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Research Article

Geometric Representations of Interacting Maps

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Tropical geometry is a kind of dynamical scale transform which connects automata with real rational dynamics. Real rational dynamics are deeply studied from global analytic viewpoints. On the other hand, automata appear in various contexts in topology, combinatorics, and integrable systems. In this paper we study the analysis of these materials passing through tropical geometry. In particular we discover a new duality on the set of automata which arise from the projective duality in algebraic geometry.

1. Introduction

Motivated by the phenomena in molecular biology, in [1] we have formulated random interaction systems by use of families of maps on intervals, which consist of infinite families of compositions between them. By wasting detailed and extracting more rough information by use of projections, one can produce automata. This construction covers some discrete integrable systems which possess solitons. This paper is a trial to represent creation of some macroscopic patterns which arise from random and micro dynamics.

Let $f : [0,1] \to [0,1]$ be a map, and consider its iteration $\{f^n(x)\}_{n=0}^{\infty}$. This will behave very randomly and will touch sensitively with respect to the initial points $x \in [0,1]$ (see [2]).

Let $X_2 = \{(a_0, a_1, \dots) : a_i \in \{0,1\}\}$ be the set of one-sided sequences, and take two maps $\{f_0, f_1\}$. Let us generalize the iteration of a single map to random compositions by two maps. Let us choose an element $\overline{k} = (k_0, k_1, \dots) \in X_2$. Then, correspondingly one obtains families of maps $\{h^n : [0,1] \to [0,1]\}_{n=0,1,\dots}$ on the interval by compositions $h^n(x) = f_{k_n} \circ \cdots f_{k_0}(x)$.

Using them, one can construct continuous maps $\Phi(x) = \Phi(f_1, f_2)(x) : X_2 \to X_2$ for each point $x \in [0,1]$, which are called the *interaction maps*. These are defined by use of projections on the interval, and so their values are determined by which subinterval $h^n(x)$

lies on for each n. Thus in order to determine a value $\Phi(x)(\bar{k})$, it is enough to know rough values of them, rather than rigorous ones on $[0,1]$, even though one is required to have information on $h^n(x)$ for all n. So $\Phi(x)$ is a map in a macroscopic scale compared with the micro interaction $\{h^n(x)\}$. This might explain one aspect of a very simple mechanism to create patterns in macro scale from random micro dynamics. This method is immediately generalized to use families of maps on intervals $\{f_i\}_i$, and one also obtains interaction maps $\Phi(f_1,\dots,f_k)$ by the same way.

So far, we have known that many important cell automata can be expressible by interaction systems as above (see [3, $1.A$]). In the first part of this paper, we study geometric properties of such cell automata which include the box and ball system (BBS), Lotka-Volterra cell automaton, and lamplighter automaton. For example, we will construct assignments from BBS flows to braid groups and extensions of BBS actions on compactified spaces passing through group actions on trees. We will see some relation between LV cell automaton and the lamplighter automaton, and generalize the latter to obtain more group actions on trees using such relation.

In the middle part, we study connections of automata with complex geometry. In real algebraic geometry, some geometric mappings from complex planes to the real ones were discovered, by taking coordinatewisely absolute values and their logarithms (see [4, 5]). This connects algebraic varieties with piecewise linear maps and is called the *tropical geometry*. On the other hand, in [6] several relations of pl maps with automata are studied. For example, the LV cell automaton is given by a family of pl maps. Passing through the above operations, in this paper we will associate a family of polynomials from an automaton. Using the projective duality on projective varieties and the above assignment, we will discover a duality between cell automata. In fact, for an automaton A, we will associate a dual one $A^\vee$, and call it as the *projectively dual automaton*. We will calculate some examples in the case of curves. Notice that duality on the Mealy automata was already introduced by exchanging the role of the exit and the transition functions mutually (see [7, 8]).

In general, automata are represented by families of interaction maps. Thus direct assignments from families of maps to these geometric objects will show geometric representations of interaction systems.

In the last part, we construct infinite families of graphs from finite families of maps, points, and symbolic sequences, and we call them the *interaction graphs*. We will construct these graphs so that they represent some dynamics of interacting states in mesoscopic scale. The central dogma in molecular biology tells us that proteins are products of various interaction systems in mesoscopic scale, beginning from DNA. From geometric point of view, one would like to obtain a space and an automorphism on it from such interacting data, which will represent more macro features. In analogy to the central dogma as above, the space will be compared with protein or polymer, and orbits of the automorphism correspond to states on it, since functions of proteins are determined by the shapes of themselves.

A *biological space-form problem* is to construct a space V and an automorphism A on it so that the orbits $\{x, A(x), A^2(x),\dots\} \subset V$ are induced from the dynamics of the interaction graphs. In this paper we will formulate and address a space-form problem from toric variety viewpoint, passing through several hierarchies of dynamics.

Throughout this paper, our basic direction is to study geometric properties of them as macro objects, which are induced by scaling transform of dynamics by compositions and iterations of families of maps as the micro one. In some particular cases of families of maps which include some piecewise linear maps, the interaction maps induce some automata.

On the other hand, in general, maps on intervals are very flexible objects and their dynamics are very random. One of our main aims in further development will be to study macro properties of these dynamics for families of maps which are near such special types of maps. Namely, let $\{f_i\}_i$ be a family as above so that the corresponding interaction system induces some automaton A. Let us take any geometric object $G = G(\{f_i\}_i)$ arising from the interaction system. We have several examples of G below. Since G passes through A, its structure will hold some rigid properties.

Let us take another family of maps $\{g_i\}_i$ which is sufficiently near the original $\{f_i\}_i$, where the corresponding interaction system for $\{g_i\}_i$ may not induce any automata in general. We would like to study geometric properties of $G(\{g_i\}_i)$ by comparing with $G(\{f_i\}_i)$. This might give one direction of study to understand mechanisms of creation of patterns. In particular, several stability properties of G under small deformation of these maps will be particularly of interest for us.

Now let us describe the contents of the paper more concretely.

The box and ball system (BBS) is a dynamical system on the set of finite subsets in $\mathbf{Z}$ (see [9]). Let us describe the dynamics shortly. For $\{i_1, \ldots, i_l\} \subset \mathbf{Z}$ with $i_1 < i_2 < \cdots < i_l$, let us imagine that a ball occupies in each position i_m for $m = 1, \ldots, l$. These balls will be moved by the following rule; the ball i_1 in the most left-hand side is moved to some $i_1 < j_1$ which is the most left-hand side in $\{i_1 + 1, i_1 + 2, \ldots\} \setminus \{i_2, \ldots, i_l\}$. We repeat the same procedure for i_k with $2 \le k \le l$. When this procedure is finished for i_l, then we are done.

Let Σ_2^0 be all of the finite subsets in $\mathbf{Z}$. Then, the above procedure is expressed as

$$T : \Sigma_2^0 \cong \Sigma_2^0. \tag{1.1}$$

Let $\Sigma_2^0(N) \subset \Sigma_2^0$ be the sets of N-subsets in $\mathbf{Z}$. Then, since BBS preserves the number of the balls, T is a map as $T : \Sigma_2^0(N) \cong \Sigma_2^0(N)$.

Following our expression of interaction, in Section 2, we will describe the BBS system by an interaction of maps for some family of continuous maps.

It is known that the BBS flows by the dynamics of T contain solitons. Among dynamical properties of solitons, the relative positions of the individual waves and how these waves pass through the others will be in the most important structures. In Section 3 we will represent such information geometrically by using the braid groups. Elements in braid groups are certainly representing such situation. In order to eliminate infinitely many ambiguities, in this paper we will use quotient groups $\overline{B}_n$ of the braid groups. There are many of them, and in a special case it is a subgroup of the mapping class group of finite index. Each σ admits an index among subsets in $\{1, \ldots, N-1\}$, which is determined by $T^t(\sigma)$ near $t = \pm\infty$. We will associate a quotient braid group with respect to individual index. Then, we obtain the canonical maps

$$B : \Sigma_2^0(N) \longrightarrow \overline{B}_N \tag{1.2}$$

which we call the *braiding map*. $B(\sigma)$ is constructed from the dynamics of iterations

$$\left\{\ldots, T^{-1}, \sigma, T(\sigma), T^2(\sigma), \ldots\right\}. \tag{1.3}$$

Scattering process of BBS is described in [10] by using the combinatorial R-matrix. Using such direction, one may obtain some invariants of the dynamics T.

Let Σ_2 be the set of two-sided sequences with the alphabets $\{0,1\}$. Then, Σ_2^0 can be regarded as a subset Σ_2. In Section 5, we study geometric properties of the BBS map T. For example, we see that it can be extended as $T : \Sigma_2 \to \Sigma_2$.

The Lotka-Volterra cell automaton is given by the equation (see [9])

$$V_n^{t+1} - V_n^t = \max(0, V_{n+1}^t - L) - \max\left(0, V_{n-1}^{t+1} - L\right). \tag{1.4}$$

This is obtained from the original Lotka-Volterra equation by making discretizations twice. It is known that this possesses solitons which are induced from the ones of the difference LV solitons (see [3]). BBS is isomorphic to the Lotka-Volterra cell automaton. In fact, there is an explicit procedure to construct such isomorphism. Thus one obtains an injection $\Sigma_2^0 \hookrightarrow \Sigma_\infty^0$ which assigns flows of the BBS to the corresponding dynamics of the LV. We will see that it can be extended on some partially compactified space. In Section 4, we study structures of *path spaces* of the set of cell automata, which arose from the relation of LV-CA with BBS.

The lamplighter group is well known as an automata group in geometric group theory (see [11]), which admits an action on the rooted binary tree. In Section 5, we find that in a special case the LV cell automaton is in fact a transition function for the lamplighter automaton.

Lemma 1.1. *Suppose that the initial sequence $(a_0, a_1, \ldots)$ consists of only $\{0,1\}$ entries. Then, the degeneration of the LV cell automaton is the same as the transition function ϕ of an automaton whose group is isomorphic to the lamplighter group.*

Thus using the LV cell automaton as transition functions, one can construct a family of automata groups which we call *LV cell automata groups*. They can act on the boundary $\widetilde{\partial} T_\infty^* \equiv \bigcup_i \partial T_i^*$ of the rooted infinite tree.

In Section 6, we study automata from complex geometry point of view. Tropical geometry connects algebraic geometry with automata. There is a kind of scale transform Φ_t with $t \in (1, \infty)$ by taking absolute values and taking the conjugation by Log_t (see [4, 12]). To a pair of parameterized polynomials (f_t^1, f_t^2), both $\lim_{t\to\infty}\Phi_t(f_t^1)$ and $\lim_{t\to\infty}\Phi_t(f_t^2)$ become piecewise linear maps. In general, the equation $\Phi_\infty(f_\infty^1) = \Phi_\infty(f_\infty^2)$ represent complicated dynamics, but in some cases it gives automata. This is the point where we find importance to study stability of the dynamics. We study this aspect in Section 6.3.2.

On the other hand, the above process is invertible in the sense that one can associate pairs of parameterized polynomials (f_t^1, f_t^2) from automata A. Now using these polynomials, one can obtain spaces as follows.

Definition 1.2. The associated affine hypersurfaces are parameterized family of hypersurfaces given by the following equations:

$$V(A)_t = \left\{ \mathbf{z} \in \mathbf{C}^N : f_t^1(\mathbf{z}) = f_t^2(\mathbf{z}) \right\}. \tag{1.5}$$

One important reason to consider such varieties comes from *projective duality* on algebraic varieties. Let us denote by $X^\vee$ the projective dual of an algebraic variety X. Suppose that the projective duals of $V(A)_t$ are also hypersurfaces. The parameterized families of polynomials $\Delta(A)_t$ which define these hypersurfaces are called *A-discriminant*.

Again by taking the above scaling limit in tropical geometry, one obtains another automaton $A^\vee$. Thus passing through duality in complex geometry, there is an assignment between automata:

$$A \longrightarrow A^\vee. \tag{1.6}$$

We call $A^\vee$ the *projectively dual automaton*.

We have calculated in the case of some curves. Let $a \geq 2$, α, and c be integers. Then, we have the following.

Proposition 1.3. *Consider the following:*

$$[\max\{au_n, \alpha + au_{n+1}\} = c]^\vee = \max\left\{\frac{a}{a-1}\left(c - \frac{\alpha}{a}\right) + \frac{a}{a-1}u_{n+1}, \frac{ac}{a-1} + \frac{a}{a-1}u_n\right\} = c. \tag{1.7}$$

In micro level, molecular interactions occur by covalent or hydrogen bonds where electrons of molecules share their orbitals. Inspired on this aspect, in Section 7 one formulates interaction systems of families of maps by constructing some graphs.

Let us take two interval maps $f_0, f_1 : [0,1] \to [0,1]$ and let $\Phi(x, f_0, f_1) : X_2 \to X_2$ be the interaction map. Let us choose another map $d : [0,1] \to [0,1]$. In the light of orbitals stated above, if the projection of the iterations $\{d^n(z)\}_n$ coincides with $\Phi(x, f, g)(\overline{k})$ in X_2, then we construct a marked oriented edge as

$$(f, x) \xrightarrow{(g,\overline{k})} (d, z). \tag{1.8}$$

Let us fix families of maps $\{f_1, \ldots, f_k\}$, points $\{x_1, \ldots, x_l\} \subset [0,1]$, and $\{\overline{a}(i,j,h)\}_{i,j,h=1}^{i,j=k,h=l} \in X_2$. By the above way, we obtain the corresponding oriented marked finite graph

$$G(\{f_i\}, \{x_j\}, \{\overline{a}(i,j,h)\}). \tag{1.9}$$

We call them the *interaction graphs*.

