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Research Article

Remarks on Generalized Derivations in Prime and Semiprime Rings

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Let R be a ring with center Z and I a nonzero ideal of R. An additive mapping $F : R \to R$ is called a generalized derivation of R if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In the present paper, we prove that if $F([x, y]) = \pm[x, y]$ for all $x, y \in I$ or $F(x \circ y) = \pm(x \circ y)$ for all $x, y \in I$, then the semiprime ring R must contains a nonzero central ideal, provided $d(I) \neq 0$. In case R is prime ring, R must be commutative, provided $d \neq 0$. The cases (i) $F([x, y]) \pm [x, y] \in Z$ and (ii) $F(x \circ y) \pm (x \circ y) \in Z$ for all $x, y \in I$ are also studied.

1. Introduction

Let R be an associative ring. The center of R is denoted by Z. For $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anticommutator $xy + yx$. We will make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. An additive mapping d from R to R is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping g from R to R is called a generalized derivation of R if there exists a derivation d from R to R such that $g(xy) = g(x)y + xd(y)$ holds for all $x, y \in R$. Obviously, every derivation is a generalized derivation of R. Thus, generalized derivation covers both the concept of derivation and left multiplier mapping. A mapping F from R to R is called centralizing on S where $S \subseteq R$, if $[F(x), x] \in Z$ for all $x \in S$.

Over the last several years, a number of authors studied the commutativity in prime and semiprime rings admitting derivations and generalized derivations. In [1], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal K and d is a derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x, y \in K$, then K is central ideal. In particular, if $K = R$, then R

is commutative. Recently, Quadri et al. [2] generalized this result replacing derivation d with a generalized derivation in a prime ring R. More precisely, they proved the following.

Let R be a prime ring and I a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that any one of the following holds: (i) $F([x,y]) = [x,y]$ for all $x, y \in I$, (ii) $F([x,y]) = -[x,y]$ for all $x, y \in I$, (iii) $F(x \circ y) = (x \circ y)$ for all $x, y \in I$; (iv) $F(x \circ y) = -(x \circ y)$ for all $x, y \in I$, then R is commutative.

In the present paper, we study all these cases in semiprime ring.

2. Main Results

We recall some known results on prime and semiprime rings.

Lemma 2.1 (see [3, Lemma 1.1.5]or [1, Lemma 2]). (a) *If R is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of R, in particular, any commutative one-sided ideal is contained in the center of R.*

(b) *If R is a prime ring with a nonzero central ideal, then R must be commutative.*

Lemma 2.2 (see [1, Lemma 1]). *Let R be a semiprime ring and I a nonzero ideal of R. If $z \in R$ and z centralizes $[I, I]$, then z centralizes I.*

Lemma 2.3 (see [4, Theorem 3]). *Let R be a semiprime ring and U a nonzero left ideal of R. If R admits a derivation d which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

Now we begin with the theorem.

Theorem 2.4. *Let R be a semiprime ring, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d of R such that $d(I) \neq 0$. If $F([x,y]) = \pm[x,y]$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. By our assumption, we have that

$$F([x,y]) = \pm[x,y] \tag{2.1}$$

for all $x, y \in I$. If $F(I) = 0$, then we find that $[x,y] = 0$ for all $x, y \in I$, that is, I is commutative. Then, by Lemma 2.1, $I \subseteq Z$ and thus we obtain our conclusion.

Next assume that $F(I) \neq 0$. Putting $y = yx$ in (2.1), we get that

$$F([x,y]x) = \pm[x,y]x. \tag{2.2}$$

Since F is a generalized derivation of R associated with a derivation d of R, (2.2) gives

$$F([x,y])x + [x,y]d(x) = \pm[x,y]x. \tag{2.3}$$

Using (2.1), it reduces to

$$[x,y]d(x) = 0 \tag{2.4}$$

for all $x, y \in I$. Now putting $y = d(x)y$ in (2.4), we get

$$0 = \left[x, d(x)y\right]d(x) = d(x)\left[x, y\right]d(x) + [x, d(x)]yd(x). \tag{2.5}$$

Using (2.4), it gives

$$0 = [x, d(x)]yd(x) \tag{2.6}$$

for all $x, y \in I$. Now we put $y = yx$ in (2.6) and obtain that

$$0 = [x, d(x)]yxd(x) \tag{2.7}$$

for all $x, y \in I$. Right multiplying (2.6) by x and then subtracting from (2.7), we get

$$0 = [x, d(x)]y[x, d(x)] \tag{2.8}$$

for all $x, y \in I$. This implies for all $x \in I$ that $([x, d(x)]I)^2 = 0$ and so $[x, d(x)]I = 0$, forcing $[x, d(x)] \in I \cap \mathrm{Ann}(I) = 0$. Then by Lemma 2.3, R contains a nonzero central ideal. □

