(4h/(k,4), k/(k,4))}{\omega(2h/(k,2), k/(k,2))} e^{-2\pi i n h/k} \\ &\times \frac{d}{dn}\left(\frac{\sinh\left(\pi\sqrt{(k,4)(8n-1)}/4k\right)}{\sqrt{8n-1}} \right). \end{aligned} \tag{1.13}$$

Equation (1.13) is the case $r = 2$ of Theorem 2.1 to be presented in the next section.

2. A Common Generalization

Let us define

$$f_r(x) := \frac{f(x)f(x^{2^r})}{f(x^2)}, \tag{2.1}$$

where r is a nonnegative integer. Thus,

$$\begin{aligned} f_0(x) &= \overline{f}(x) = \sum_{n=0}^{\infty} \overline{p}(n)x^n, \\ f_1(x) &= f(x) = \sum_{n=0}^{\infty} p(n)x^n, \\ f_2(x) &= g(x) = \sum_{n=0}^{\infty} pod(n)x^n. \end{aligned} \tag{2.2}$$

Let $p_r(n)$ denote the coefficient of x^n in the expansion of $f_r(n)$, that is,

$$f_r(x) = \sum_{n=0}^{\infty} p_r(n)x^n. \tag{2.3}$$

Notice that $f_r(x)$ can be represented by several forms of equivalent infinite products, each of which has a natural combinatorial interpretation:

$$f_r(x) = \prod_{m=1}^{\infty} \frac{1+x^m}{1-x^{2^r m}} \tag{2.4}$$

$$= \prod_{m=1}^{\infty} \frac{1}{\left(1-x^{2m-1}\right)\left(1-x^{2^r m}\right)} \tag{2.5}$$

$$= \prod_{m=1}^{\infty} \frac{1}{1-x^{2^{r-1}m}} \prod_{\lambda=1}^{2^{r-1}-1} \left(1+x^{2^{r-1}m+\lambda}\right). \tag{2.6}$$

Thus, $p_r(n)$ equals each of the following:

(i) the number of overpartitions of n where nonoverlined parts are multiples of 2^r (by (2.4));

(ii) the number of partitions of n where all parts are either odd or multiples of 2^r (by (2.5)), provided $r \geqq 1$;

(iii) the number of partitions of n where nonmultiples of 2^{r-1} are distinct (by (2.6)), provided $r \geqq 1$.

Theorem 2.1. *For $r = 0, 1, 2, 3, 4$,*

$$\begin{aligned}
p_r(n) &= \frac{2^{(r+1)/2}\sqrt{3}}{\pi} \sum_{\substack{k\geqq 1\\ (k,2^{\max(r,1)})=1}} \sqrt{k} \\
&\quad\times \sum_{\substack{0\leqq h<k\\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)} \\
&\qquad\times \frac{d}{dn}\left\{ \frac{\sinh\left(\pi\sqrt{(24n-2^r+1)(1+2^{r-1})}/(2^{r/2}\cdot 6k)\right)}{\sqrt{24n-2^r+1}} \right\} \\
&\quad+ \frac{\sqrt{3}}{\pi} \sum_{j=1+\lfloor r/2\rfloor}^{r} 2^{(2-j+r)/2} \sum_{\substack{k\geqq 1\\ (k,2^r)=2^j}} \sqrt{k} \\
&\quad\times \sum_{\substack{0\leqq h<k\\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(h,k/2)} \\
&\qquad\times \frac{d}{dn}\left\{ \frac{\sinh\left(\pi\sqrt{(24n-2^r+1)(-1+2^{2j-r})}/6k\right)}{\sqrt{24n-2^r+1}} \right\}.
\end{aligned} \tag{2.7}$$

3. A Proof of Theorem 2.1

The method of proof is based on Rademacher's proof of (1.6) in [48] with the necessary modifications. Additional details of Rademacher's proof of (1.6) are provided in [49], [50, Chapter 14], and [51, Chapter 5].

Of fundamental importance is the path of integration to be used. In [48], Rademacher improved upon his original proof of (1.6) given in [3], by altering his path of integration from a carefully chosen circle to a more complicated path based on Ford circles, which in turn led to considerable simplifications later in the proof.

3.1. Farey Fractions

The sequence $\mathcal{F}_N$ of *proper Farey fractions of order* N is the set of all h/k with $(h,k) = 1$ and $0 \leqq h/k < 1$, arranged in increasing order. Thus, for example, $\mathcal{F}_4 = \{0/1, 1/4, 1/3, 1/2, 2/3, 3/4\}$.

For a given N, let h_p, h_s, k_p, and k_s be such that h_p/k_p is the immediate predecessor of h/k and h_s/k_s is the immediate successor of h/k in $\mathcal{F}_N$. It will be convenient to view each $\mathcal{F}_N$ cyclically, that is, to view $0/1$ as the immediate successor of $(N-1)/N$.