One can interpret these graphs to represent states of the system consisted by the triple $(\{f_i\}, \{x_j\}, \{\overline{a}(i,j,h)\})$.

Let us choose (f_i, f_j, x_h). Then, we have the corresponding interaction map $\Phi(f_i, f_j, x_h) : X_2 \to X_2$. In particular one can obtain other elements as follows:

$$\overline{a}(i,j,h)_2 \equiv \Phi(f_i, f_j, x_h)(\overline{a}(i,j,h)) \in X_2. \tag{1.10}$$

By this way one can obtain another interaction graph:

$$G(\{f_i\}, \{x_j\}, \{\overline{a}(i,j,h)_2\}). \tag{1.11}$$

Let us denote by $\mathfrak{G}(\{f_i\},\{x_j\})$ the set of interaction graphs with fixed families of maps and points. The numbers of vertices are all the same in any element in this. Notice that this is a finite set. Then, by the above procedure one has obtained the following map:

$$\Phi_* : \mathfrak{G}(\{f_i\},\{x_j\}) \longrightarrow \mathfrak{G}(\{f_i\},\{x_j\}). \tag{1.12}$$

By iterating this procedure, one obtains an infinite family of interaction graphs $G(\{f_i\},\{x_j\},\{\overline{a}(i,j,h)\})$, $G(\{f_i\},\{x_j\},\{\overline{a}(i,j,h)_2\})$, $G(\{f_i\},\{x_j\},\{\overline{a}(i,j,h)_3\}),\ldots$. One can regard that this family of interaction graphs might represent dynamics of states of the micro interaction systems, and according to our principle at the beginning of the introduction, one may induce some macro patterns from them. In this paper we will induce some *hierarchies* of combinatoric objects arising from such family of graphs.

Let $\mathfrak{G}$ be the set of finite graphs, and let $F : \mathfrak{G}(\{f_i\},\{x_j\}) \to \mathfrak{G}$ be the forgetful map. Passing through F, one obtains a family of finite graphs:

$$G_1 \equiv F\left(\overline{G}_1\right), G_2 \ldots \subset \mathfrak{G}, \qquad \overline{G}_t = G\left(\{f_i\},\{x_j\},\{\overline{a}(i,j,h)_t\}\right), \tag{1.13}$$

which we call just the *associated graphs*. Any G_i has the same number of vertices N. Thus there is a finite number of finite graphs $\{H_1,\ldots,H_m\}$ so that each G_i coincides with one of $\{H_j\}$.

We say that a family of finite graphs is *strongly regular* if they have the same number of edges as others.

Let G be a finite graph. Then, the associated *configuration* $\overline{a} \in \mathbf{Z}^m$ in combinatorics is determined. Thus one obtains another family of configurations:

$$\overline{a}_1, \overline{a}_2, \overline{a}_3, \ldots \subset \mathbf{Z}^m, \tag{1.14}$$

which we call the *transcripted configurations*.

For each configuration $\overline{a} \in \mathbf{Z}$, one obtains an ideal $I_{\overline{a}} \subset \mathbf{C}[y_1,\ldots,y_m]$ which is called the *toric ideal*. Thus associated to the family of configurations, one obtains the corresponding family of ideals:

$$I_1, I_2, \ldots, I_j \subset \mathbf{C}\left[y_1,\ldots,y_{m_j}\right], \tag{1.15}$$

which we call the *associated ideals*.

For each ideal $I \subset \mathbf{C}[y_1,\ldots,y_m]$, one obtains a complete fan over $\mathbf{R}^m$ and the corresponding toric variety $X_I \subset \mathbf{CP}^{m-1}$. The fan is called the *Gröbner fan*.

Thus corresponding to the associate ideals, one obtains a family of toric varieties:

$$X_1, X_2, \ldots \subset \mathbf{CP}^{m-1}, \quad m = \sup\{m_1, m_2, \ldots\}. \tag{1.16}$$

We call the sequence as the *translated toric variety*. When the associated graphs are strongly regular, then all X_j have the same dimension.

Let us have more abstract setting. Let V be an algebraic variety with the affine coordinates V_i defined by an ideal J_i. Let us take an automorphism A on V. We say that an affine coordinate $\{(V_i, J_i)\}_{i=1}^m$ is an (stable) *algebraic Markov partition* for A if, for each i, there is some j so that

$$A(V_i) \subset V_j \tag{1.17}$$

holds.

We say that the associated ideals $\{I_i\}_i$ are *regular* if they have the same dimension as others. Let us put $\bigcup_i I_i = \{J_1, \ldots, J_k\}$. Now we have one realization of the space-form problem.

Definition 1.4. Let $(\{f_i\}_i, \{x_j\}_j, \{\overline{a}(i,j,h)\})$ be interaction data, and suppose that the associated ideals $I = (I_0, I_1, \ldots)$ are regular. The sequence is called a symbolic flow of an automorphism if there is an algebraic Markov partition for (V, A) with an affine coordinate $\{(V_i, J_i)\}_{i=1}^k$ and some $x \in V$ so that its orbit $\{A^n(x)\}_{n=0,1,\ldots}$ corresponds to the sequence.

We would like to call such pair (V, A) a *prohedron* (which comes from "proteiform"). We will see by an easy example that combinatorics of the interaction graphs will reflect existence of such pairs.

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2. Automata and Interaction by Families of Maps

2.1. Classes of Automata

In this paper we treat some automata whose classes we will clarify below.

Let A be a finite set called the *alphabet*, and let Q be a set called the *state set*.

Let us introduce the most general form of automata here. Let $m, l \geq 0$ be integers. A *bounded automaton* $\mathbf{A}$ over A is given by two functions:

$$\varphi : Q \times A^{m+l+2} \longrightarrow A, \qquad \phi : Q \times A \longrightarrow Q. \tag{2.1}$$

A flow of the sets of strings of one-sided infinite length

$$\left\{\overline{k}_t = \left(k_0^t, k_1^t, \ldots\right) : k_i^t \in A\right\}_{t \geq 0} \tag{2.2}$$

is determined by A if $\{k_N^t\}_{N,t\geq 0}$ satisfy the following relations:

$$k_i^{t+1} = \varphi\left(q_i^{t+1}, k_{i-l}^t, \ldots, k_{i+m}^t, k_{i-1}^{t+1}\right), \qquad q_i^t = \phi\left(q_{i-1}^t, k_i^t\right). \tag{2.3}$$

We regard that they grow time evolutionally as $\overline{k}_0 \to \overline{k}_1 \to \cdots \to \overline{k}_t \to \cdots$.

One can similarly consider the two-sided case

$$\left\{\overline{k}_t = \left(\ldots, k_{-n}^t, \ldots, k_0^t, k_1^t, \ldots\right) : k_i^t \in A\right\}_{t \geq 0}. \tag{2.4}$$

2.1.1. Mealy Automaton (see [7, 8])

A *Mealy automaton* **A** over X is given by two functions:

$$\psi : Q \times A \longrightarrow A, \qquad \phi : Q \times A \longrightarrow Q. \tag{2.5}$$

(A) One-Sided Case

Let

$$\mathrm{X}_A = \left\{ \overline{k} = (k_0, k_1, \ldots) : k_i \in A \right\} \tag{2.6}$$

be all of the set of strings of one-sided infinite length. Then, each $q \in Q$ induces a continuous map

$$A_q : \mathrm{X}_A \longrightarrow \mathrm{X}_A \tag{2.7}$$

given by $A_q(k_0, k_1, \ldots) = (k'_0, k'_1, \ldots)$, where k'_i are inductively defined with $q_{-1} = q$ as follows:

$$k'_i = \psi(q_i, k_i), \qquad q_i = \phi(q_{i-1}, k_i). \tag{2.8}$$

One can check easily that Mealy automata can be obtained by choosing $\psi : Q \times A \to A$ by $k_i^{t+1} = \psi(q_i^{t+1}, k_i^t)$ and ϕ are the same.

Let $\mathrm{X}_A^N = \{\overline{k}^* = (k_0, k_1, \ldots, k_N) : k_i \in A\}$ be the set of words of length N with alphabet X. Then, A_q restricts the action as $A_q : \mathrm{X}_A^N \to \mathrm{X}_A^N$ for all $N \in \{0, 1, 2, \ldots\}$.

Let $m = |A|$ and let T_m be the rooted m-tree. The set of all vertices of T_m can be identified with $\mathrm{X}_A^\infty \equiv \bigcup_N \mathrm{X}_A^N$. Thus A_q gives the following action:

$$A_q : T_m \longrightarrow T_m. \tag{2.9}$$

Let us say that A is invertible (see [7]) if $\psi(q,) : A \cong A$ are one-to-one onto for all $q \in Q$. An invertible automaton A gives automorphisms $A_q : T_m \cong T_m$, and the group generated by the set of states is denoted by $G(A)$:

$$G(A) = \mathrm{gen}\left\{ A_q : T_m \cong T_m : q \in Q \right\}. \tag{2.10}$$

(B) Two-Sided Case

Let $A = \{0, 1, \ldots, L-1\}$ and let S be a finite set. Let us consider the two maps

$$\psi : S \times A \longrightarrow A, \qquad \phi : S \times A \longrightarrow S \tag{2.11}$$

equipped with the initial state $q \in S$. These data give a structure of Mealy automaton **A**.

Let us put the set of the two-sided sequences by $\Sigma_L = \{(\ldots, v_{-n}, \ldots, v_0, \ldots) : v_i \in A\}$. Let $\sigma = (\ldots, v_{-n}, \ldots, v_0, \ldots)$ be an infinite sequence by $\{0, \ldots, L-1\}$ such that $v_n = v_{-n} = 0$ holds for all sufficiently large $n \gg 0$. Let us denote the set of such sequences by

$$\Sigma_L^0 = \{(\ldots, v_{-n}, \ldots, v_0, v_1, \ldots) : v_i \in A,\ v_n = 0\ \forall \text{ large } |n| \gg 0\}. \tag{2.12}$$

Notice that Σ_L^0 are shift invariant.

Let us say that $\mathbf{A}$ is *semiproper* if $\psi(q, 0) = 0$ is satisfied.

If $\mathbf{A}$ is semi-proper, then it induces a continuous map

$$A_q : \Sigma_L^0 \longrightarrow \Sigma_L, \qquad A_q(\ldots, v_n, v_{n+1}, \ldots) = (\ldots, v'_n, v'_{n+1}, \ldots) \tag{2.13}$$

given by the same rule as that of the one-sided case. Namely, let us take $(\ldots, v_n, v_{n+1}, \ldots) \in \Sigma_L^0$, and choose sufficiently small m_0 so that $v_m = 0$ holds for all $m \leq m_0$. Then, we define inductively as

$$v'_{m_0+k} = \psi(s_k, v_{m_0+k}), \qquad s_k = \phi(s_{k-1}, v_{m_0+k}), \tag{2.14}$$

where we put $v'_m = 0$ for all $m \leq m_0$. This is independent of choice of m_0 and gives an assignment

$$A_q : (\ldots, v_m, v_{m+1}, \ldots) \longrightarrow (\ldots, v'_m, v'_{m+1}, \ldots). \tag{2.15}$$

Let $\mathbf{A}$ be a Mealy automaton which is semi-proper. If it induces a continuous map

$$A_q : \Sigma_L^0 \longrightarrow \Sigma_L^0, \tag{2.16}$$

then we say that $\mathbf{A}$ is *proper*.

2.1.2. Cellular Automaton

The cell automata we treat here require only the initial states. Let $A = Q$, and take a function $\psi : A^3 \to A$. Then, each $q \in Q = A$ induces a continuous map

$$A_q : X_A \longrightarrow X_A \tag{2.17}$$

given by $A_q(k_0, k_1, \ldots) = (k'_0, k'_1, \ldots)$, where k'_i are inductively defined with $k'_{-1} = q$:

$$k'_i = \psi(k_i, k_{i+1}, k'_{i-1}). \tag{2.18}$$

So the flows $\{\overline{k}_t\}_t$ satisfy the relations $k_i^{t+1} = \psi(k_i^t, k_{i+1}^t, k_{i-1}^{t+1})$.

2.2. Interacting Maps

Let us take the two interval maps

$$f_0, f_1 : [0,1] \longrightarrow [0,1] \tag{2.19}$$

and consider their iterations as

$$O_0(x) = \left\{ f_0^k(x) \right\}_{k=0,1,\ldots}, \qquad O_1(x) = \left\{ f_1^k(x) \right\}_{k=0,1,\ldots}. \tag{2.20}$$

We call them the *oscillations* (see [3]).

Let us define *interaction* of these orbits below. For this, let X_2 be the set of one-sided sequences with two alphabets $\{0,1\}$ as follows:

$$\mathrm{X}_2 = \{(k_0, k_1, \ldots) : k_i \in \{0,1\}\}. \tag{2.21}$$

For each element $\overline{k} = (k_0, k_1, \ldots) \in \mathrm{X}_2$, we associate a family of maps $\{h^m\}_{m=0,1,\ldots}$ with $h^m : [0,1] \to [0,1]$ by the random iterations

$$h^m(x) \equiv f_{k_m} \circ f_{k_{m-1}} \circ \cdots \circ f_{k_0}(x). \tag{2.22}$$

Let

$$\pi : [0,1] \longrightarrow \{0,1\} \tag{2.23}$$

be the measurable map given by $\pi([0,1/2)) \equiv 0$ and $\pi((1/2,1]) \equiv 1$. Then, for each $x \in [0,1]$, one can compose $\{h^m(x)\}_m$ with π and obtain another element for a.e. x as follows:

$$\overline{k}' \equiv \pi\Big(\Big(h^0(x), h^1(x), \ldots\Big)\Big) \equiv \Big(\pi \circ h^0(x), \pi \circ h^1(x), \ldots\Big) \in \mathrm{X}_2. \tag{2.24}$$

This assignment gives a map from each element $\overline{k}$ to $\overline{k}'$. We denote this map

$$\Phi(x, f_0, f_1) : \mathrm{X}_2 \longrightarrow \mathrm{X}_2 \tag{2.25}$$

by $\Phi(x, f, g)(\overline{k}) = \overline{k}' \equiv \pi((h^0(x), h^1(x), \ldots))$ and call it the *interaction map*.