Corollary 2.5. *Let R be a prime ring, I a nonzero ideal of R and F a generalized derivation of R. If $F([x, y]) = \pm[x, y]$ for all $x, y \in I$, then R is commutative or $F(x) = \pm x$ for all $x \in I$.*

Proof. Let d be the associated derivation of F. By Theorem 2.4, we conclude that either $d(I) = 0$ or R is commutative. Assume that R is not commutative. Then $d(I) = 0$. Since R is a prime ring, $d(I) = 0$ implies $d(R) = 0$ and hence $F(xy) = F(x)y$ for all $x, y \in R$. Set $G(x) = F(x) \mp x$ for all $x \in R$. Then $G(xy) = G(x)y$ for all $x \in R$. Now, our assumption $F([x, y]) = \pm[x, y]$ gives $F(x)y - F(y)x = \pm(xy - yx)$, that is, $G(x)y - G(y)x = 0$ for all $x, y \in I$. Thus using $G(x)y = G(y)x$, we have $G(x)yz = G(y)xz = G(xz)y = G(x)zy$, that is, $G(x)[y, z] = 0$ for all $x, y, z \in I$. Thus $0 = G(I)[I, I] = G(IR)[I, I] = G(I)R[I, I]$. Since R is prime, this implies $G(I) = 0$ or I is commutative. By Lemma 2.1, I commutative implies that R is commutative, a contradiction. Thus $G(I) = 0$ which gives $G(x) = F(x) \mp x = 0$ for all $x \in I$. □

Theorem 2.6. *Let R be a semiprime ring, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d of R such that $d(I) \neq 0$. If $F(x \circ y) = \pm(x \circ y)$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. If $F(I) = 0$, then by our assumption we have that $x \circ y = 0$, that is, $xy + yx = 0$ for all $x, y \in I$. This implies that $x(yz) = -(yz)x = -y(zx) = y(xz) = (yx)z = -(xy)z$ for all $x, y, z \in I$ and so $2I^3 = 0$, forcing $2I = 0$. Therefore, for all $x, y \in I$, $xy + yx = 0$ gives $xy = yx$, that is, I is commutative. Then by Lemma 2.1, $I \subseteq Z$ and thus we obtain our conclusion. □

Next assume that $F(I) \neq 0$. Then for any $x, y \in I$, we have

$$F(xy + yx) = \pm(xy + yx). \tag{2.9}$$

Since F is a generalized derivation associated with a derivation d, above expression yields

$$F(x)y + xd(y) + F(y)x + yd(x) = \pm(xy + yx). \tag{2.10}$$

Putting $y = yx$ in (2.10), we have

$$F(x)yx + x(d(y)x + yd(x)) + (F(y)x + yd(x))x + yxd(x) = \pm\left(xyx + yx^2\right). \tag{2.11}$$

Right multiplying (2.10) by x and then subtracting from (2.11), we get

$$xyd(x) + yxd(x) = 0 \tag{2.12}$$

for all $x, y \in I$. Replacing y with $d(x)y$ in (2.12) and then again using (2.12) we find that

$$[x, d(x)]yd(x) = 0. \tag{2.13}$$

Again replacing y with yx in (2.13) and then using (2.13) we obtain

$$[x, d(x)]y[x, d(x)] = 0 \tag{2.14}$$

for all $x, y \in I$, which is the same identity as (2.8) in the proof of Theorem 2.4. Thus by the same argument as in the proof of Theorem 2.4, we conclude that R contains a nonzero central ideal.

Corollary 2.7. *Let R be a prime ring, I a nonzero ideal of R and F a generalized derivation of R. If $F(x \circ y) = \pm(x \circ y)$ for all $x, y \in I$, then R is commutative or $F(x) = \pm x$ for all $x \in I$.*

Proof. Let d be the associated derivation of F. By Theorem 2.6, we conclude that either $d(I) = 0$ or R is commutative. If R is not commutative, then $d(I) = 0$. Since R is a prime ring, $d(I) = 0$ implies $d(R) = 0$ and hence $F(xy) = F(x)y$ for all $x, y \in R$. Set $G(x) = F(x) \mp x$ for all $x \in R$. Then $G(xy) = G(x)y$ for all $x \in R$. Now, our assumption $F(x \circ y) = \pm(x \circ y)$ gives $F(x)y + F(y)x = \pm(xy + yx)$, that is, $G(x)y + G(y)x = 0$ for all $x, y \in I$. Thus using $G(x)y = -G(y)x$, we have $G(x)yz = -G(y)xz = G(xz)y = G(x)zy$, that is, $G(x)[y, z] = 0$ for all $x, y, z \in I$. Thus $0 = G(I)[I, I] = G(IR)[I, I] = G(I)R[I, I]$. Since R is prime, this implies $G(I) = 0$ or I is commutative. By Lemma 2.1, I commutative implies that R is commutative, a contradiction. Therefore, $G(I) = 0$ and hence $G(x) = F(x) \mp x = 0$ for all $x \in I$. □