3.2. Ford Circles and the Rademacher Path

Let h and k be integers with $(h,k)=1$ and $0 \leqq h < k$. The *Ford circle* [52] $C(h,k)$ is the circle in $\mathbb{C}$ of radius $1/2k^2$ centered at the point

$$\frac{h}{k}+\frac{1}{2k^2}i. \tag{3.1}$$

The *upper* arc $\gamma(h,k)$ *of the Ford circle* $C(h,k)$ is those points of $C(h,k)$ from the initial point

$$\alpha_I(h,k) := \frac{h}{k}-\frac{k_p}{k\left(k^2+k_p^2\right)}+\frac{1}{k^2+k_p^2}i \tag{3.2}$$

to the terminal point

$$\alpha_T(h,k) := \frac{h}{k}+\frac{k_s}{k\left(k^2+k_s^2\right)}+\frac{1}{k^2+k_s^2}i, \tag{3.3}$$

traversed in the clockwise direction.

Note that we have

$$\alpha_I(0,1)=\alpha_T(N-1,N). \tag{3.4}$$

Every Ford circle is in the upper half plane. For $h_1/k_1, h_2/k_2 \in \mathcal{F}_N$, $C(h_1,k_1)$ and $C(h_2,k_2)$ are either tangent or do not intersect.

The *Rademacher path* $P(N)$ of order N is the path in the upper half of the τ-plane from i to $i+1$ consisting of

$$\bigcup_{h/k\in\mathcal{F}_N}\gamma(h,k) \tag{3.5}$$

traversed left to right and clockwise. In particular, we consider the left half of the Ford circle $C(0,1)$ and the corresponding upper arc $\gamma(0,1)$ to be translated to the right by 1 unit. This is legal given then periodicity of the function which is to be integrated over $P(N)$.

3.3. Set Up the Integral

Let n and r be fixed, with $n>(2^r-1)/24$.

Since

$$f_r(x)=\sum_{n=0}^{\infty}p_r(n)x^n, \tag{3.6}$$

Cauchy's residue theorem implies that

$$p_r(n) = \frac{1}{2\pi i}\int_{C} \frac{f_r(x)}{x^{n+1}}dx, \tag{3.7}$$

where C is any simply closed contour enclosing the origin and inside the unit circle. We introduce the change of variable

$$x = e^{2\pi i\tau} \tag{3.8}$$

so that the unit disk $|x| \leqq 1$ in the x-plane maps to the infinitely tall, unit-wide strip in the τ-plane where $0 \leqq \Re\tau \leqq 1$ and $\Im\tau \geqq 0$. The contour C is then taken to be the preimage of $P(N)$ under the map $x \mapsto e^{2\pi i\tau}$.

Better yet, let us replace x with $e^{2\pi i\tau}$ in (3.7) to express the integration in the τ-plane:

$$\begin{aligned} p_r(n) &= \int_{P(N)} f_r\left(e^{2\pi i\tau}\right)e^{-2\pi in\tau}d\tau \\ &= \sum_{h/k\in\mathcal{F}_N}\int_{\gamma(h,k)} f_r\left(e^{2\pi i\tau}\right)e^{-2\pi in\tau}d\tau \\ &= \sum_{k=1}^{N}\sum_{\substack{0\leqq h<k \\ (h,k)=1}}\int_{\gamma(h,k)} f_r\left(e^{2\pi i\tau}\right)e^{-2\pi in\tau}d\tau. \end{aligned} \tag{3.9}$$

3.4. Another Change of Variable

Next, we change variables again, taking

$$\tau = \frac{iz+h}{k}, \tag{3.10}$$

so that

$$z = -ik\left(\tau - \frac{h}{k}\right). \tag{3.11}$$

Thus $C(h,k)$ (in the τ-plane) maps to the clockwise-oriented circle $K_k^{(-)}$ (in the z-plane) centered at $1/2k$ with radius $1/2k$.

So we now have

$$p_r(n) = i\sum_{k=1}^{N}k^{-1}\sum_{\substack{0\leqq h<k \\ (h,k)=1}} e^{-2\pi inh/k}\int_{\substack{z_I(h,k) \\ \text{arc of } K_k^{(-)}}}^{z_T(h,k)} e^{2n\pi z/k}f_r\left(e^{2\pi ih/k-2\pi z/k}\right)dz, \tag{3.12}$$

where $z_I(h,k)$ (resp., $z_T(h,k)$) is the image of $\alpha_I(h,k)$ (see (3.2)) (resp. $\alpha_T(h,k)$ [see 3.2]) under the transformation (3.11).