Let us introduce a generalization of the interaction. Let us fix a set of alphabets $\{0,1,\ldots,L\}$ and choose a family of maps $\{f_{i,j}\}_{i,j=0,1,\ldots,L}$ and $\overline{k} \in X_{L+1}$. Then, we define inductively

$$h^m(x) = f_{k_m,k_{m+1}} \circ h^{m-1}(x), \qquad h^{-1}(x) \equiv x. \tag{2.26}$$

We say that $\{h^m(x)\}_m$ is a *two-step* interaction.

Let $\pi : [0,1] \to \{0,1,\ldots,L\}$ be the projection. Then, as before one obtains a map

$$\Phi(x) : X_{L+1} \longrightarrow X_{L+1}, \qquad \Phi(x)\left(\overline{k}\right) = \overline{k}', \\ \overline{k}'_m = \pi \circ h^m(x). \tag{2.27}$$

Let $\Phi : X_{L+1} \to X_{L+1}$ be a map, and take an initial condition

$$\left(k_0^0, k_1^0, k_2^0, \ldots\right) \tag{2.28}$$

and a boundary condition

$$\left(k_0^1, k_0^2, k_0^3, \ldots\right). \tag{2.29}$$

Then, we inductively put

$$(k_0^t, k_1^t, \ldots) = \Phi\left(k_0^{t-1}, k_1^{t-1}, \ldots\right) = \Phi^t\left(k_0^0, k_1^0, k_2^0, \ldots\right) \in X_{L+1}. \tag{2.30}$$

We call $(k_0^t, k_1^t, \ldots)$ as the *flow* of Φ.

In Section 6, we consider the flow of the Lotka-Volterra cell automaton by the iteration of the interaction map for some family of interval maps. LV cell automaton is isomorphic to the BBS system described below. We will study the isomorphism in Section 4.

2.3. BBS System

Let Σ_2 be the set of two-sided sequences with two alphabets, and consider its subsets $\Sigma_2^0 \subset \Sigma_2$:

$$\Sigma_2^0 = \left\{(\ldots, v_{-n}, \ldots, v_0, v_1, \ldots) \in \Sigma_2 : v_n = 0 \ \forall \text{ large } |n| \gg 0\right\}. \tag{2.31}$$

Notice that it is shift invariant.

Let us describe the dynamical system $T : \Sigma_2^0 \cong \Sigma_2^0$ called BBS below (see [9]). One can canonically identify Σ_2^0 with the set of ordered integers

$$\Sigma_2^0 \cong O = \{i_1 < i_2 < \cdots < i_m : i_k \in \{\ldots, 0, 1, \ldots\}, m \in \mathbb{N}\} \tag{2.32}$$

by $(\ldots, v_{-n}, \ldots, v_n, \ldots) \to \{n : v_n = 1\}$ as in the introduction.

Let us choose an element $\sigma \in \Sigma_2^0$ and let $(i_1 < i_2 < \cdots < i_m)$ be the corresponding ordered integers. Let

$$T(\sigma)_1 = \left(\ldots, v_{-m}^1, \ldots, v_0^1, v_1^1, \ldots\right) \in \Sigma_2^0 \tag{2.33}$$

be another element defined as follows: let $j_1 \geq i_1$ be the smallest index with $v_{j_1} = 0$. Then, we put

$$v_l^1 = \begin{cases} v_l, & l \neq i_1, j_1, \\ 0, & l = i_1, \\ 1, & l = j_1. \end{cases} \tag{2.34}$$

Next we do the same thing for $v_{i_2}^1 = v_{i_2}$ in $T(\sigma)_1$ and find another smallest index $j_2 \geq i_2$ with $v_{j_2}^1 = 0$. Then, we exchange 0 and 1 in $v_{i_2}^1$ and $v_{j_2}^1$ as above. The result is denoted by $T(\sigma)_2$.

We continue this process for $i_3, i_4, \ldots$ until i_m, and finally one obtains the desired $T(\sigma) \equiv T(\sigma)_m \in \Sigma_2^0$.

Thus one has obtained a continuous bijective map

$$T : \Sigma_2^0 \cong \Sigma_2^0 \tag{2.35}$$

which is calld the *box and ball system (BBS)* (see [13]).

Let $O_N = \{i_1 < i_2 < \cdots < i_N : i_k \in \{\ldots, 0, 1, \ldots\}\} \subset O$ be the subset consisted by the set of exactly N indices. Let $\Sigma_2^0(N) \subset \Sigma_2^0$ be the corresponding subsets. Then, the BBS system preserves them and induces bijections $T : \Sigma_2^0(N) \cong \Sigma_2^0(N)$.

2.3.1. BBS as a Mealy Automaton

Let us describe the BBS map T by an interaction of a family of interval maps. The basic method will take two steps. Firstly we will describe it by a proper Mealy automaton, and then we write down the automaton by an interaction of a family of maps. Here we will use some modified way in order to express it by a family of continuous maps. For the presentation, see also Section 4.1.

Firstly we will construct an automaton A. It will induce a map $M : \Sigma_2^0(N) \cong \Sigma_2^0(N)$ which corresponds to $T(\sigma)_1$ in Section 2.3. Then, the BBS map $T : \Sigma_2^0(N) \cong \Sigma_2^0(N)$ is given by $T \mid \Sigma_2^0(N) = M^N$.

Lemma 2.1. *For the state set $S = \{0, 1, 2, 3, 4\}$ with the initial state 0, there is a proper Mealy automaton $\mathbf{A}$ which induces $M : \Sigma_2^0 \cong \Sigma_2^0$.*

Proof. Let us define $\mathbf{A}$ by

$$\varphi : S \times \{0, 1\} \longrightarrow \{0, 1\}, \qquad \phi : S \times \{0, 1\} \longrightarrow S, \tag{2.36}$$

Table 1: Values of ϕ.

	0	1	2	3	4
0	0	2	4	2	4
1	1	3	4	3	4

where

$$\psi(a,) = \begin{cases} \epsilon, & a = 1,2, \\ \mathrm{id}, & a = 0,3,4 \end{cases} \tag{2.37}$$

($\epsilon \in S_2$ is the nontrivial element in the permutation group on $\{0,1\}$).

It is immediate to check the conclusion. This completes the proof. □

Let us consider expressing BBS by an interaction of maps. Let us denote $\phi = \{\phi_0, \phi_1\}$ as in Table 1 and $\psi = \{\psi_0, \psi_1\}$, where $\phi_i : S \to S$ and $\psi_i : S \to \{0,1\}$ for $i = 0,1$. Let $\pi : [0,1] \to \{0,1\}$ and $\pi_5 : [0,1] \to \{0,1,2,3,4\}$ be the canonical projections.

If the automaton above is described by compositions of a family of continuous maps $\{\alpha_i\}_{i=0,1}$ and $\{f_i\}_{i=0,1}$ as $\phi_i = \pi_5 \circ \alpha_i$ and $\psi_i = \pi \circ f_i$, then we say that the automaton is given by an *interaction of maps on intervals* (see [6]).

It is immediate to see that the above **A** cannot be expressible by a family of continuous maps as above. So we will modify as follows. Let us choose two permutations $\epsilon_0, \epsilon_1 : S \cong S$. Then, we say that A is given by a *modified interaction of maps* with respect to (ϵ_0, ϵ_1), if there are families of continuous maps $\{\alpha_i\}_{i=0,1}$ and $\{f_i\}_{i=0,1}$ so that the following hold:

$$\begin{aligned} \phi_i &= \epsilon_i \circ \pi_5 \circ \alpha_i, \\ \psi_i &= \pi \circ f_i. \end{aligned} \tag{2.38}$$

When these maps can be chosen piecewisely linearly, then tropical geometry in Section 6 can be applied, since piecewise linear functions can have representations by the relative arithmetics of $(\max, +)$. BBS is also the following case.

Lemma 2.2. *There are families of continuous maps and permutations so that they give the modified interaction of maps representing the proper Mealy automaton* **A** *above. In particular the BBS automaton is given by* $M^N = M \circ \cdots \circ M$ *on* $\Sigma_2^0(N)$ *for* $N = 0,1,2,\ldots$.

Proof. Let us choose permutations as

$$\begin{aligned} \epsilon_0 &: (0,1,2,3,4) \longrightarrow (0,2,4,1,3), \\ \epsilon_1 &: (0,1,2,3,4) \longrightarrow (0,1,3,4,2). \end{aligned} \tag{2.39}$$

Then, we can choose piecewise linear functions α_i and f_i so that they satisfy the following properties:

$$\begin{gathered}
\alpha_0 \mid \left[\frac{i}{5}, \frac{i+1}{5}\right] \subset \left[\frac{i}{5}, \frac{i+1}{5}\right], \quad i = 0,1,2, \\
\alpha_0 \mid \left[\frac{3}{5}, \frac{4}{5}\right] \subset \left[\frac{1}{5}, \frac{2}{5}\right], \qquad \alpha_0 \mid \left[\frac{4}{5}, 1\right] \subset \left[\frac{2}{5}, \frac{3}{5}\right], \\
\alpha_1 \mid \left[\frac{i}{5}, \frac{i+1}{5}\right] \subset \left[\frac{i+1}{5}, \frac{i+2}{5}\right], \quad i = 0,1,2, \\
\alpha_1 \mid \left[\frac{i}{5}, \frac{i+1}{5}\right] \subset \left[\frac{i-1}{5}, \frac{i}{5}\right], \quad i = 3,4, \\
f_0 \mid \left[\frac{j}{5}, \frac{j+1}{5}\right] \subset \left[0, \frac{1}{2}\right], \quad j = 0,3,4, \qquad f_0 \mid \left[\frac{j}{5}, \frac{j+1}{5}\right] \subset \left[\frac{1}{2}, 1\right], \quad j = 1,2, \\
f_1 \mid \left[\frac{j}{5}, \frac{j+1}{5}\right] \subset \left[0, \frac{1}{2}\right], \quad j = 1,2, \qquad f_1 \mid \left[\frac{j}{5}, \frac{j+1}{5}\right] \subset \left[0, \frac{1}{5}\right], \quad j = 0,3,4.
\end{gathered} \tag{2.40}$$

This completes the proof. □

3. Quotient of the Braid Groups and BBS System

3.1. Quotient of the Braid Groups

Let B_n be the braid group with n-strands. It has the following presentation:

$$B_n = \left\{t_1, \ldots, t_{n-1} : t_i t_j = t_j t_i,\ \left|i-j\right| > 1,\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}\right\}. \tag{3.1}$$

It has a canonical element representing the half-twist, that is,

$$\Omega_n \equiv (t_1 \ldots t_{n-1})(t_1 \ldots t_{n-2}) \cdots (t_1 t_2) t_1 \in B_n. \tag{3.2}$$

The quotient group by Ω_n^2 is isomorphic to a subgroup of the mapping class group on $(n+1)$-punctured sphere, which is consisted by groups which fix ∞ point and index $n+1$ (see [14]):

$$B_n / \langle \Omega_n^2 \rangle \subset \mathrm{MPG}_{n+1}. \tag{3.3}$$

Let us generalize it and have quotient braid groups.

Let $i = 1, \ldots, n-1$ and $k = 0, \ldots, n-1$, and denote subsets by $[i,k] \equiv (i, i+1, \ldots, i+k) \subset \{1, \ldots, n-1\}$. Let $B_{[i,k]} \subset B_n$ be the subgroup generated by $\{t_i, \ldots, t_{i+k}\}$. By the same way, each $B_{[i,k]}$ contains the following corresponding canonical elements:

$$\Omega_{[i,k]} \in B_{[i,k]}. \tag{3.4}$$

We will define two types of quotients of the braid groups using these canonical elements.

Definition 3.1. The mod 2 braid group $M_2\overline{B}_n$ is given by the following:

$$\begin{aligned} M_2\overline{B}_n &= \left\{t_1,\dots,t_{n-1} : t_it_j = t_jt_i, \left|i-j\right| > 1,\ t_it_{i+1}t_i = t_{i+1}t_it_{i+1}, t_i^2\right\} \\ &= B_n/\mathrm{gen}\left\langle \bigcup_{i=1}^{n-1} t_i^2 \right\rangle. \end{aligned} \tag{3.5}$$

For example, $M_2\overline{B}_2 = \mathbf{Z}_2$ and $M_2\overline{B}_3$ is an infinite group which has a presentation given by $\mathbf{Z}_2 * \mathbf{Z}_2/\langle t_1t_2t_1^2t_2t_1\rangle$.

Let us denote the quotient map by

$$\pi : B_n \longrightarrow M_2\overline{B}_n. \tag{3.6}$$

Lemma 3.2. *Let $\pi(\Omega^2_{[i,k]}) = 1 \in M_2\overline{B}_n$ for any i,k. Then, in particular one has*

$$M_2\overline{B}_n = B_n/\mathrm{gen}\left\langle \bigcup_{[i,k]\subset\{1,\dots,n-1\}} \Omega^2_{[i,k]} \right\rangle. \tag{3.7}$$

Proof. We check for $n = 4$, and the general case will follow from this immediately.