Theorem 2.8. *Let R be a semiprime ring with center $Z \neq \{0\}$, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d of R. If $F([x, y]) \pm [x, y] \in Z$ for all $x, y \in I$, then $Id(Z) \subseteq Z$.*

Proof. We have

$$F([x, y]) \pm [x, y] \in Z \tag{2.15}$$

for all $x, y \in I$. Since $Z \neq \{0\}$, we may choose $0 \neq z \in Z$. Then $yz \in I$ for any $y \in I$. Now we replace y with yz in (2.15) and then we get

$$\begin{aligned} F([x,y]z) \pm [x,y]z &= F([x,y])z + [x,y]d(z) \pm [x,y]z \\ &= \{F([x,y]) \pm [x,y]\}z + [x,y]d(z) \in Z. \end{aligned} \tag{2.16}$$

By (2.15), we have $[x,y]d(z) \in Z$ for all $x, y \in I$. Since $d(z) \in Z$, this gives that for any $r \in R$, $[r,[x,y]d(z)] = 0$ which implies $[rd(z),[x,y]] = 0$ for all $x, y \in I$. By Lemma 2.2, $[rd(z),x] = 0$ for all $x \in I$. Since $d(z) \in Z$, this gives $[r, xd(z)] = 0$ for all $r \in R$ and for all $x \in I$. Thus, $xd(z) \in Z$, that is, $Id(Z) \subseteq Z$. □

Corollary 2.9. *Let R be a prime ring with center $Z \neq \{0\}$, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d. If $d(Z) \neq \{0\}$ and $F([x,y]) \pm [x,y] \in Z$ for all $x, y \in I$, then R is commutative.*

Proof. Since $d(Z) \subseteq Z$ and Z contains no nonzero elements which are zero divisors, we have from Theorem 2.8 that $I \subseteq Z$. Then by Lemma 2.1(b), we obtain our conclusion. □

Theorem 2.10. *Let R be a semiprime ring with center $Z \neq \{0\}$, I a nonzero ideal of R and F a generalized derivation of R associated with a derivation d of R. If $F(x \circ y) \pm (x \circ y) \in Z$ for all $x, y \in I$, then $Id(Z) \subseteq Z$.*

Proof. We have

$$F(x \circ y) \pm (x \circ y) \in Z \tag{2.17}$$

for all $x, y \in I$. Since $Z \neq \{0\}$, we choose $0 \neq z \in Z$. Then $yz \in I$ for any $y \in I$. Now we replace y with yz in (2.17) and then we get

$$\begin{aligned} F((x \circ y)z) \pm (x \circ y)z &= F((x \circ y))z + (x \circ y)d(z) \pm (x \circ y)z \\ &= \{F(x \circ y) \pm (x \circ y)\}z + (x \circ y)d(z) \in Z. \end{aligned} \tag{2.18}$$

By (2.17), we have $(x \circ y)d(z) \in Z$ that is $(xy + yx)d(z) \in Z$ for all $x, y \in I$. Now putting $y = yr$ and $x = rx$, $r \in R$, respectively, we obtain that $(xyr + yrx)d(z) \in Z$ and $(rxy + yrx)d(z) \in Z$. Subtracting these two results yields $[xyd(z), r] \in Z$ for all $x, y \in I$ and for all $r \in R$. This gives

$$[[xyd(z), r], s] = 0 \tag{2.19}$$

for all $x, y \in I$ and for all $r, s \in R$. We know the Jacobian identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for any $x, y, z \in R$. Using this identity, it follows that

$$0 = [[xyd(z), r], s] = -[[r,s], xyd(z)] - [[s, xyd(z)], r]. \tag{2.20}$$

By using (2.19), it reduces to

$$[[r,s],xyd(z)] = 0 \tag{2.21}$$

for all $r,s \in R$ and for all $x,y \in I$. By Lemma 2.2, this implies that $[xyd(z),r] = 0$, that is, $[I^2d(z),R] = 0$. Thus $[[I,I],Id(z)] = 0$ and then again by Lemma 2.2, $[I,Id(z)] = 0$. This yields $0 = [IR,Id(z)] = I[R,Id(z)]$ which implies $Id(z) \subseteq Z$, since $[R,Id(z)] \subseteq I \cap \mathrm{Ann}(I) = 0$. Since z is any nonzero element in Z, we conclude that $Id(Z) \subseteq Z$. □

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