So the transformation (3.11) maps the upper arc $\gamma(h,k)$ of $C(h,k)$ in the τ-plane to the arc on $K_k^{(-)}$ which initiates at

$$z_I(h,k) = \frac{k}{k^2+k_p^2} + \frac{k_p}{k^2+k_p^2}i \tag{3.13}$$

and terminates at

$$z_T(h,k) = \frac{k}{k^2+k_s^2} - \frac{k_s}{k^2+k_s^2}i. \tag{3.14}$$

3.5. Exploiting a Modular Transformation

From the theory of modular forms, we have the transformation formula [2, page 93, Lemma 4.31]:

$$\begin{aligned} &f\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k}\right)\right) \\ &\quad = \omega(h,k)\exp\left(\frac{\pi\left(z^{-1}-z\right)}{12k}\right)\sqrt{z}f\left(\exp\left(2\pi i\frac{iz^{-1}+H}{k}\right)\right), \end{aligned} \tag{3.15}$$

where $\sqrt{z}$ is the principal branch, $(h,k)=1$, and H is a solution to the congruence

$$hH \equiv -1(\operatorname{mod} k). \tag{3.16}$$

From (3.15), we deduce the analogous transformation for $f_r(x)$.

The transformation formula is a piecewise defined function with $r+1$ cases corresponding to $j=0,1,2,\ldots,r$, where $(k,2^r)=2^j$:

$$\begin{aligned} &f_r\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k}\right)\right) \\ &\quad = \frac{\omega(h,k)\omega\left(2^{r-j}h, 2^{-j}k\right)}{\omega\left(2h/(2-\delta_{j0}), k/(2-\delta_{j0})\right)}\exp\left(\frac{\pi\left(2+2^{2j-r+1}-(2-\delta_{j0})^2\right)}{24kz} + \frac{\pi(1-2^r)z}{12k}\right) \\ &\quad\times\sqrt{z\;\;2^{r-j-1}(2-\delta_{j0})} \\ &\quad\times\frac{f\left(\exp(-2\pi/kz+2H_j\pi i/k)\right)f\left(\exp\left(-2^{2j-r+1}\pi/kz+2^{2j-r+1}H_j\pi i/k\right)\right)}{f\left(\exp\left(-\pi(2-\delta_{j0})^2/kz+H_j\pi(2-\delta_{j0})^2 i/k\right)\right)}, \end{aligned} \tag{3.17}$$

where H_j is divisible by 2^{r-j} and is a solution to the congruence $hH_j \equiv -1 (\bmod k)$, and

$$\delta_{j0} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0 \end{cases} \tag{3.18}$$

is the Kronecker δ-function.

Notice that in particular, for $\lfloor r/2 \rfloor \leqq j \leqq r$, (3.17) simplifies to

$$\begin{aligned} & f_r\left(\exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k}\right)\right) \\ & \quad = \frac{\omega(h,k)\omega(2^{r-j}h, 2^{-j}k)}{\omega(h,k/2)} \exp\left(\frac{\pi}{12k}\left[\left(2^{2j-r}-1\right)z^{-1} + (1-2^r)z\right]\right) \\ & \quad \times \sqrt{z2^{r-j}}\, f_{2j-r}\left(\exp\left(\frac{-2\pi}{kz} + \frac{2H_j\pi i}{k}\right)\right). \end{aligned} \tag{3.19}$$

Since the $r = 0$ case was established by Zuckerman, and the $r = 1$ case by Rademacher, we will proceed with the assumption that $r > 1$.

Apply (3.17) to (3.12) to obtain

$$\begin{aligned} p_r(n) = {} & i\sum_{j=0}^{r} \sum_{\substack{k=1 \\ (k,2^r)=j}}^{N} k^{-1} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \\ & \times \int_{\substack{z_I(h,k) \\ \text{arc of } K_k^{(-)}}}^{z_T(h,k)} \frac{\omega(h,k)\omega(2^{r-j}h, 2^{-j}k)}{\omega(2h/(2-\delta_{j0}), k/(2-\delta_{j0}))} \\ & \qquad \times \exp\left(\frac{\pi\left(2 + 2^{2j-r+1} - (2-\delta_{j0})^2\right)}{24kz} + \frac{\pi(24n+1-2^r)z}{12k}\right) \\ & \qquad \times \sqrt{z2^{r-j-1}(2-\delta_{j0})} \\ & \qquad \times \frac{f(\exp(-2\pi/kz + 2H_j\pi i/k))\, f(\exp(-2^{2j-r+1}\pi/kz + 2^{2j-r+1}H_j\pi i/k))}{f\left(\exp\left(-\pi(2-\delta_{j0})^2/kz + H_j\pi(2-\delta_{j0})^2 i/k\right)\right)}\, dz. \end{aligned} \tag{3.20}$$