In fact $\pi(\Omega_4^2) = (t_1t_2t_3t_1t_2t_1)(t_1t_2t_3t_1t_2t_1) = t_1t_2t_3t_1t_2t_1^2t_2t_3t_1t_2t_1 = t_1t_2t_3t_1t_2^2t_3t_1t_2t_1 = t_1t_2t_3t_1t_3t_1t_2t_1 = t_1t_2t_3t_3t_1t_1t_2t_1 = 1$.

This completes the proof. □

Let $\mathbf{I} \subset \{1,\dots,n-1\}^2$ be a set of subsets, and denote by $\mathrm{gen}\langle\Omega_I : I \in \mathbf{I}\rangle$ the group generated by elements Ω_I. Then, we define another type of quotient braid groups by

$$\overline{B}_n(\mathbf{I}) = B_n/\mathrm{gen}\langle\Omega_I : I \in \mathbf{I}\rangle. \tag{3.8}$$

Notice that we do not include elements like $(i,0)$ which correspond to one string. We will denote $\pi_{\mathbf{I}} : B_n \to \overline{B}_n(\mathbf{I})$ for the projection.

Later we will have geometric meaning to divide by the twists in terms of the BBS system. In particular we will see in the following section that solitary behaviour in BBS system can be represented as an element in the quotient braid groups.

3.2. Solitary Flow

Any $\sigma \in \Sigma_2^0$ can be uniquely expressed by a finite set

$$\{I_1,\dots,I_l\} \equiv \{(i_1,N_1),\dots,(i_l,N_l)\}, \tag{3.9}$$

where $N_j \geq 1$ and $l \geq 0$. They satisfy the following:

(1) $v_j = 1$ for all $j \in \bigcup_{k=1,\ldots,l}\{i_k, i_k + 1, \ldots, i_k + N_k - 1\}$,

(2) $v_j = 0$ holds for some j in $\{i_k + N_k, \ldots, i_{k+1} - 1\}$, $k = 1, \ldots, l$.

We call $(N_1, \ldots, N_l)$ the *index* of σ. Thus one may regard each I_j as one soliton.

Let $\sigma \in \Sigma_2^0$ and consider the BBS map T. The indices of $T^t(\sigma)$, $(S_1, \ldots, S_m)$ are all constant for sufficiently small $t \ll 0$, and also for all sufficiently large $t \gg 0$, they are also constant $(T_1, \ldots, T_{m'})$, where $\Sigma S_i = \Sigma T_j$ holds. Here inequalities hold as follows:

$$S_1 \geq S_2 \geq \cdots \geq S_m, \qquad T_1 \leq T_2 \leq \cdots \leq T_{m'}. \tag{3.10}$$

We say that σ has its *type*

$$\left(\{S_i\}_{i=1}^{m}, \{T_j\}_{j=1}^{m'}\right). \tag{3.11}$$

If σ is a soliton, then the equality

$$(T_1, \ldots, T_{m'}) = (S_m, S_{m-1}, \ldots, S_1) \tag{3.12}$$

holds. More precisely, let $\{\sigma^t\}_{t\in\mathbf{Z}} \subset \Sigma_2^0$ be a flow. We say that it is *solitary* if there are a set $\{M_1, \ldots, M_m\} \subset \mathbf{N}$, $1 \leq M_1 < M_2 < \cdots < M_m$, and families $\{i_1^t, \ldots, i_m^t\} \subset \mathbf{Z}$ such that, for all sufficiently large $t \gg 0$,

$$\begin{aligned} \sigma^{-t} &= \left\{\left(i_1^{-t}, M_1\right), \left(i_2^{-t}, M_2\right), \ldots, \left(i_m^{-t}, M_m\right)\right\}, \\ \sigma^{t} &= \left\{\left(i_1^{t}, M_m\right), \left(i_2^{t}, M_{m-1}\right), \ldots, \left(i_m^{t}, M_1\right)\right\}, \end{aligned} \tag{3.13}$$

where

(1) $i_1^t < i_2^t < \cdots < i_m^t$ and $i_m^{-t} < i_{m-1}^{-t} < \cdots < i_1^{-t}$ hold,

(2) $|i_j^{\pm t} - i_{j+1}^{\pm t}| \to \infty$ as $t \to \infty$ for all $j = 1, \ldots, m-1$.

Thus for any $\sigma \in \Sigma_2^0$, the corresponding flow $\{T^t(\sigma)\}_{t\in\mathbf{Z}}$ is solitary.

3.3. Assignment of Braid Elements

Let us take $\sigma \in \Sigma_2^0(N)$, and let T be the BBS map. Let us take a large $t_0 \gg 0$ so that $T^{-t_0}(\sigma)$ and $T^{t_0}(\sigma)$ have indices $(S_1, \ldots, S_m)$ and $(T_1, \ldots, T_{m'})$, respectively. Notice the equality $N \equiv \Sigma_{i=1}^m S_i = \Sigma_{j=1}^{m'} T_j$.

Let us put $\sigma_0 \equiv T^{-t_0}(\sigma)$, and we consider the step $T(\sigma_0)_i$, $i = 1, \ldots, m$, in the definition of the map T in Section 2.2. For the step from σ_0 to $T(\sigma_0)_1$, let us assign a natural element

$$b_1 \in B_N \tag{3.14}$$

below, where each string corresponds to an element 1 in σ_0. Namely, when the most left-hand side 1 moves into a position passing through another r number of 1's, then $t_{r-1}\cdots t_1$ is assigned. For example, if $\sigma_0 = (\ldots,0,1,1,0,\ldots)$ moves as $(\ldots,0,0,1,1,\ldots)$, then one generating element t_1 is assigned. Similarly, if $\sigma_0 = (\ldots,1,1,1,0,\ldots)$ which moves as $(\ldots,0,0,1,1,1,\ldots)$, then t_2t_1 is assigned.

Next for the second step from $T(\sigma_0)_1$ to $T(\sigma_0)_2$, one assigns another element $b_2 \in B_N$ by the same way. Continuing, one obtains another $b_3,\ldots,b_m$.

By this way one has assigned an element

$$b = b(\sigma_0, t_0) = b_m b_{m-1}\cdots b_1 \in B_N \tag{3.15}$$

which we call the *braiding element*.

3.3.1. Braiding Maps

Notice that b depends on the choice of t_0 and so σ_0 in the above. In fact there will be infinitely many elements which are different from each other with respect to the choice of t_0. The ambiguities arise from choices of the starting point $\sigma_0 = T^{-t_0}(\sigma)$ and the ending point $T^{t_0}(\sigma)$. Let σ have the type $(\{S_i\}_{i=1}^m, \{T_j\}_{j=1}^{m'})$. They are essentially given by the twists of the canonical elements

$$\Omega_{[1,S_1-1]}, \Omega_{[S_1+1,S_2-1]}, \Omega_{[S_1+S_2+1,S_3-1]}, \ldots, \Omega_{[S_1+\cdots+S_{i-1}+1,S_i-1]}, \tag{3.16}$$

for the former and given for the latter by

$$\Omega_{[T_1+\cdots+T_j+1,T_{j+1}-1]}, \Omega_{[T_1+\cdots+T_j+T_{j+1}+1,T_{j+2}-1]}, \ldots, \Omega_{[N-T_{m'},T_{m'}-1]}, \tag{3.17}$$

where the rest indices all correspond to one string:

$$S_k = 1 \quad (k = i+1,\ldots,m), \qquad T_l = 1 \quad (l = 1,2,\ldots,j). \tag{3.18}$$

Let us denote the set

$$\begin{aligned}\mathbf{I}(\sigma) = \{&[1,S_1-1],[S_1+1,S_2-1],\ldots,[S_1+\cdots+S_{i-1}+1,S_i-1],\\ &\left[T_1+\cdots+T_j+1,T_{j+1}-1\right],\left[T_1+\cdots+T_{j+1}+1,T_{j+2}-1\right],\ldots,[N-T_{m'},T_{m'}-1]\}.\end{aligned} \tag{3.19}$$

We say that $\mathbf{I}(\sigma)$ is the *index* of the BBS flow for σ.

Let $\mathbf{I} \subset \{1,\ldots,n-1\}$ be a set of subsets. Then, we put

$$\Sigma_2^0(\mathbf{I}) \equiv \left\{\sigma \in \Sigma_2^0 : \mathbf{I}(\sigma) = \mathbf{I}\right\}. \tag{3.20}$$

Let $\pi_{\mathbf{I}}$ be the projection as before. Then, we define the following *restricted braiding map with respect to* $\mathbf{I}$ as

$$B(\mathbf{I}) : \Sigma_2^0(\mathbf{I}) \longrightarrow \overline{B}_N(\mathbf{I}) \tag{3.21}$$

by assigning $\pi_{\mathbf{I}}(b)$, $b = b(\sigma_0, t_0))$. It is independent of choice of t_0 and gives a single map.

The above map depends on $\mathbf{I}$. Below we will have another braiding map from Σ_2^0 as a multivalued one. The target space is also obtained by the quotient of the braid group.

Let $\pi : B_N \to M_2\overline{B}_N$ be the projection. Thus $\pi(b) \in M_2\overline{B}_N$ have ambiguity at most finitely many elements with respect to t_0. Now we define the mod 2 *braiding map*:

$$B^2 : \Sigma_2^0(N) \longrightarrow M_2\overline{B}_N \tag{3.22}$$

as the images of all various values of t_0 for sufficiently large $|t_0| \gg 0$, given by $\pi(b) \equiv \overline{b} \in M_2\overline{B}_N$, where b is as above. Thus B^2 is a finite multivalued map

We call $\overline{b}$ the mod 2 *braiding element*.

3.4. Connected Braiding Maps

Let us take two elements as follows:

$$\sigma^k = \left(\dots, 0, a_{i_1,k}, \dots, a_{i_{m_k},k}, 0, \dots\right) \in \Sigma_2^0, \quad k = 1, 2. \tag{3.23}$$

Choose sufficiently large $M \gg 0$. Then, we define the *connected sum* of σ^1 with σ^2 of length M by

$$\sigma_1 \#_M \sigma_2 = \left(\dots, 0, a_{i_1,1}, \dots, a_{i_{m_1},1}, 0, \dots, 0, a_{i_2,2}, \dots, a_{i_{m_2},2}, 0, \dots\right) \in \Sigma_2^0, \tag{3.24}$$

where 0 appears M times in the middle.

The index of the *connected sum* and their union are given by

$$\begin{gathered} \mathbf{I}\left(\sigma^1, \sigma^2; M\right) = \mathbf{I}\left(\sigma^1\right) \cup \mathbf{I}\left(\sigma^2\right) \cup \mathbf{I}(\sigma_1 \#_M \sigma_2) \subset \{1, \dots, m_1 + m_2 - 1\}, \\ \mathbf{I}\left(\sigma^1, \sigma^2\right) \equiv \bigcup_{M \gg 0} \mathbf{I}(\sigma_1 \#_M \sigma_2). \end{gathered} \tag{3.25}$$

In fact $\mathbf{I}(\sigma^1, \sigma^2; M)$ is completely determined by the triple $(\mathbf{I}(\sigma^1), \mathbf{I}(\sigma^2), M)$.

Let $\mathbf{I}_N$ be all of the set of subsets in $\{1, \dots, N-1\}$ such that each element can be an index for some $\sigma \in \Sigma_2^0$. Then, for large $M \gg 0$, there are maps

$$H_M : \mathbf{I}_N \times \mathbf{I}_{N'} \longrightarrow \mathbf{I}_{N+N'} \tag{3.26}$$

which give the indices of connected sums.

Thus these induce a family of maps

$$H(\mathbf{I},\mathbf{I}';M) : \overline{B}_N(\mathbf{I}) \times \overline{B}_{N'}(\mathbf{I}') \longrightarrow \overline{B}_{N+N'}(H_M(\mathbf{I},\mathbf{I}')) \tag{3.27}$$

satisfying

$$H(\mathbf{I},\mathbf{I}';M)\left(B(\mathbf{I})\left(\sigma^1\right), B(\mathbf{I}')\left(\sigma^2\right)\right) = B(H_M(\mathbf{I},\mathbf{I}'))\left(\sigma^1 \#_M \sigma^2\right) \tag{3.28}$$

for $\mathbf{I} \in \mathbf{I}_N$ and $\mathbf{I}' \in \mathbf{I}_{N'}$. We will say that $H(\mathbf{I},\mathbf{I}';M)$ is a *connected braiding map*.

Proposition 3.3. *$H(\mathbf{I},\mathbf{I}';M)$ is eventually period with respect to M.*

This follows since the target space is divided by twists in braid groups.

4. Transformations on Cell Automata

4.1. From BBS to LV

The Lotka-Volterra cell automaton is given by the equation

$$V_n^{t+1} - V_n^t = \max(0, V_{n+1}^t - L) - \max\left(0, V_{n-1}^{t+1} - L\right). \tag{4.1}$$

This is obtained from the original Lotka-Volterra equation by taking differentiations twice. It is known that this possesses solitons which are induced from the ones of the difference LV solitons.

There is a procedure to transform BBS equation to LV cell automaton and vice versa (see [9, 13, 15]):

$$B_n^{t+1} = \min\left\{1 - B_n^t, \Sigma_{i=-\infty}^{n-1}\left(B_i^t - B_i^{t+1}\right)\right\} \Longleftrightarrow V_n^{t+1} - V_n^t = \max\{L, V_{n+1}^t\} - \max\left\{L, V_{n-1}^{t+1}\right\}, \tag{4.2}$$

by changes of variables.

Let

$$O_0 = \text{LV cell automaton} \longrightarrow O_1 \longrightarrow \cdots \longrightarrow O_k = \text{BBS} \tag{4.3}$$

be a procedure of transformations. We say that it is *invertible*, if the procedure has its inverse $O_k \to O_{k-1} \to \cdots \to O_0$. We say that O_0 and O_k can be connected by an invertible procedure.

