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Research Article

New Examples of Einstein Metrics in Dimension Four

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We construct new examples of four-dimensional Einstein metrics with neutral signature and two-dimensional holonomy Lie algebra.

1. Introduction

The holonomy group of a metric g at a point p of a manifold M is the group of all linear transformations in the tangent space of p defined by parallel translation along all possible loops starting at p [1]. It is obvious that a connection can only be the Levi-Civita connection of a metric g if the holonomy group is a subgroup of the generalized orthogonal group corresponding to the signature of g [1–3]. At any point $p \in M$, and in some coordinate system about p, the set of matrices of the form

$$R^i_{jkl}X^kY^l, R^i_{jkl;m}X^kY^lZ^m, R^i_{jkl;mn}X^kY^lZ^mW^n, \dots, \tag{1.1}$$

where $X, Y, Z, W \in T_pM$ and semicolon denotes covariant derivative, forms a Lie subalgebra of the Lie algebra of $M_n(\mathbb{R})$ of $\mathrm{GL}(n, \mathbb{R})$ called the infinitesimal holonomy algebra of M at p. Up to isomorphism the latter is independent of the coordinate system chosen. The corresponding uniquely determined connected subgroup of $\mathrm{GL}(n, \mathbb{R})$ is called the infinitesimal holonomy group of M at p.

A metric tensor g is a nondegenerate symmetric bilinear form on each tangent space T_pM for all $p \in M$. The signature of a metric g is the number of positive and negative eigenvalues of the metric g. The signature is denoted by an ordered pair of positive integers

(p, q), where p is the number of positive eigenvalues and q is the number of negative eigenvalues. If $p = q$, we say that the metric is of neutral signature. In this article, we are interested in four-dimensional metrics with neutral signature.

If the metric g satisfies the condition

$$R_{ij} = \frac{R}{4} g_{ij}, \tag{1.2}$$

where R_{ij} are the components of the Ricci tensor and R is the scalar curvature, then we say that g is an Einstein metric and the pair (M, g) is an Einstein space.

In [4], Ghanam and Thompson studied and classified the holonomy Lie subalgebras of neutral metrics in dimension four. In this paper, we will focus on one of the subalgebras presented in [4], namely, A_{17}. For this subalgebra we will show that the metric presented in [4] will lead us to the construction of Einstein metrics. In Section 3, we will give the metrics explicitly, and in Section 4, we will show that these Einstein metrics produce A_{17} at their two-dimensional holonomy.

As a final remark regarding our notation, we will use subscripts for partial derivatives. For example, the partial derivative of a function a with respect to x will be denoted by a_x.

2. The Subalgebra A_{17} As a Holonomy

In this section we will consider the Lie algebra A_{17}; it is a 2-dimensional Lie subalgebra of the Lie algebra of the generalized orthogonal group $O(2,2)$ of neutral signature [4, 5]. A basis for A_{17} is given by

$$e_1 = \begin{bmatrix} -J & J \\ -J & J \end{bmatrix}, \qquad e_2 = \begin{bmatrix} J & L \\ L & J \end{bmatrix}, \tag{2.1}$$

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.2}$$

We turn now to a theorem of Walker [6] that will be a key to the existence of a metric g that produces A_{17} as a two-dimensional holonomy.

Theorem 2.1 (Walker [6]). *Let (M, g) be a pseudo-Riemannian manifold of class C^∞. If g admits a parallel, null r-distribution, then there is a system of coordinates (x^i) relative to which g assumes the following form:*

$$g_{ij} = \begin{bmatrix} 0 & 0 & I \\ 0 & A & H \\ I & H^t & B \end{bmatrix}, \tag{2.3}$$

where I is the $r \times r$ identity matrix and $A, B, H,$ and H^t are matrix functions of the same class as M, satisfying the following conditions but otherwise arbitrary.

(1) *A and B are symmetric; A is of order $(n-2r) \times (n-2r)$ and nonsingular, B is of order $r \times r$, H is of order $(n-2r) \times r$, and H^t is the transpose of H.*

(2) *A and H are independent of the coordinates $x^1, x^2, \ldots, x^r$.*

Now we show that A_{17} is a holonomy Lie algebra of a four-dimensional neutral metric.