3.6. Normalization

Next, introduce a normalization $\zeta = zk$ (this is not strictly necessary, but it will allow us in the sequel to quote various useful results directly from the literature):

$$\begin{aligned}
p_r(n) = i\sum_{j=0}^{r} \sum_{\substack{k=1\\(k,2^r)=j}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}
&\times \int_{\substack{\zeta_I(h,k)\\ \text{arc of } K^{(-)}}}^{\zeta_T(h,k)} \frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(2h/(2-\delta_{j0}),k/(2-\delta_{j0}))} \\
&\times \exp\left(\frac{\pi\left(2+2^{2j-r+1}-(2-\delta_{j0})^2\right)}{24\zeta}+\frac{\pi(24n+1-2^r)\zeta}{12k^2}\right) \\
&\times \sqrt{\zeta 2^{r-j-1}(2-\delta_{j0})} \\
&\times \frac{f(\exp(-2\pi/\zeta+2H_j\pi i/k))f(\exp(-2^{2j-r+1}\pi/\zeta+2^{2j-r+1}H_j\pi i/k))}{f\left(\exp\left(-\pi(2-\delta_{j0})^2/\zeta+H_j\pi(2-\delta_{j0})^2 i/k\right)\right)}d\zeta,
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
\zeta_I(h,k) &= \frac{k^2}{k^2+k_p^2}+\frac{kk_p}{k^2+k_p^2}i, \\
\zeta_T(h,k) &= \frac{k^2}{k^2+k_s^2}-\frac{kk_s}{k^2+k_s^2}i.
\end{aligned} \tag{3.22}$$

Let us now rewrite (3.21) as

$$\begin{aligned}
p_r(n) = i\sum_{j=0}^{r} \sum_{\substack{k=1\\(k,2^{\max(r,1)})=2^j}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(2h/(2-\delta_{j0}),k/(2-\delta_{j0}))} \\
\times (\mathcal{I}_{j,1}+\mathcal{I}_{j,2}),
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
\mathcal{I}_{j,1} := \int_{\substack{\zeta_I(h,k)\\ \text{arc}}}^{\zeta_T(h,k)} &\exp\left(\frac{\pi\left(2+2^{2j-r+1}-(2-\delta_{j0})^2\right)}{24\zeta}+\frac{\pi(24n+1-2^r)\zeta}{12k^2}\right) \\
&\times \sqrt{\zeta 2^{r-j-1}(2-\delta_{j0})} \\
&\times \left\{-1+\frac{f(\exp(-2\pi/\zeta+2H_j\pi i/k))f(\exp(-2^{2j-r+1}\pi/\zeta+2^{2j-r+1}H_j\pi i/k))}{f\left(\exp\left(-\pi(2-\delta_{j0})^2/\zeta+H_j\pi(2-\delta_{j0})^2 i/k\right)\right)}\right\}d\zeta,
\end{aligned}$$

$$\mathcal{I}_{j,2} := \int_{\substack{\zeta_I(h,k) \\ \text{arc}}}^{\zeta_T(h,k)} \exp\left(\frac{\pi\left(2+2^{2j-r+1}-(2-\delta_{j0})^2\right)}{24\zeta} + \frac{\pi(24n+1-2^r)\zeta}{12k^2}\right) \times \sqrt{\zeta 2^{r-j-1}(2-\delta_{j0})}\,d\zeta. \tag{3.24}$$

3.7. Estimation

It will turn out that as $N \to \infty$, only $\mathcal{I}_{j,2}$ for $j = 0$ and $\lfloor r/2 \rfloor < j \leqq r$ ultimately make a contribution. Note that all the integrations in the ζ-plane occur on arcs and chords of the circle K of radius $1/2$ centered at the point $1/2$. So, inside and on K, $0 < \Re\zeta \leqq 1$ and $\Re(1/\zeta) \geqq 1$.

3.7.1. Estimation of $\mathcal{I}_{j,2}$ for $1 \leqq j \leqq \lfloor r/2 \rfloor$

The regularity of the integrand allows us to alter the path of integration from the arc connecting $\zeta_I(h,k)$ and $\zeta_T(h,k)$ to the directed segment. By [51, page 104, Theorem 5.9], the length of the path of integration does not exceed $2\sqrt{2}k/N$, and on the segment connecting $\zeta_I(h,k)$ to $\zeta_T(h,k)$, $|\zeta| < \sqrt{2}k/N$. Thus, the absolute value of the integrand is

$$\begin{aligned}
&\left|\exp\left(\frac{\pi(2^{2j-r}-1)}{12\zeta} + \frac{\pi(24n+1-2^r)\zeta}{12k^2}\right)\sqrt{\zeta 2^{r-j}}\right| \\
&= |\zeta|^{1/2} 2^{(r-j)/2} \exp\left(\frac{(24n+1-2^r)\pi\Re\zeta}{12k^2}\right)\exp\left(\frac{\pi(2^{2j-r}-1)}{12}\Re\frac{1}{\zeta}\right) \\
&\leqq |\zeta|^{1/2} 2^{r/2} \exp(2\pi n).
\end{aligned} \tag{3.25}$$

Thus, for $1 \leqq j \leqq \lfloor r/2 \rfloor$,

$$\left|\mathcal{I}_{j,2}\right| \leqq \frac{2\sqrt{2}k}{N}\left(\frac{\sqrt{2}k}{N}\right)^{1/2} 2^{r/2} e^{2\pi n} \leqq C_j' k^{3/2} N^{-3/2} \tag{3.26}$$

for a constant C_j' (recalling that n and r are fixed).