Lemma 4.1. *LV cell automaton and BBS can be connected by an invertible procedure as follows:*

$$O_0 = \{V_n^t\} \longrightarrow O_1 = \{U_n^t\} \longrightarrow O_2 = \{S_n^t\} \longrightarrow O_3 = \{B_n^t\}. \tag{4.4}$$

Proof. They can be connected by three steps as follows:

$$S_{n+1}^{t+1} - S_n^t = \min\left\{0, 1 - S_{n+1}^t + S_n^{t+1}\right\}, \quad S_n^t = \Sigma_{i=-\infty}^n B_i^t,$$
$$U_{n+1}^{t+1} - U_n^t = \max\left\{0, U_n^{t+1} - 1\right\} - \max\{0, U_{n+1}^t - 1\}, \quad U_n^t = S_{n+1}^t - S_n^{t+1}, \tag{4.5}$$
$$V_n^{t+1} - V_n^t = \max\{1, V_{n+1}^t\} - \max\left\{1, V_{n-1}^{t+1}\right\}, \quad V_{t-n}^n = U_n^t.$$

For the first transformation, one has the relations $B_n^t = S_n^t - S_{n-1}^t$. For the second, notice that $S_{n-a}^{t+a+1} = 0$ for all sufficiently large a. Then, we have the relations

$$\Sigma_{x=0}^{\infty} U_{n-x}^{t+x} = S_{n+1}^t - \lim_{a\to\infty} S_{n-a}^{t+a+1} = S_{n+1}^t. \tag{4.6}$$

This completes the proof. □

Remark 4.2. It has been shown in [3] that LV cell automaton is given by a family of PL maps and projections on the interval $[0,1]$. We have seen that BBS is given by a family of PL maps, projections, and permutations in Section 2.3.

4.1.1. Deformation by Commutators

Let us consider the second step in the proof of Lemma 4.1. We express the linear transformation $U_n^t = S_{n+1}^t - S_n^{t+1}$ as

$$U = \alpha(S), \tag{4.7}$$

where we mean that $\alpha(S)_n^t \equiv U_n^t$. By this way, let us express others by

$$f(S), \quad f(S)_n^t = S_n^t - S_{n-1}^{t-1},$$
$$\beta(S), \quad \beta(S)_n^t = S_{n+1}^t - S_n^{t+1}, \tag{4.8}$$
$$\gamma(S), \quad \gamma(S)_n^t = \min\left\{0, 1 - S_{n-1}^{t-1}\right\}.$$

Now the second step is expressed as

$$U = \alpha(S), \tag{4.9}$$
$$f(U) = \beta \circ f(S), \tag{$*$}$$

and the defining equation becomes

$$f(S) = \gamma \circ \alpha(S). \tag{$**$}$$

Thus combining with $(*)$ and $(**)$, after the transformation, the equation changes as below:

$$f(U) = \beta \circ \gamma(U). \tag{$***$}$$

Let us consider deforming the transformations. Let h be a transformation. Then, we say that it *commutes* with f if

$$f \circ h = h \circ f \tag{4.10}$$

holds.

Example 4.3. Let $U = \{U_n^t\}_{t,n}$ satisfy $\lim_{i \to \infty} U_{n-i}^{t-i} = 0$ for each t and n.

Let h be given as

$$h(U)_t^n = \Sigma_{i=-\infty}^{0} U_{n-i}^{t-i}, \tag{4.11}$$

and let f be $f(U)_n^t = U_n^t - U_{n-1}^{t-1}$ as above. Then, h commutes with f.

Let us consider transformations given by $(*)$ and $(**)$ above. Then, a *deformation of the transformation* by a commutator h is another one given below:

$$\begin{aligned} W &= h \circ \alpha(S), \\ f(W) &= h \circ \beta \circ f(S), \end{aligned} \tag{4.12}$$

which follows from $(*)$. When one considers the equation of the form

$$f(S) = \gamma \circ h \circ \alpha(S), \tag{4.13}$$

then it is changed to $f(W) = h \circ \beta \circ \gamma(W)$.

We denote all of the set of commutators with respect to f by

$$C(f) = \{h : [h, f] = 0\}. \tag{4.14}$$

4.2. Spaces of Cell Automata

So far, we have considered certain types of automato in Section 2.1, mainly those whose defining equations are given by max-plus equations. These are not closed under change of variables, and so here we will generalize the classes of functions we treat.

Let us denote the set of integer-valued maps

$$\mathrm{PL}_n = \left\{ f : \mathbf{Z}^{3n+1} \longrightarrow \mathbf{Z}, f\left(x_{-n}, \ldots, x_{-1}, y_{-n}, \ldots, y_0, y_1, \ldots, y_n\right) \in \mathbf{Z} \right\}. \tag{4.15}$$

Notice that there are canonical extensions of any $f \in \mathrm{PL}_n$ to piecewisely linear maps $f : \mathbf{R}^{3n+1} \to \mathbf{R}$ just by connecting images of f piecewisely linearly. Sometimes we will identify both of the maps. Such viewpoints will become important in Section 6.

A *general automaton* is given by an element of the direct limit

$$f \in \mathrm{PL}_\infty = \lim_{n\to\infty} \mathrm{PL}_n, \tag{4.16}$$

where the defining equation of the automaton is given by

$$T_k^t = f\left(\left\{\ldots, T_{k-n}^t, T_{k-n+1}^t, \ldots, T_{k-1}^t\right\}, \left\{\ldots, T_{k-n}^{t-1}, \ldots, T_k^{t-1}, \ldots\right\}\right). \tag{4.17}$$

If f is of the form $f(x_{-N}, \ldots, x_{-1}, \ldots, y_{-n}, \ldots, y_1, \ldots, y_n, \ldots)$, for some N, then we say that it is of (N, ∞) type.

If f is of the form $f(\ldots, x_{-n}, \ldots, x_{-1}, y_{-M}, \ldots, y_1, \ldots, y_n, \ldots)$, for some M, then we say that it is of (∞, M) type.

For simplicity, we will say that they are of finite-∞ or ∞-finite types, respectively. If both cases hold, then we say that it is of (N, M) or of finite-finite type.

Notice that the Lotka-Volterra cell automaton is of finite-finite type and the BBS is of infinite-infinite type.

Let us denote all of the set of general automata and its subsets consisted by those of $*$-$*'$ types by

$$\mathbf{GAut}_{*\text{-}*'} \subset \mathbf{GAut}, \tag{4.18}$$

where $*, *'$ = finite or ∞. Notice that $\mathbf{GAut}_{\infty\text{-}\infty} = \mathbf{GAut}$.

We will also denote the set of general automata presented by the max-plus equations and its subsets by

$$\mathbf{MP}_{*-*'} \subset \mathbf{MP}. \tag{4.19}$$

One has the following inclusions:

$$\mathrm{BBS} \in \mathbf{MP} \subset \mathbf{GAut} \supset \mathbf{GAut}_{\text{f-f}} \supset \mathbf{MP}_{\text{f-f}} \ni \text{L-V CA}. \tag{4.20}$$

4.3. Loop Groupoid

Let us choose two general automata $O = \{S_n^t\}$ and $O' = \{T_n^t\}$ in $\mathbf{GAut}$. A *transformation* from O to O' is the one given by a change of variables:

$$T_k^t = F\left(\{\ldots, S_{k-n}^t, S_{k-n+1}^t, \ldots, S_{k-1}^t\}, \left\{\ldots, S_{k-n}^{t-1}, \ldots, S_k^{t-1}, \ldots\right\}\right) \tag{4.21}$$

for some linear function F. We denote the transformation by

$$O \longrightarrow O'. \tag{4.22}$$

They are mutually *invertible* if both $O \to O'$ and $O' \to O$ hold.

Let us consider two invertible paths $O_0 = O \to O_1 \to \cdots \to O_m = O'$ and $O'_0 = O' \to O'_1 \to \cdots \to O'_n = O$ in **GAut**. Then, the composition

$$O = O_0 \longrightarrow \cdots \longrightarrow O_m \longrightarrow O'_1 \longrightarrow O'_2 \cdots \longrightarrow O'_n = O \tag{4.23}$$

gives a loop with the origin O. Clearly two loops with the origin O admit a natural composition, and by this operation, the set of loops

$$\Omega_O = \left\{ \overline{O} \equiv O = O_0 \longrightarrow O_1 \cdots \longrightarrow O_m = O : \text{invertible paths} \right\} \tag{4.24}$$

admits a group structure.

Definition 4.4. The loop groupoid Ω is given by

$$\Omega(O, O') = \{ O = O_0 \longrightarrow O_1 \longrightarrow \cdots \longrightarrow O_n : \text{invertible paths} \}. \tag{4.25}$$

Therefore, $\Omega(O) \equiv \Omega(O, O)$ is called the loop group.

Let O_0 be the LV cell automaton, and denote all of the set of the solutions of the equation

$$S_{\mathrm{LV}} = \left\{ \{v_n^t\}_{t,n} : \text{solutions of the LV cell automaton} \right\}. \tag{4.26}$$

Let us take an element $\overline{O} \in \Omega_{O_0}$. Then correspondingly, there is a bijective map between solutions of the equations:

$$\overline{O}_* : S_{\mathrm{LV}} \cong S_{\mathrm{LV}}. \tag{4.27}$$

We call it as an *induced map* associated with $\overline{O} \in \Omega_{O_0}$.

In Section 6 we construct and study an involution on some classes of **GAut** by use of tropical geometry and projective duality in affine algebraic varieties.

Example 4.5. Let us use the notations in Section 4.1. Let $\{B_n^t\}$ and $\{V_n^t\}$ be the BBS and LV cell automaton, respectively. In Section 4.1, one has obtained a path $\overline{O} = \{O_0 = \{V_n^t\} \to O_1 = \{U_n^t\} \to O_2 = \{S_n^t\} \to O_3 = \{B_n^t\}\}$.

Let us put a general automaton

$$T_n^t = B_{n+1}^t - B_n^{t+1}. \tag{4.28}$$

Then, $O'_1 \equiv \{T_n^t\} \in \mathbf{GAut}$ is linearly invertible with $\{U_n^t\}$ by the relations

$$U_n^t = \Sigma_{i=-\infty}^n T_i^t \Longleftrightarrow T_n^t = U_n^t - U_{n-1}^t. \tag{4.29}$$

Thus one has obtained another path $\overline{O}' = \{O_0 \to O'_1 \to O_2 \to O_3\}$.

5. Actions on the Boundary of Trees

5.1. Actions on the Boundary and Compactification

For any graph with marking on each edge, we denote by $m(e)$ the marking at the edge e. A tree T is said to be *biinfinite* if, for any vertex $v \in T$, it contains a geodesic real line $v \in \mathbf{R} \subset T$.

Let T_2 be the marked biinfinite binary tree with marking $\{0,1\}$ so that it contains a base path $l_0 : \mathbf{R} \to T_2$ with $m(l_0(t)) = 0$, $t \in \mathbf{Z}$. Let $\mathbf{P} = \{\mathbf{R} \to T_2\}$ be all of the set of geodesics. Then, there is a canonical inclusion

$$\mathbf{P} \subset \Sigma_2, \tag{5.1}$$

where Σ_2 is the set of two-sided sequences with the alphabets $\{0,1\}$. Thus Σ_2 can be regarded as a compactification of the set of all geodesics in T_2.

Let us put

$$\mathbf{P}^0 = \left\{ l \in \mathbf{P} : m(l(t)) = 0 \ \forall \text{ sufficiently large } |t| \gg 0 \right\}. \tag{5.2}$$

Then, we have the following proper inclusions:

$$\mathbf{P}^0 \subset \mathbf{P} \subset \Sigma_2. \tag{5.3}$$

It is easy to see that $\mathbf{P}^0 \subset \Sigma_2$ is dense. In fact $\Sigma_2 \setminus \mathbf{P}^0$ consist of elements $l = (\dots, a_{-1}, a_0, a_1, \dots)$ such that the sets $\{i : m(a_i) = 1\}$ are unbounded at least for one direction.

Now we have a natural identification (Section 2.2)

$$\Sigma_2^0 \cong \mathbf{P}^0 \subset \Sigma_2. \tag{5.4}$$

Then, the BBS system is described by an isomorphism

$$\Phi : \mathbf{P}^0 \cong \mathbf{P}^0. \tag{5.5}$$

Notice that the only fixed point is $(\dots, 0, 0, \dots)$.

Proposition 5.1. *Φ can be naturally extended to the continuous map*

$$\Phi : \Sigma_2 \longrightarrow \Sigma_2, \tag{5.6}$$

where

$$\Phi((\dots, 1, 1, \dots)) = (\dots, 0, 0, \dots). \tag{5.7}$$

Thus the extended map is not an isomorphism. More generally one has noninjective points

$$\Phi((\dots, 0, 1, 0, \dots)) = (\dots, 0, 0, 1, \dots) = \Phi((\dots, 0, 1, 1, 0, 1, \dots)). \tag{5.8}$$

The extension also has one fixed point $(\dots, 0, \dots)$.

In Section 4.1, we have assigned the following isomorphic procedures:

$$O_0 = \text{BBS} \longrightarrow O_1 \longrightarrow O_2 \longrightarrow O_3 = \text{LV cell automaton}, \tag{5.9}$$

where we denote each step by $F_i : O_i \to O_{i+1}$, $i = 0, 1, 2$. Each O_i admits an $\mathbf{N}$ action by the flow. By the construction, we have that F_0 and F_1 both are equivariant with respect to the flow, but F_2 is not.