Proposition 2.2. *A_{17} is a holonomy algebra.*

Proof. In this case, we have an invariant null 2-distribution, and so by Walker's theorem, there exists a coordinate system, say (x, y, z, w), such that the metric g is of the form

$$g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix}, \tag{2.4}$$

where a, b, c are smooth functions in (x, y, z, w). Since the invariant distribution contains a parallel null vector field, we must have

$$a_x = b_x = c_x = 0. \tag{2.5}$$

It was shown in [4] that, in order for g to produce A_{17} as its holonomy algebra, the functions $a, b,$ and c must satisfy the following conditions:

$$b_{yy} = 0, \qquad c_{yy} = 0, \qquad c_{yw} - b_{yz} = 0. \tag{2.6}$$

□

3. New Einstein Metrics

In Section 2, we obtained a metric of the form

$$g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix}, \tag{3.1}$$

where $a, b,$ and c are smooth functions in y, z, w and they satisfy (2.5) and (2.6). We solve these conditions to obtain

$$\begin{aligned} b(y, z, w) &= m(z, w)y + n(z, w), \\ c(y, z, w) &= r(z, w)y + s(z, w) \end{aligned} \tag{3.2}$$

with

$$r_w = m_z, \tag{3.3}$$

where m, n, r, s are smooth functions in z, w.

The nonzero components of the Ricci tensor for g are

$$\begin{aligned} R_{33} &= -\frac{y}{2} m a_{yy} - \frac{1}{2} n a_{yy} + a_{yw} - r_z - \frac{1}{2} m a_y + \frac{1}{2} r^2, \\ R_{34} &= \frac{1}{2}(r_w - m_z). \end{aligned} \tag{3.4}$$

The Ricci scalar is

$$R = 0. \tag{3.5}$$

Because of (3.3), we obtain

$$R_{34} = 0. \tag{3.6}$$

In this case, the Einstein condition $R_{ij} = (R/4) g_{ij}$ becomes

$$R_{ij} = 0. \tag{3.7}$$

Hence, in order to have an Einstein metric, we must have

$$R_{33} = 0, \tag{3.8}$$

and so we obtain the following partial differential equation (PDE):

$$-\frac{y}{2} m a_{yy} - \frac{1}{2} n a_{yy} + a_{yw} - r_z - \frac{1}{2} m a_y + \frac{1}{2} r^2 = 0. \tag{3.9}$$

Since we are interested in finding at least one solution, we take the following special values in (3.9):

$$m = 1, \qquad n = 0, \qquad r = 0, \qquad s = 0 \tag{3.10}$$

to obtain the following PDE:

$$y a_{yy} - 2 a_{yw} + a_y = 0. \tag{3.11}$$

To solve (3.11), we use the method of separation. For example, assume that $a(y,z,w)$ is of the form

$$a(y,z,w) = f(y)g(z)h(w), \tag{3.12}$$

where f, g, and h are smooth functions in y, z, and w, respectively. We substitute (3.12) in (3.11) to obtain

$$yf''(y)g(z)h(w) - 2f'(y)g(z)h'(w) - f'(y)g(z)h(w) = 0. \tag{3.13}$$

We assume that $g(z)$ is nowhere zero to obtain

$$yf''(y)h(w) - 2f'(y)h'(w) - f'(y)h(w) = 0, \tag{3.14}$$

and so

$$yf''(y)h(w) - f'(y)\left(2h'(w) - h(w)\right) = 0. \tag{3.15}$$

Hence

$$yf''(y)h(w) = f'(y)\left(2h'(w) - h(w)\right). \tag{3.16}$$

Dividing both sides by $h(w)f'(y)$ gives

$$\frac{yf''(y)}{f'(y)} = \frac{2h'(w) - h(w)}{h(w)} = c, \tag{3.17}$$

where c is a constant.

We now solve (3.17) and for that we will consider three cases.

(1) If $c = 0$, then f is a linear function given by

$$f(y) = c_1 y + c_2, \tag{3.18}$$

and the condition on h becomes

$$2h' - h = 0, \tag{3.19}$$

which gives

$$h(w) = c_3 e^{w/2}. \tag{3.20}$$

The solution $a(y,z,w)$ to the PDE equation (3.11) is

$$a(y,z,w) = (c_1 y + c_2) c_4 e^{w/2} g(z) = (c_1 y + c_2) e^{w/2} g(z), \tag{3.21}$$

where c_1, c_2 are constants and $g(z)$ is a smooth nowhere zero function.