3.7.2. Estimation of $\mathcal{I}_{j,1}$ for $1 \leqq j \leqq \lfloor r/2 \rfloor$

We have the absolute value of the integrand:

$$\left|\sqrt{\zeta 2^{r-j}} \exp\left(\frac{\pi(2^{2j-r}-1)}{12\zeta} + \frac{\pi(24n+1-2^r)\zeta}{12k^2}\right)\right| \times \left|-1 + \frac{f\left(\exp(-2\pi/\zeta + 2H_j\pi i/k)\right) f\left(\exp(-2^{2j-r+1}\pi/\zeta + 2^{2j-r+1}H_j\pi i/k)\right)}{f\left(\exp\left(\dfrac{-4\pi}{\zeta} + \dfrac{4H_j\pi i}{k}\right)\right)}\right|$$

$$
\begin{aligned}
&= \left| \sqrt{\zeta 2^{r-j}} \exp\left(\frac{\pi\left(2^{2j-r}-1\right)}{12\zeta} + \frac{\pi(24n+1-2^r)\zeta}{12k^2} \right) \right| \\
&\quad \times \left| -1 + f_{2j-r}\left(\exp\left(\frac{-2\pi}{\zeta} + \frac{2H_j\pi i}{k} \right) \right) \right| \\
&= |\zeta|^{1/2} 2^{(r-j)/2} \exp\left(\frac{(24n+1-2^r)\pi\Re\zeta}{12k^2} \right) \exp\left(\frac{\left(2^{2j-r}-1\right)\pi}{12} \Re\frac{1}{\zeta} \right) \\
&\quad \times \left| \sum_{m=1}^{\infty} p_{2j-r}(m) \exp\left(\frac{-2\pi m}{\zeta} + \frac{2H_j\pi i m}{k} \right) \right| \\
&\leqq |\zeta|^{1/2} 2^{(r-j)/2} \exp\left(\frac{(24n+1-2^r)\pi}{12} \right) \sum_{m=1}^{\infty} p_{2j-r}(m) \exp\left(-\frac{\pi}{12}\left(24m - 2^{2j-r}+1\right) \right) \\
&\leqq |\zeta|^{1/2} 2^{r/2} e^{2\pi n} \sum_{m=1}^{\infty} p_0(m) e^{-2\pi m} \\
&= c_j |\zeta|^{1/2}
\end{aligned}
\tag{3.27}
$$

for a constant c_j. So, for $1 \leqq j \leqq \lfloor r/2 \rfloor$,

$$
\left| \mathcal{I}_{j,1} \right| \leqq \frac{2\sqrt{2}k}{N} \left(\frac{\sqrt{2}k}{N} \right)^{1/2} c_j < C_j k^{3/2} N^{-3/2} \tag{3.28}
$$

for a constant C_j.

3.7.3. Estimation of $I_{0,1}$

Let $p_r^*(x)$ be defined by

$$
\sum_{n=0}^{\infty} p_r^*(n) x^n = \frac{f\left(x^{2^r}\right) f(x)}{f\left(x^{2^{r-1}}\right)}. \tag{3.29}
$$

Again, the regularity of the integrand allows us to alter the path of integration from the arc connecting $\zeta_I(h,k)$ and $\zeta_T(h,k)$ to the directed segment.

With this in mind, we estimate the absolute value of the integrand:

$$
\begin{aligned}
&\left|\exp\left(\frac{\pi\left(1+2^{1-r}\right)}{24\zeta}+\frac{\pi(24n+1-2^{r})\zeta}{12k^{2}}\right)\sqrt{\zeta 2^{r-1}}\right| \\
&\qquad\times\left|-1+\frac{f\left(\exp(-2\pi/\zeta+2H_0\pi i/k)\right)f\left(\exp\left(-2^{1-r}\pi/\zeta+2^{1-r}H_0\pi i/k\right)\right)}{f\left(\exp(-\pi/\zeta+H_0\pi i/k)\right)}\right| \\
&\quad=\left|\exp\left(\frac{\pi\left(1+2^{1-r}\right)}{24\zeta}+\frac{\pi(24n+1-2^{r})\zeta}{12k^{2}}\right)\sqrt{\zeta 2^{r-1}}\right| \\
&\qquad\times\left|\sum_{m=1}^{\infty}p^{*}(m)\exp\left(\frac{-2^{1-r}\pi m}{\zeta}+\frac{2^{1-r}H_0\pi i m}{k}\right)\right| \\
&\quad=\exp\left(\frac{\pi\left(1+2^{1-r}\right)}{24}\Re\frac{1}{\zeta}\right)\exp\left(\frac{\pi(24n+1-2^{r})\Re\zeta}{12k^{2}}\right)|\zeta|^{1/2}2^{(r-1)/2} \\
&\qquad\times\left|\sum_{m=1}^{\infty}p^{*}(m)\exp\left(\frac{-2^{1-r}\pi m}{\zeta}\right)\exp\left(\frac{2^{1-r}H_0\pi i m}{k}\right)\right| \\
&\quad\leqq e^{2\pi n}|\zeta|^{1/2}2^{(r-1)/2} \\
&\qquad\times\sum_{m=1}^{\infty}\left|p^{*}(m)\right|\exp\left(\frac{\pi\left(1+2^{1-r}\right)}{24}\Re\frac{1}{\zeta}-2^{1-r}\pi m\Re\frac{1}{\zeta}\right) \\
&\quad=e^{2\pi n}|\zeta|^{1/2}2^{(r-1)/2}\sum_{m=1}^{\infty}\left|p^{*}(m)\right|\exp\left(-\frac{\pi}{24\cdot 2^{r-1}}\Re\frac{1}{\zeta}\left(24m-1-2^{r-1}\right)\right) \\
&\quad\leqq e^{2\pi n}|\zeta|^{1/2}2^{(r-1)/2}\sum_{m=1}^{\infty}\left|p^{*}(m)\right|\exp\left(-\frac{\pi}{24}\left(24m-1-2^{r-1}\right)\right) \\
&\quad< e^{2\pi n}|\zeta|^{1/2}2^{(r-1)/2}\sum_{m=1}^{\infty}\left|p^{*}\left(24m-1-2^{r-1}\right)\right|y^{24m-1-2^{r-1}}\quad\left(\text{where } y=e^{-\pi/24}\right) \\
&\quad=c_0|\zeta|^{1/2}
\end{aligned}
\tag{3.30}
$$