Let us put $\Sigma_\infty^0 \equiv \bigcup_{m=1}^\infty \Sigma_m^0$, and Σ_∞ is defined similarly. Then, by the above procedure $F : O_0 \to O_3$, we have an injection

$$F : \mathbf{P}^0 \hookrightarrow \Sigma_\infty^0 \tag{5.10}$$

which assigns the solution of the BBS to the one of LV.

Notice that F extends as $F : \mathbf{P} \to \Sigma_\infty$ with respect to the compactification of $\mathbf{P}^0$ defined above; on the other hand, it cannot be extended as $F : \Sigma_2 \to \Sigma_\infty$.

Lemma 5.2. *$F : \mathbf{P} \to \Sigma_\infty$ is not a surjection to the set of all solutions of the LV cell automaton.*

Proof. Recall that, in the transformations above, F_0 and $F_1 : O_1 \to O_2$ are equivariant, and F_2 is just change of indices. Then, the result follows since there are at least two fixed points for O_2 as

$$\{(\ldots, 0, 0, \ldots), (\ldots, 1, 1, \ldots)\}. \tag{5.11}$$

□

Question 1. Let us choose a large $N \gg 0$. Describe compactification of $F(\mathbf{P}^0) \cap \Sigma_N \subset \Sigma_N^0$.

5.2. Embedding of the Lamplighter Group

Let S and A be finite sets. An *automata group* G is an infinite group acting on the rooted tree T_m^*, $m = \#A$, which is determined by a transition function $\phi : S \times A \to S$ and an exit function $\psi : S \times A \to A$ such that $\psi(a) \in S_m$ is an element of the permutation group for each $a \in A$.

Let T_2 be as in Section 5.1. Recall that the lamplighter group is an automata group which acts on the rooted binary tree T_2^*, and it has a presentation (see [11])

$$G = \left\langle a, \gamma : \gamma^2 = 1, \left[\gamma^{a^i}, \gamma^{a^j}\right] = 1, \ i, j \in \mathbf{Z} \right\rangle. \tag{5.12}$$

The action of G preserves each level set of T_2^*, since it is constructed using an automaton. The automaton is given in Table 2:

$$\psi(a) = \epsilon, \qquad \psi(b) = \text{id}, \tag{5.13}$$

where $\epsilon \in S_2$ is the nontrivial permutation on $\{0, 1\}$. We call it the *lamplighter automaton*.

Table 2: Values of ϕ.

	a	b
0	a	a
1	b	b

Let us construct another group $\widetilde{G}$ which contains G and acts on T_2. Recall that T_2 contains a line $(\ldots,0,0,\ldots) \subset T_2$, and choose a base vertex $*$ on the line. By identifying $*$ with the root in T_2^*, one can embed T_2^* into T_2:

$$T_2^* \hookrightarrow T_2. \tag{5.14}$$

Now passing through this embedding, one can make G act on T_2 by letting the same action as the lamplighter group on $T_2^* \hookrightarrow T_2$ and by putting the identity on $T_2 \setminus T_2^*$.

Now Aut $T_2 \supset \widetilde{G} \supset G$ is generated by G and another element τ. Let us describe τ. Notice that each edge of T_2 is assigned by one of $\{0,1\}$. We say that an automorphism g on T_2 *preserves the marking* if $g(e)$ has the same marking as the one of e in $\{0,1\}$. Now τ is an automorphism preserving the marking and uniquely defined by the property that it shifts i-zero in $((\ldots,0,0,\ldots))$ to $i+1$-one. Thus τ preserves the line. Then, $\widetilde{G}$ is generated by a, γ, and τ.

The action of $\widetilde{G}$ on $\widetilde{T}_2$ has no fixed point. Moreover it is finitely generated and has quotient isomorphic to $\mathbf{Z}$. Thus the Bass-Serre theory suggests the following.

Question 2. Is $\widetilde{G}$ an amalgam? If so, write down explicitly G_1, G_2, and A with an isomorphism

$$\widetilde{G} \cong G_1 *_A G_2. \tag{5.15}$$

5.3. Actions on Tree by Cellular Automata

The action of the automata group on T_2^* is determined by the levelwise way. In fact it is an action on each vertex of T_2^*. This is not the case for general cellular automata, like LV cell automaton. In fact the action is determined for each path in T_2^*, rather than points in T_2^*. More precisely, for many automata including LV case, the image of a vertex v by an element $g \in G$ is determined by a *neighbourhood* of v.

5.3.1. LV Cell Automaton as a Transition Function

Let us consider a cell automaton

$$\Phi : \mathbf{N} \times \mathbf{N}^2 \longrightarrow \mathbf{N}, \tag{5.16}$$

where each U_n^t is determined inductively by $U_n^{t+1} = \Phi(U_{n-1}^{t+1}, U_n^t, U_{n+1}^t)$.

We say that $\{U_n^t\}$ *degenerates* with respect to Φ if there is another function G so that it satisfies the following relation:

$$U_n^{t+1} = \Phi\left(G\left(U_n^t, U_{n+1}^t\right), U_n^t, U_{n+1}^t\right). \tag{5.17}$$

Recall that the lamplighter automaton has two states $\{a, b\}$ and the alphabets $\{0, 1\}$. In order to compare the LV cell automaton with the transition function of it, we regard the flow $\{U_t^0\}_t \to \{U_t^1\}_t$ as an output of a transition function. By this way let us regard that the initial flow $(U_0^0, U_1^0, \ldots)$, $U_i^0 \in \{0, 1\}$ is a sequence of the alphabets and $(U_0^1, U_1^1, \ldots)$ is another sequence of the states, where one identifies a and b with 0 and 1, respectively.

Lemma 5.3. *Suppose an initial sequence $(a_0, a_1, \ldots)$ consists of only $\{0, 1\}$ entries.*

Then, the degeneration of the LV cell automaton by $G(x, y) = x$ is the same as the transition function ϕ of the lamplighter automaton.

When one expresses the flow of Φ by an automaton, one puts an exit function by:

$$\psi(a, i) = 0, \quad \psi(b, i) = 1, \quad i = 0, 1. \tag{5.18}$$

In particular, the corresponding continuous map on the rooted tree is not an automorphism (it is not one to one).

In general flows of the LV cell automaton take integer elements $U_n^t \in \mathbf{N}$. In order to treat these cases, let us put a sequence of states

$$(a_0, a_1, \ldots), \tag{5.19}$$

where $a_0 = a$ and $a_1 = b$. Let S_∞ be the group of compactly supported permutation on $\{0, 1, \ldots\}$. Then, consider an exit function

$$\psi : \mathbf{N} \longrightarrow S_\infty \tag{5.20}$$

with $\psi(0) = \epsilon \in S_2 \subset S_\infty$ and $\psi(1) = \mathrm{id}$.

In general let $\Phi : \mathbf{N} \times \mathbf{N}^2 \to \mathbf{N}$ be a transition function, and choose an initial sequence $(i_0, i_1, \ldots) \subset \mathbf{N}$ and any $U_0^1 \in \mathbf{N}$.

Now we define the *generalized automaton* (Φ, ψ) as follows. As an output, we will exit another sequence $(i_0^1, i_1^1, \ldots) \subset \mathbf{N}$ as follows.

Firstly determine the sequence of the states $(U_0^1, U_1^1, \ldots)$ inductively by Φ:

$$U_n^1 = \Phi\left(U_{n-1}^1, i_n^t, i_{n+1}^t\right). \tag{5.21}$$

Then, inductively we obtain the exits by

$$i_n^1 = \psi\left(U_n^1\right)\left(i_n^0\right). \tag{5.22}$$

By this way one obtains an assignment which is in fact an isomorphism:

$$g_{U_0^1} : X_\infty \cong X_\infty, \tag{5.23}$$

where X_N are the sets of one-sided sequences with N alphabets and X_∞ is their union. The group $G(\Phi, \psi)$ generated by g_m, $m = 0, 1, \ldots$, is also called the *generalized automata group* given by (Φ, ψ).

In the case when Φ_0 is the LV cell automaton, the equation of the transition function becomes

$$U_{t+1}^1 = i_n^t + \max\left(L, i_{n+1}^t\right) - \max\left(L, U_{n-1}^{t+1}\right). \tag{5.24}$$

Definition 5.4. The LV cell automata group is a group acting on the boundary $\widetilde{\partial} T_\infty^* \equiv \bigcup_i \partial T_i^*$ of the rooted infinite tree defined by the transition function Φ_0 and an exit function ψ as above.

A generalized automata group is of *bounded type* by N if it induces an action between $X_N = \partial T_N^*$.

5.4. Quasi Actions on Trees

Let G be a group acting on the boundary of the rooted tree T_2^*.

We say that an element $\gamma \in G$ is a k_0-*quasi action* on T_2^* if there is some $k_0 \geq 0$ such that, if we write $\gamma(a_0, a_1, \ldots) = (a_0', a_1', \ldots)$, then a_M' is determined by the data $(a_0, a_1, \ldots, a_{M+k_0})$ for each $M = 0, 1, \ldots$. We denote

$$\gamma(a_0, a_1, \ldots)_M = \left(a_0', \ldots, a_M'\right) \in F_2, \tag{5.25}$$

where F_2 is the free group generated by $\{\alpha, \beta\}$. We denote the word length by $|g|$.

Suppose that γ is a 1-quasi action. Then, we have a map

$$\begin{gathered} F_\gamma : F_2 \longrightarrow F_2, \\ F_\gamma(g) = \left(\gamma(g\alpha)_{|g|}\right)^{-1} \gamma(g\beta)_{|g|} \in F_2, \end{gathered} \tag{5.26}$$

which we call the *differential* of γ. Notice that, when γ is an automorphism on the tree, then F is the identity map.

We say that a quasi action by γ is (k_0, l_0)-*semi Markov* if there is some $l_0 \geq 0$ such that a_M' is determined by $(a_{M-l_0}, \ldots, a_{M+k_0})$ for each $M = 0, 1, \ldots$.

Let γ has a 1-quasi action on T_2^*. We say that it is *bounded* if there is a bounded function $B : F_2 \to F_2$ so that the differential satisfies the following equality:

$$F_\gamma(g) = B(g\alpha)^{-1} B(g\beta). \tag{5.27}$$

Lemma 5.5. *A bounded quasi action γ is semi-Markov.*

Proof. In fact one has the equality

$$\gamma(g\alpha)_{|g|}B(g\alpha)^{-1} = \gamma(g\beta)_{|g|}B(g\beta)^{-1}. \tag{5.28}$$

This implies the equality

$$\gamma(g\alpha)_{|g|-N} = \gamma(g\beta)_{|g|-N}, \tag{5.29}$$

where $N = \sup\{|B(g)| : g \in F_2\}$. This completes the proof. □

6. Associated Algebraic Varieties

6.1. Tropical Algebra

Maslov introduced a kind of scale transform called the *dequantization* of the real line **R**. It is given by deformations of the arithmetics over the real number **R**, which are parameterized by a family of semirings R_t for $t > 1$.

The multiplications and the additions are, respectively, given by

$$x \bigoplus_t y = \log_t(t^x + t^y), \qquad x \bigotimes_t y = x + y. \tag{6.1}$$

Notice the following particular property:

$$x \bigoplus_\infty y \equiv \lim_{t\to\infty} x \bigoplus_t y = \max\{x, y\}. \tag{6.2}$$

This is called the *tropical semiring*.

Corresponding to polynomials over the usual real numbers, one obtains R_t*-polynomials* as

$$\varphi_t(x) = \left(\alpha_1 + j^1 x\right) \bigoplus_t \cdots \bigoplus_t \left(\alpha_k + j^k x\right) \quad \left(x \in \mathbf{R}^n,\ j^l \in \mathbf{Z}^n\right) \tag{6.3}$$

whose limit $t \to \infty$ satisfies the following max-plus equation:

$$\varphi_\infty(x) = \max\left(\alpha_1 + j^1 x, \ldots, \alpha_k + j^k x\right). \tag{6.4}$$

Let $\mathrm{Log}_t : (\mathbf{C}^*)^n \to \mathbf{R}^n$ be defined as

$$\mathrm{Log}_t(z_1, \ldots, z_n) \equiv \left(\log|z_1|, \ldots, \log|z_n|\right). \tag{6.5}$$

The following can be checked easily by direct calculations, but the conclusions are highly nontrivial and interesting.

Proposition 6.1 (see [4, 12]). *$f_t \equiv (\log_t)^{-1} \circ \varphi_t \circ \mathrm{Log}_t : \mathbf{R}_+^n \to \mathbf{R}_+$ is in fact a polynomial map $f_t(z) = \Sigma_j t^{\alpha_j} z^j$.*

Thus R_t-polynomials are conjugate to the standard real polynomials by $\log_t$.

Conversely let $\varphi : \mathbf{R}_+^n \to \mathbf{R}_+$ be a piecewisely linear map equipped with the presentation by a max-plus equation. Then, one can associate a parameterized R_t and $\mathbf{R}$ polynomials, respectively, since their presentations are determined by the coefficients $\alpha_1, \dots, \alpha_k, j^1, \dots, j^k$. Later on, we will denote them, respectively, by φ_t and f_t and call the *associated R_t and $\mathbf{R}$ polynomials* with respect to φ.

6.2. Tropical Maps in LV Cell Automaton

It is immediate to generalize Section 6.1 to the relative case [16].

Let $F_t : \mathbf{C}^N \to \mathbf{C}$ be a family of rational functions given by

$$F(z_1, \dots, z_N) = \frac{\Sigma_l t^{a_l} z^{j^l}}{\Sigma_m t^{b_m} z^{j^m}}, \tag{6.6}$$

where $j = (j_1, \dots, j_N) \in \mathbf{Z}^N$ and $z = (z_1, \dots, z_N) \in \mathbf{C}^N$. Then, we define the corresponding *tropical polynomial* $f = f_{\mathrm{tr}}$ as a piecewise linear map on $\mathbf{R}^N$ by

$$f_{\mathrm{tr}}(x_1, \dots, x_N) = \max_l \left\{ a_l + j^l x \right\} - \max_m \{ b_m + j^m x \}. \tag{6.7}$$

where $x = (x_1, \dots, x_N) \in \mathbf{R}^N$. Such functions are called the *relative* $(\max, +)$ *functions*.