(2) If $c \neq 0, -1$, then the differential equations (3.17) become

$$\begin{gathered} y f'' - c f' = 0, \\ 2h' - (1+c) h = 0. \end{gathered} \tag{3.22}$$

The solutions to (3.22) are

$$\begin{gathered} f(y) = c_1 y^{c+1} + c_2, \\ h(w) = c_3 e^{(1+c)w/2}, \end{gathered} \tag{3.23}$$

and so

$$a(y,z,w) = \left(c_1 y^{c+1} + c_2 \right) e^{(1+c)w/2} g(z), \tag{3.24}$$

where $g(z)$ is a no-where zero smooth function in z.

(3) If $c = -1$, then the differential equations (3.17) become

$$\begin{gathered} y f'' + f' = 0, \\ h' = 0. \end{gathered} \tag{3.25}$$

The solutions to (3.25) are

$$\begin{gathered} f(y) = c_1 \ln(y) + c_2, \\ h(w) = c_3, \end{gathered} \tag{3.26}$$

and so

$$a(y,z,w) = (c_1 \ln(y) + c_2) c_3 g(z) = (c_1 \ln(y) + c_2) g(z), \tag{3.27}$$

where c_1, c_2 are constants and $g(z)$ is a no-where zero smooth function in z.

4. The Holonomy of the New Metrics

In this section we compute the infinitesimal holonomy algebra and make sure that it produces a two-dimensional algebra. To do so, we consider our metric g given by

$$g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(y,z,w) & 0 \\ 0 & 1 & 0 & y \end{bmatrix}. \tag{4.1}$$

The only nonzero components of the curvature are

$$\begin{aligned} R_{2323} &= -\frac{1}{2}a_{yy}, \\ R_{2334} &= \frac{1}{2}a_{wy} - \frac{1}{4}a_y, \\ R_{3434} &= -\frac{1}{2}a_{ww} + \frac{y}{4}a_y - \frac{1}{4}a_w. \end{aligned} \tag{4.2}$$

The holonomy matrices are

$$\begin{gathered} R_{ij23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}a_{yy} & 0 \\ 0 & -\frac{1}{2}a_{yy} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ R_{ij34} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}a_{wy} - \frac{1}{4}a_y & 0 \\ 0 & \frac{1}{2}a_{wy} - \frac{1}{4}a_y & 0 & -\frac{1}{2}a_{ww} + \frac{y}{4}a_y - \frac{1}{4}a_w \\ 0 & 0 & -\frac{1}{2}a_{ww} + \frac{y}{4}a_y - \frac{1}{4}a_w & 0 \end{bmatrix}, \end{gathered} \tag{4.3}$$

Now, in order for the metric to produce two-dimensional holonomy, we must have

$$a_{yy} \neq 0, \tag{4.4}$$

$$-\frac{1}{2}a_{ww} + \frac{y}{4}a_y - \frac{1}{4}a_w \neq 0. \tag{4.5}$$

We have to check these equations for the three cases discussed in Section 3.

(1) We consider

$$a(y,z,w) = (c_1 y + c_2) e^{w/2} g(z). \tag{4.6}$$

Then

$$a_{yy} = 0, \tag{4.7}$$

and we obtain a one-dimensional holonomy. Therefore we must exclude this case.

(2) We consider

$$a(y,z,w) = \left(c_1 y^{c+1} + c_2\right) e^{(1+c)w/2} g(z), \tag{4.8}$$

where $c \neq 0, -1$. In this case

$$a_{yy} = c_1 c(c+1) y^{c-1} e^{(1+c)w/2} g(z) \neq 0. \tag{4.9}$$

The second condition is that (4.5) gives

$$\frac{e^{(c+1)w/2}}{8}\left(2c_1 y^{c+1} c^2 + 5c_1 y^{c+1} c + 3c_1 y^{c+1} + 2c_2 c^2 + 3c_2 c + c_2\right) \neq 0. \tag{4.10}$$

This shows that the metric we constructed in Section 3 is an Einstein metric with a two-dimensional holonomy. In fact, its holonomy Lie algebra is A_{17}.

(3) We consider

$$a(y,z,w) = (c_1 \ln(y) + c_2) g(z). \tag{4.11}$$

In this case the first condition is that (4.4) gives

$$a_{yy} = -\frac{c_1 g(z)}{y^2} \neq 0, \tag{4.12}$$

and the second condition is that (4.5) gives

$$\frac{c_1 g(z)}{4} \neq 0. \tag{4.13}$$

And once again we obtain an Einstein metric with A_{17} as its two-dimensional holonomy Lie algebra.

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