for a constant c_0. So,

$$
|\mathcal{I}_{0,1}|\leqq\frac{2\sqrt{2}k}{N}\left(\frac{\sqrt{2}\mathrm{k}}{N}\right)^{1/2}c_0<C_0k^{3/2}N^{-3/2}
\tag{3.31}
$$

for a constant C_0.

3.7.4. Estimation of $\mathcal{I}_{j,1}$ for $1 + \lfloor r/2 \rfloor \leqq j \leqq r \leqq 4$

Again, the regularity of the integrand allows us to alter the path of integration from the arc connecting $\zeta_I(h,k)$ and $\zeta_T(h,k)$ to the directed segment.

With this in mind,

$$\begin{aligned}
&\left|\exp\left(\frac{\pi(2^{2j-r}-1)}{12\zeta}+\frac{\pi(24n+1-2^r)\zeta}{12k^2}\right)\sqrt{\zeta 2^{r-j}}\right| \\
&\quad\times\left|-1+f_{2j-r}\left(\exp\left(\frac{-2\pi}{\zeta}+\frac{2H_j\pi i}{k}\right)\right)\right| \\
&=\left|\exp\left(\frac{\pi(2^{2j-r}-1)}{12\zeta}\right)\exp\left(\frac{\pi(24n+1-2^r)\zeta}{12k^2}\right)\sqrt{\zeta 2^{r-j}}\right| \\
&\quad\times\left|\sum_{m=1}^{\infty}p_{2j-r}(m)\exp\left(\frac{-2\pi m}{\zeta}\right)\exp\left(\frac{2H_j\pi i m}{k}\right)\right| \\
&=\exp\left(\frac{\pi(2^{2j-r}-1)}{12}\Re\frac{1}{\zeta}\right)\exp\left(\frac{\pi(24n+1-2^r)\Re\zeta}{12k^2}\right)|\zeta|^{1/2}2^{(r-j)/2} \qquad (3.32)\\
&\quad\times\left|\sum_{m=1}^{\infty}p_{2j-r}(m)\exp\left(-2\pi m\Re\frac{1}{\zeta}\right)\exp\left(\frac{2H_j\pi i m}{k}\right)\right| \\
&\leqq e^{2\pi n}|\zeta|^{1/2}2^{r/2}\sum_{m=1}^{\infty}p_{2j-r}(m)\exp\left(-\frac{\pi}{12}\Re\frac{1}{\zeta}\left(24m-2^{2j-r}+1\right)\right) \\
&\leqq e^{2\pi n}|\zeta|^{1/2}2^{r/2}\sum_{m=1}^{\infty}p_{0}(m)\exp\left(-\frac{\pi}{12}\Re\frac{1}{\zeta}(24m-2^{r}+1)\right) \\
&=c_j|\zeta|^{1/2}
\end{aligned}$$

for a constant c_j. So,

$$\left|\mathcal{I}_{j,1}\right|\leqq\frac{2\sqrt{2}k}{N}\left(\frac{\sqrt{2}k}{N}\right)^{1/2}c_j<C_jk^{3/2}N^{-3/2} \qquad (3.33)$$

for a constant C_j, when $1+\lfloor r/2\rfloor\leqq j\leqq r$.