Let us recall the Lotka-Volterra cell automaton in Section 4. We describe it as flows of the generalized interactions in Section 2.2.

Proposition 6.2 (see [3]). *There is a family of smooth maps $\{f_{i,j}\}_{i,j=0,\dots,L}$ so that the corresponding flow $\Phi(x)^t$, $t = 0, 1, \dots$, determined by $\{f_{i,j}\}_{i,j}$ gives the solutions of the Lotka-Volterra cell automaton.*

Proof. Let us construct the map

$$\begin{gathered} \tilde{\Phi} : \{0, \dots, L\} \times \{0, \dots, L\}^2 \longrightarrow \{0, \dots, L\}, \\ \tilde{\Phi}\left(k_{n-1}^t, k_n^{t-1}, k_{n+1}^{t-1}\right) = k_n^t \in \{0, , \dots, L\}. \end{gathered} \tag{6.8}$$

Let us put a piecewise linear map f_{LV} by

$$\begin{aligned} f_{\mathrm{LV}}(x_1, x_2, x_3) &= x_1 + \max\{L_0, x_2\} - \max\{L_0, x_3\} \\ &= x_1 + \max\{0, x_2 - L_0\} - \max\{0, x_3 - L_0\}. \end{aligned} \tag{6.9}$$

Then, we renormalize them and put $f_{i,j} : [0,1] \to [0,1]$ by

$$f_{i,j}(x) = \frac{1}{L} f_{\mathrm{LV}}(i, j, Lx) \in [0,1]. \tag{6.10}$$

Let us divide the interval $[0,1]$ into $L+1$ intervals, and denote them as $I_0 < I_1 < \cdots < I_L$. Then, $f_{i,j}(I_l) \subset I_{\tilde{\Phi}(l,i,j)}$ are satisfied for all $i, j, l \in \{0, 1, \ldots, L\}$. This is the desired family of maps.

This completes the proof. □

6.3. Associated Hypersurfaces

In order to avoid minus sign in the defining equations, one can transpose some terms there. Then, the equation can be expressible by max-plus equations in both sides. So when a bounded automaton is determined by a relative $(\max, +)$ function φ, then it associates the pair of $(\max, +)$ functions (φ_1, φ_2).

For example, the LV-CA can be immediately rewritten as

$$V_{n+1}^{t+1} + \max\left\{L, V_n^{t+1}\right\} = V_{n+1}^{t} + \max\left\{L, V_{n+2}^{t}\right\}. \tag{6.11}$$

Let us consider a bounded automaton defined by a relative $(\max, +)$ function:

$$V_{n+1}^{t+1} = \varphi\left(V_{n-k}^{t-l}, \ldots, V_n^{t+1}\right). \tag{6.12}$$

Let φ_1 and φ_2 be two max-plus equations with respect to φ. Then, the above equation can be rewritten as

$$\varphi_1\left(V_{n-k}^{t-l}, \ldots, V_{n+1}^{t+1}\right) = \varphi_2\left(V_{n-k}^{t-l}, \ldots, V_n^{t+1}\right), \quad n = 0, 1, \ldots, \tag{6.13}$$

for some k and l. Notice that φ_1 can be rewritten as

$$\varphi_1\left(V_{n-k}^{t-l}, \ldots, V_{n+1}^{t+1}\right) = V_{n+1}^{t+1} + \tilde{\varphi}_1\left(V_{n-k}^{t-l}, \ldots, V_n^{t+1}\right) \tag{6.14}$$

for another some $(\max, +)$ function $\tilde{\varphi}_1$, and they satisfy the relation

$$\begin{aligned}\varphi\left(V_{n-k}^{t-l}, \ldots, V_n^{t+1}\right) &= \varphi_2\left(V_{n-k}^{t-l}, \ldots, V_n^{t+1}\right) - \tilde{\varphi}_1\left(V_{n-k}^{t-l}, \ldots, V_n^{t+1}\right) \\ &= \varphi_2\left(V_{n-k}^{t-l}, \ldots, V_n^{t+1}\right) - \varphi_1\left(V_{n-k}^{t-l}, \ldots, V_{n+1}^{t+1}\right) + V_{n+1}^{t+1}.\end{aligned} \tag{6.15}$$

One may assume that both $\tilde{\varphi}_1$ and φ_2 have the same N variables.

Let f_t^1 and f_t^2 be the associated $\mathbf{R}$ polynomials and assign the complex variables $\mathbf{z} = (z_1, z_2, \ldots, z_{N+1})$ with respect to $(V_{n-k}^{t-l}, \ldots, V_{n+1}^{t+1})$.

Definition 6.3. The associated affine hypersurfaces are a parameterized family of hypersurfaces given by the following equations:

$$V(A)_t = \left\{ \mathbf{z} \in \mathbf{C}^{N+1} : f_t^1(\mathbf{z}) = f_t^2(\mathbf{z}) \right\}. \tag{6.16}$$

Example 6.4. Let us consider the LV cell automaton. Let us put a parameterized polynomial of degree 2 by $f_t(z,w) = t^L z + zw$. Then, the corresponding equation becomes $f_t(z_4, z_3) = f_t(z_1, z_2)$, where each z_i corresponds to $z_1 \leftrightarrow V_{n+1}^t$, $z_2 \leftrightarrow V_{n+2}^t$, $z_3 \leftrightarrow V_n^{t+1}$, and $z_4 \leftrightarrow V_{n+1}^{t+1}$. Thus the associated hypersurfaces $V(\mathrm{LV})_t \subset \mathbf{C}^4$ are a family defined by

$$V(\mathrm{LV})_t = \left\{ (z_1, z_2, z_3, z_4) \in \mathbf{C}^4 : f_t(z_4, z_3) = f_t(z_1, z_2) \right\}. \tag{6.17}$$

6.3.1. *The Associated Complex Dynamics*

Let us consider a bounded automaton A given by a relative $(\max, +)$ function φ. As above, it is given by a pair of $(\max, +)$ functions (φ_1, φ_2). Let f_t and (f_t^1, f_t^2) be the associated rational and the pair of polynomial functions with respect to φ and (φ_1, φ_2) mutually. Notice that they are related in the following way:

$$f_t(z_1, z_2, \ldots, z_N) = \frac{z_{N+1} f_t^2(z_1, z_2, \ldots, z_N)}{f_t^1(z_1, z_2, \ldots, z_N, z_{N+1})}. \tag{6.18}$$

For the LV cell automaton, f_t above is given by

$$f_t(z_1, z_2, z_3) = \left(t^L + z_3\right)^{-1} \left(t^L z_1 + z_1 z_2\right). \tag{6.19}$$

Now for $t = 0, 1, \ldots$, let us consider a flow given by infinite sequences of complex numbers $(z_1^t, z_2^t, z_3^t, \ldots) \in \mathbf{C}^\infty$ which obey the equation

$$z_{n+1}^{t+1} = f_t\left(z_{n-k}^{t-l}, \ldots, z_n^{t+1}\right). \tag{6.20}$$

Notice that, if $(z_{n-k}^{t-l}, \ldots, z_n^{t+1}) \in \mathbf{R}_{>0}^N$, then $z_{n+1}^{t+1} \in (0, \infty)$.

Since it also satisfies the equation

$$f_t^1\left(z_{n-k}^{t-l}, \ldots, z_{n+1}^{t+1}\right) = f_t^2\left(z_{n-k}^{t-l}, \ldots, z_n^{t+1}\right), \tag{6.21}$$

each building block $\mathbf{z}_n^t = (z_{n-k}^{t-l}, \ldots, z_n^{t+1})$ lies on $V(A)_t$ for all $n \geq k$ and $t \geq l$. We say that the sequences

$$\left\{ \mathbf{z}_n^t \right\}_{n \geq k,\, t \geq l} \subset V(A)_t \tag{6.22}$$

are the *induced dynamics* on $V(A)_t$.

6.3.2. Stability

So far, we have discussed by fixing the parameter t. Here we consider the asymptotics of flows as $t \to \infty$.

Let (φ_1, φ_2) be a bounded automaton given by two max-plus functions with the $N+1$ variables $(V_{n-k}^{t-l}, \ldots, V_{n+1}^{t+1})$ as above. Let us denote the associated polynomials φ_t^i and f_t^i, $i = 1, 2$.

Lemma 6.5. *Let $z^t \in \mathbf{R}_{>0}^{N+1}$ be a family of points. Suppose that the convergent condition*

$$\mathbf{x} \equiv \lim_{t\to\infty} \operatorname{Log}_t(\mathbf{z}^t) \tag{6.23}$$

is satisfied. Then, $x \in \mathbf{R}^{N+1}$ satisfies the following equation:

$$\varphi_1(\mathbf{x}) = \varphi_2(\mathbf{x}). \tag{6.24}$$

Proof. We recall the equality $f_t = \log_t^{-1} \circ \varphi_t \circ \operatorname{Log}_t$ on the positive real numbers. Thus they satisfy the following equation:

$$\varphi_t^1 \circ \operatorname{Log}_t(\mathbf{z}^t) = \varphi_t^2 \circ \operatorname{Log}_t(\mathbf{z}^t). \tag{6.25}$$

Then the conclusion holds, since φ_t^i converge to the original max-plus functions φ^i. This completes the proof. □

Let $[v] \in \mathbf{Z}$ be the integer part of $v \in \mathbf{R}$. For a sequence $\mathbf{v} = (v_1, v_2, \ldots)$, we denote by $[\mathbf{v}] \equiv ([v_1], [v_2], \ldots)$.

Definition 6.6. Let A be a bounded automaton, and choose parameterized families of sequences $\{\mathbf{z}_n^t\}_{n\geq k,\, t\geq l} \subset \mathbf{C}^{N+1}$.

We say that they are stable, if there is t_0 so that, for all $t \geq t_0$,

$$\{[\operatorname{Log}_t(\mathbf{z}_n^t)]\}_{n\geq k,\, t\geq l} \subset \mathbf{Z}^{N+1} \tag{6.26}$$

gives a flow of the solutions of the original automaton A.

Let A be the LV cell automaton. Then, as in Section 6.2, there is a family of interval maps $\{f_{i,j}\}_{i,j}$ so that the flow of the solutions of LV-CA can be represented by interactions of maps between $\{f_{i,j}\}$. Thus existence of stable families above will be heavily influenced by stability of dynamical properties of the family $\{f_{i,j}\}$ under continuous deformations of these maps.

6.4. Cell Automatic Varieties

Let A_1 and A_2 be two cell automata, and consider an integer-valued flow

$$\{\mathbf{V}^t = (V_1^t, V_2^t, \ldots)\}_{t\geq 0} \subset \mathbf{Z}^\infty. \tag{6.27}$$

If it satisfies both equations for A_1 and A_2, then we say that $\{\mathbf{V}^t\}_t$ is a *flow* for A_1 and A_2.

Let us denote the set of such flows by

$$\mathbf{F}(A_1, A_2) = \left\{ \{\mathbf{V}^t\}_{t=0,1,2,\ldots} \subset \mathbf{Z}^\infty : \text{flows for both } A_1, A_2 \right\}. \tag{6.28}$$

We say that $\mathbf{F}(A_1, A_2)$ is a *cell automatic variety* for A_1 and A_2.

Let $A_1, \ldots, A_m$ be a family of cell automata. By considering flows for them, one also obtains the cell automatic variety $\mathbf{F}(A_1, \ldots, A_m)$.

6.4.1. Compatible Automata

Let A_1 and A_2 be two cell automata given by φ and φ', respectively:

$$V_{n+1}^{t+1} = \varphi(')\left(\{V_{n-k}^t, \ldots, V_{n+l}^t\}, \left\{V_{n-k'}^{t+1}, \ldots, V_n^{t+1}\right\}\right) \quad [*]('). \tag{6.29}$$

Let us denote the following sets:

$$\mathbf{V}_n^t \equiv \left(V_{n-k}^t, \ldots, V_{n+l}^t, V_{n-k'}^{t+1}, \ldots, V_n^{t+1}\right) \in \mathbf{Z}^N. \tag{6.30}$$

We say that A_2 is *compatible* with A_1 if the following holds; suppose that V_n^t satisfies both $[*]$ and $[*]'$. Let V_{n+1}^t be determined by $[*]'$. Then, it also satisfies $[*]$.

In the case of compatible automata (A_1, A_2), the cell automatic variety $\mathbf{F}(A_1, A_2)$ will be nonempty, which does not hold in general.

Example 6.7. Let A_1 and A_2 be both automata given by linear maps as follows:

$$\begin{aligned} u_{n+1} &= \alpha u_n + \beta u_{n-1} + \gamma, \quad (A_1), \\ u_{n-1} &= a u_n + b, \quad (A_2). \end{aligned} \tag{6.31}$$

If these coefficients satisfy the following relations:

$$(1 - \beta)a = \alpha, \qquad (1 - \beta)b = \gamma, \tag{6.32}$$

then A_2 is compatible with A_1.