3.7.5. Combining the Estimates

One has

$$\begin{aligned}
&\Bigg| i\sum_{\substack{k=1\\(k,2^r)=1}}^{N} k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)}\mathcal{I}_{0,1}\\
&+i\sum_{j=1}^{\lfloor r/2\rfloor}\sum_{\substack{k=1\\(k,2^r)=2^j}}^{N} k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(h,k/2)}\left(\mathcal{I}_{j,1}+\mathcal{I}_{j,2}\right)\\
&+i\sum_{j=1+\lfloor r/2\rfloor}^{r}\sum_{\substack{k=1\\(k,2^r)=2^j}}^{N} k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(h,k/2)}\mathcal{I}_{j,1}\Bigg|\\
&\quad<\sum_{j=0}^{r}\sum_{k=1}^{N}\sum_{h=0}^{k-1}C_j k^{-1}N^{-3/2}+\sum_{j=1}^{\lfloor r/2\rfloor}\sum_{k=1}^{N}\sum_{h=0}^{k-1}C'_j k^{-1}N^{-3/2}\\
&\quad\leqq C''N^{-\frac{3}{2}}\sum_{k=1}^{N}1,\quad\left(\text{where } C''=\sum_{j=0}^{\mathrm{r}}\mathrm{C}_j+\sum_{j=1}^{\lfloor \mathrm{r}/2\rfloor}\mathrm{C}'_j\right)\\
&\quad=O\left(N^{-1/2}\right).
\end{aligned}\tag{3.34}$$

Thus, we may revise (3.21) to

$$\begin{aligned}
p_r(n)&=i\sum_{k=1}^{N}k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)}\mathcal{I}_{0,2}\\
&\quad+i\sum_{j=1+\lfloor r/2\rfloor}^{r}\sum_{k=1}^{N}k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(h,k/2)}\mathcal{I}_{j,2}+O\left(N^{-1/2}\right).
\end{aligned}\tag{3.35}$$

3.8. Evaluation of $\mathcal{I}_{j,2}$ *for* $j=0$ *and* $1+\lfloor r/2\rfloor\leqq j\leqq r$

Write $\mathcal{I}_{j,2}$ as

$$\mathcal{I}_{j,2}=\int_{K^{(-)}}\exp\left(\frac{\pi\left(2^{2j-r}-1\right)}{12\zeta}+\frac{\pi(24n+1-2^r)\zeta}{12k^2}\right)\sqrt{\zeta 2^{r-j}}\,d\zeta-\mathcal{I}_{j,3}-\mathcal{I}_{j,4},\tag{3.36}$$

where

$$\mathcal{I}_{j,3} := \int_0^{\zeta_I(h,k)}, \quad \mathcal{I}_{j,4} := \int_{\zeta_T(h,k)}^0, \tag{3.37}$$

and $\mathcal{I}_{\mathrm{j},3}$ and $\mathcal{I}_{j,4}$ have the same integrand as (3.36) (and analogously for $I_{0,2}$).

3.8.1. Estimation of $\mathcal{I}_{j,3}$ and $\mathcal{I}_{j,4}$

We note that the length of the arc of integration in $\mathcal{I}_{j,3}$ is less than $\pi k/\sqrt{2}N$, and on this arc $|\zeta| < \sqrt{2}k/N$. [50, page 272]. Also, $\Re 1/\zeta = 1$ on K [50, page 271, equation (120.2)]. Further, $0 < \Re\zeta < 2k^2/N^2$ [50, page 271, equation (119.6)]. The absolute value of the integrand is thus

$$\begin{aligned} &\left|2^{r-j}\zeta\right|^{1/2}\exp\left(\frac{(24n+1-2^r)\pi\Re\zeta}{12k^2}+\frac{(2^{2j-r}-1)\pi}{12}\Re\frac{1}{\zeta}\right) \\ &\quad< 2^{(r-j)/2}2^{1/4}k^{1/2}N^{-1/2}\exp\left(\frac{(24n+1-2^r)\pi}{6N^2}+\frac{(2^{2j-r}-1)\pi}{12}\right) \end{aligned} \tag{3.38}$$

so that

$$\begin{aligned} \left|\mathcal{I}_{j,2}\right| &< \pi k 2^{-1/2}N^{-1}2^{(r-j)/2}2^{1/4}k^{1/2}N^{-1/2}\exp\left(\frac{(24n+1-2^r)\pi}{6N^2}+\frac{(2^{2j-r}-1)\pi}{12}\right) \\ &= \pi k^{3/2}N^{-3/2}2^{(2r-2j-1)/4}\exp\left(\frac{(24n+1-2^r)\pi}{6N^2}+\frac{(2^{2j-r}-1)\pi}{12}\right) \\ &= O\left(k^{3/2}N^{-3/2}\exp\left(\frac{(24n+1-2^r)\pi}{6N^2}\right)\right). \end{aligned} \tag{3.39}$$

By the same reasoning, $|\mathcal{I}_{j,3}| = O(k^{3/2}N^{-3/2}\exp((24n+1-2^r)\pi/6N^2))$.

We may therefore revise (3.35) to

$$\begin{aligned} p_r(n) &= i\sum_{k=1}^{N}k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}}e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)} \\ &\quad\times\int_{K^{(-)}}\sqrt{\zeta 2^{r-1}}\exp\left\{\frac{\pi(2^{1-r}+1)}{24\zeta}+\frac{\pi(24n+1-2^r)\zeta}{12k^2}\right\}d\zeta \\ &\quad+i\sum_{j=1+\lfloor r/2\rfloor}^{r}\sum_{k=1}^{N}k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}}e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(h,k/2)} \\ &\quad\times\int_{K^{(-)}}\sqrt{\zeta 2^{r-j}}\exp\left\{\frac{\pi(2^{2j-r}-1)}{12\zeta}+\frac{\pi(24n+1-2^r)\zeta}{12k^2}\right\}d\zeta+O\left(N^{-1/2}\right) \end{aligned} \tag{3.40}$$

and upon letting N tend to infinity, obtain

$$\begin{aligned}
p_r(n) &= i\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)} \\
&\quad\times \int_{K^{(-)}} \sqrt{\zeta 2^{r-1}}\exp\left\{\frac{\pi\left(2^{1-r}+1\right)}{24\zeta}+\frac{\pi\left(24n+1-2^r\right)\zeta}{12k^2}\right\}d\zeta \\
&\quad+ i\sum_{j=1+\lfloor r/2\rfloor}^{r}\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega\left(2^{r-j}h,2^{-j}k\right)}{\omega(h,k/2)} \\
&\quad\times \int_{K^{(-)}} \sqrt{\zeta 2^{r-j}}\exp\left\{\frac{\pi\left(2^{2j-r}-1\right)}{12\zeta}+\frac{\pi\left(24n+1-2^r\right)\zeta}{12k^2}\right\}d\zeta.
\end{aligned} \tag{3.41}$$

3.9. The Final Form

We may now introduce the change of variable

$$\zeta = \frac{\pi\left(2+2^{2j-r+1}-\left(2-\delta_{j0}\right)^2\right)}{24t} \tag{3.42}$$

(where the first summation in (3.41) is the $j=0$ term separated out for clarity), which allows the integral to be evaluated in terms of $I_{3/2}$, the Bessel function of the first kind of order 3/2 with purely imaginary argument [53, page 372, §17.7] when we bear in mind that a "bent" path of integration is allowable according to the remark preceding Equation (8) on page 177 of [54]. See also [51, page 109]. The final form of the formula is then obtained by using the fact that Bessel functions of half-odd integer order can be expressed in terms of elementary functions.

We therefore have, for $c = \pi\left(2+2^{2j-r+1}-\left(\delta_{j0}\right)^2\right)/24$,

$$\begin{aligned}
p_r(n) &= i\sum_{k=1}^{\infty} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)} 2^{(r-1)/2}\left(\pi\frac{2^{1-r}+1}{24}\right)^{3/2} \\
&\quad\times \int_{c-i\infty}^{c+i\infty} t^{-5/2}\exp\left\{t+\frac{\left(1+2^{1-r}\right)\left(24n+1-2^r\right)\pi^2}{288k^2t}\right\}dt \\
&\quad+ i\sum_{j=\,1+\,\lfloor r/2\rfloor}^{r}\sum_{k=1}^{\infty} k^{-5/2} \\
&\quad\times \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)\omega\left(2^{r-j}h,2^{-j}k\right)}{\omega(h,k/2)}\times 2^{(r-j)/2}\left(\pi\frac{2^{2j-r}-1}{12}\right)^{3/2} \\
&\quad\times \int_{c-i\infty}^{c+i\infty} t^{-5/2}\exp\left\{t+\frac{\left(2^{2j-r}-1\right)\left(24n-2^r+1\right)\pi^2}{144k^2t}\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{(24n-2^r+1)^{3/4}} \\
&\quad \times \left\{ \frac{(1+2^{r-1})^{3/4}}{2^{(r-2)/4}} \sum_{\substack{k\geqq 1 \\ (k,\, 2^{\max(r,1)})=1}} k^{-1} \right. \\
&\qquad \times \sum_{\substack{0\leqq h<k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)\omega(2^r h,k)}{\omega(2h,k)} I_{3/2}\left(\frac{\pi\sqrt{(24n-2^r+1)(1+2^{r-1})}}{2^{r/2}\cdot 6k} \right) \\
&\qquad + \sum_{j=1+\lfloor r/2\rfloor}^{r} \left(2^{2j-r}-1\right)^{3/4} 2^{(2-j+r)/2} \sum_{\substack{k\geqq 1 \\ (k,2^r)=2^j}} k^{-1} \\
&\qquad \left. \times \sum_{\substack{0\leqq h<k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)\omega(2^{r-j}h,2^{-j}k)}{\omega(h,k/2)} I_{3/2}\left(\frac{\pi\sqrt{(24n-2^r+1)(1+2^{2j-r})}}{6k} \right) \right\},
\end{aligned} \tag{3.43}
$$

which, after application of the formula [51, page 110]

$$
I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz}\left(\frac{\sinh z}{z}\right), \tag{3.44}
$$

is equivalent to Theorem 2.1.

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