6.4.2. Associated Varieties

Let A_1 and A_2 be two CA. Let us denote the associated $\mathbf{R}$-polynomials by f_t^i and g_t^i for $i = 1, 2$, respectively. Then, one obtains the *parameterized affine algebraic varieties* given by the following equations:

$$V(A_1, A_2)_t = \left\{ \mathbf{z} \in \mathbf{C}^N : f_t^1(\mathbf{z}) = f_t^2(\mathbf{z}), g_t^1(\mathbf{z}) = g_t^2(\mathbf{z}) \right\}. \tag{6.33}$$

Let us consider the associated systems of complex dynamics. Let $(z_1^t, z_2^t, z_3^t, \ldots) \in \mathbf{C}^\infty$ which obey the following system of the equations:

$$f_t^1\left(z_{n-k}^{t-l}, \ldots, z_{n+1}^{t+1}\right) = f_t^2\left(z_{n-k}^{t-l}, \ldots, z_n^{t+1}\right),$$
$$g_t^1\left(z_{n-k}^{t-l}, \ldots, z_{n+1}^{t+1}\right) = g_t^2\left(z_{n-k}^{t-l}, \ldots, z_n^{t+1}\right). \tag{6.34}$$

Again each building block $\mathbf{z}_n^t = (z_{n-k}^{t-l}, \ldots, z_n^{t+1})$ lies on $V(A_1, A_2)_t$, and we say that $\{\mathbf{z}_n^t\}_{n,t}$ is the *induced dynamics* on $V(A_1, A_2)_t$.

Let $A_1, \ldots, A_m$ be a family of CA. Then, by the same way as above, one obtains the following parameterized associated affine algebraic variety:

$$V(A_1, \ldots, A_m)_t = \left\{\mathbf{z} \in \mathbf{C}^N : \left(f_t^k\right)^1(\mathbf{z}) = \left(f_t^k\right)^2(\mathbf{z}),\ k = 1, \ldots, m\right\}, \tag{6.35}$$

where $(f_t^k)^i$, $i = 1, 2$, are the associated $\mathbf{R}$-polynomials with respect to A_k.

6.5. Duality

Here we introduce a new duality on the set of automata. It passes through the *projective duality* between algebraic varieties [17].

Let V be a complex n-dimensional vector space and let $P(V)$ be its projective space. There is a natural isomorphism $P(V) \cong P^*(V^*)$, where $P^*(W)$ is the set of all hyperplanes in W and V^* is the dual space to V.

Projective duality generalizes this operation to varieties. Let us quickly describe its construction. Let $X \subset P(V)$ be an algebraic variety. Then, one can associate another variety $X^\vee \subset P(V^*)$ as follows. A hyperplane $H \subset P(V)$ is said to be tangent to X if there exists a smooth point $x \in H \cap X$ and the tangent space of X at x is contained in H. Let $X^* \subset P^*(V)$ be all of the set of tangent hyperplanes, and passing through the above isomorphism, one obtains a set $X^\vee \subset P(V^*)$ which is the desired one. It is called the *projective dual variety*. In the case when $X^\vee \subset P(V^*)$ is a hypersurface, then its defining polynomial Δ_X is called the *X-discriminant*.

Let $\{A_1, \ldots, A_m\}$ be a family of automata. Thus one obtains a parameterized family of projective varieties $V(\{A_i\})_t \subset \mathbf{CP}^N$ by taking closure of the associated affine varieties.

Let $\widetilde{V}(\{A_i\})_t^\vee \subset \mathbf{CP}^N$ be the corresponding parameterized projective dual varieties.

Suppose that these are hypersurfaces, and denote the defining functions by $\Sigma_j t^{\alpha_j} a^j z^j$. When one can modify the polynomial as $\Sigma_j t^{\alpha_j} w^j$ by change of variables $a_l z_l = w_l$, then we call $\Sigma_j t^{\alpha_j} w^j$ the *$\{A_i\}$-discriminant* and denote it by

$$\Delta(\{A_i\})_t. \tag{6.36}$$

We will denote the corresponding modified varieties by

$$V(\{A_i\})_t^\vee \tag{6.37}$$

and call them as the *associated dual varieties* with respect to $\{A_i\}_i$. They are isomorphic with $\widetilde{V}(\{A_i\})_t^\vee$.

Definition 6.8. Let φ_t be the associated R_t-polynomials to $\Delta(\{A_i\})_t)$. The automaton defined by $\varphi_\infty = \lim_{t\to\infty}\varphi_t$ is called the projectively dual automaton.

We denote the projectively dual automaton by

$$\{A_i\}^\vee. \tag{6.38}$$

6.5.1. Curves in $\mathbf{CP}^2$

In general it is not easy to calculate the defining equations of the projective varieties. However as far as the simplest case is that of the curves in $\mathbf{CP}^2$, one can write down them explicitly. Using this fact, we verify the following.

Proposition 6.9. *Consider*

$$\begin{aligned}&[\max\{au_n, \alpha + au_{n+1}\} = c]^\vee \\ &\quad = \max\left\{\frac{a}{a-1}\left(c - \frac{\alpha}{a}\right) + \frac{a}{a-1}u_{n+1}, \frac{ac}{a-1} + \frac{a}{a-1}u_n\right\} = c.\end{aligned} \tag{6.39}$$

Proof. Let $X \subset \mathbf{CP}^2$ be an irreducible curve. Then, it is known that $X^\vee \subset \mathbf{CP}^2$ is also another irreducible one. In the affine coordinate, if X has a parameterization $x = x(s)$ and $y = y(s)$, $s \in C$, then $X^\vee$ has a parameterization given by the following (see [17]):

$$p(s) = \frac{-y'(s)}{x'(s)y(s) - x(s)y'(s)}, \qquad q(s) = \frac{x'(s)}{x'(s)y(s) - x(s)y'(s)}. \tag{6.40}$$

Using this, let us consider the very simple case above. Let $a \geq 2$, let α and c be integers, and consider an automaton A given by

$$\max\{au_n, \alpha + au_{n+1}\} = c. \tag{6.41}$$

The associated polynomial and the associated varieties are given by

$$X = \left\{(x,y) \in \mathbf{C}^2 \subset \mathbf{CP}^2 : x^a + t^\alpha y^a = t^c\right\}. \tag{6.42}$$

Choosing a parameterization as

$$x = s, \qquad y = t^{-\alpha/a}(t^c - s^a)^{1/a}, \tag{6.43}$$

one can immediately obtain the parameterization of $X^\vee$ as

$$t^{(a/(a-1))(c-\alpha/a)}q^{a/(a-1)} + t^{ac/(a-1)}p^{a/(a-1)} = t^c \quad (*) \tag{6.44}$$

which gives the dual varieties

$$V(A)_t^\vee = \left\{(p,q) \in \mathbf{C}^2 \subset \mathbf{CP}^2 : (*)\right\}. \tag{6.45}$$

Thus the projectively dual automaton admits the desired presentation. This completes the proof. □

6.6. Transformations

Let us take two automata O and O', and let $\overline{O} = \{O = O_0 \rightarrow O_1 \rightarrow \cdots O_m = O'\} \in \Omega(O,O')$ be an invertible path, as in Section 4.3, whose associated $\mathbf{R}$-polynomials are given as the pairs $(f_t^1, f_t^2)_i$, $i = 0,\ldots,m$. Then, correspondingly the associated hypersurfaces $\{V(O_i)\}_{i=0,\ldots,m} \subset \mathbf{C}^N$ are obtained. We call this a *lifting* of $\overline{O} \in \Omega_O$.

Question 3. What are geometric structure of such liftings?

Recall that we have obtained other cell automata O_1 and O_2 during the process of transforming from BBS to LV; see Section 4.1. Similarly, as above we rewrite these as

$$\begin{aligned} U_{n+1}^{t+1} + \max\{1, U_{n+1}^t\} &= U_n^t + \max\left\{1, U_n^{t+1}\right\}, \\ S_{n+1}^{t+1} + \max\left\{S_n^{t+1} + 1, S_{n+1}^t\right\} &= S_n^t + S_n^{t+1} + 1. \end{aligned} \tag{6.46}$$

Let us assign the variables as

$$z_1 \longleftrightarrow X_n^t, \qquad z_2 \longleftrightarrow X_{n+1}^t, \qquad z_3 \longleftrightarrow X_n^{t+1}, \qquad z_4 \longleftrightarrow X_{n+1}^{t+1} \tag{6.47}$$

for both $X = S$ and $X = U$. Then, we have the corresponding polynomial pairs as

$$\begin{aligned} &f_t^1(z_2, z_4) = z_4(t + z_2), \quad f_t^2(z_1, z_3) = z_1(t + z_3), \quad \text{for } U, \\ &f_t^1(z_2, z_3, z_4) = z_4(tz_3 + z_2), \quad f_t^2(z_1, z_3) = tz_3 z_1, \quad \text{for } S. \end{aligned} \tag{6.48}$$

Thus one has obtained three associated hypersurfaces as follows:

$$V(\mathrm{LV}), V(U), V(S) \subset \mathbf{C}^4. \tag{6.49}$$

Below we study one aspect of Question 3 for this case.

6.6.1. Representatives by Homogeneous Maps

Let us consider a bounded automaton A given by a relative $(\max,+)$ function φ. As in Section 6.3.1, it is given by a pair of $(\max,+)$ functions (φ_1, φ_2). Let (f_t^1, f_t^2) be the associated polynomials with respect to (φ_1, φ_2).

If both f_t^1 and f_t^2 are homogeneous polynomials of the same degree, then we say that A is a *homogeneous* cell automaton.

The LV-CA is not homogeneous; see Section 6.3. However one can deform it so that the result becomes homogeneous as below.

Let O be a cell automaton. We say that O admits a *homogeneous representative* if there is an invertible transformation $O = O_0 \to O_1 \to \cdots \to O_k$ so that O_k is homogeneous.

Lemma 6.10. *LV cell automaton admits a homogeneous representative.*

Proof. Let $O_0 = \mathrm{LV} \to O_1 \to O_2 \to O_3 = \mathrm{BBS}$ be the invertible path in Lemma 4.1. Then, certainly O_2 is homogeneous as above.

This completes the proof. □

Let O be a homogeneous cell automaton. Then, one obtains the *associated projective hypersurface*

$$V(O) \subset \mathbf{CP}^{N-1}. \tag{6.50}$$

We say that $V(O)$ is the *associated projective hypersurface*.

7. Interaction Graphs

Let $f, g : [0,1] \to [0,1]$ be two interval maps and let $\Phi(x, f, g) : X_2 \to X_2$ be the interaction map. Recall that $\pi : [0,1] \setminus 1/2 \to \{0,1\}$ is the projection. Let us choose another map $d : [0,1] \to [0,1]$.

Suppose that, for a point $z \in [0,1]$ and some $\overline{k} \in X_2$, the following equality holds:

$$\Phi(x, f, g)\left(\overline{k}\right) = \pi\left(\left(d(z), d^2(z), \ldots\right)\right) \equiv \left(\pi(d(z)), \pi\left(d^2(z)\right), \ldots\right). \tag{7.1}$$

Then, we express this by a marked oriented edge as

$$(f, x) \xrightarrow{(g,\overline{k})} (d, z). \tag{7.2}$$

Let us choose families of maps $\{f_0, \ldots, f_k\}$ and points $\{x_0, \ldots, x_l\}$. For each $(i, j, x) \in \{0, \ldots, k\}^2 \times \{x_0, \ldots, x_l\}$, let us assign an element $\overline{k}(i, j, x) \in X_2$. Thus we obtain another family $\{\overline{k}(i, j, x_h)\}_{i,j,h=0}^{i,j=k,h=l} \subset X_2$. Then, we put the following two sets:

$$\begin{aligned} V &= \left\{\left(f_i, x_j\right) : 0 \le i \le k, 0 \le j \le l\right\} \quad \text{(the set of vertices)}, \\ E &= \left\{e_{i,j,k} : \left(f_i, x_h\right) \xrightarrow{\left(f_j, \overline{k}\left(i,j,x_h\right)\right)} \left(f_k, x_v\right) :\right\} \quad \text{(the set of edges)}. \end{aligned} \tag{7.3}$$

Definition 7.1. An interaction graph is a marked oriented graph, where the sets of vertices V and edges E are given as above. We denote it by

$$G\left(\{f_i\}_i^k; \{x_j\}_j^l; \left\{\overline{k}(i,j,x_h)\right\}_{i,j,h=0}^{i,j=k,h=l}\right). \tag{7.4}$$

We will denote the set of interaction graphs arising from $\{f_0,\ldots,f_k\}$ and $\{x_0,\ldots,x_l\}$ by

$$\mathfrak{G}\left(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^l\right). \tag{7.5}$$

Notice that this is a finite set.

Let us put

$$X_2^{k,l} \equiv X_2^{k^2+l} = X_2 \times X_2 \times \cdots \times X_2. \tag{7.6}$$

Then, any element in $X_2^{k,l}$ can be written as $\overline{k}(i,j,x)$ as above. Then, the family of the interaction map gives a map

$$\Phi : X_2^{k,l} \longrightarrow X_2^{k,l}, \tag{7.7}$$

where

$$\begin{gathered}\Phi\left(\left\{\overline{k}(i,j,x)\right\}\right) = \left\{\overline{k}'(i,j,x)\right\},\\ \overline{k}'(i,j,x) \equiv \Phi(f_i,f_j,x)\left(\overline{k}(i,j,x)\right).\end{gathered} \tag{7.8}$$

This induces a map on the set of the interaction graph as

$$\Phi_* : \mathfrak{G}\left(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^l\right) \longrightarrow \mathfrak{G}\left(\{f_i\}_{i=0}^k; \{x_j\}_{j=0}^l\right) \tag{7.9}$$

by

$$\Phi_*\left(G\left(\{f_i\}_i^k; \{x_j\}_j^l; \left\{\overline{k}(i,j,x_h)\right\}\right)\right) = G\left(\{f_i\}_i^k; \{x_j\}_j^l; \Phi\left(\left\{\overline{k}(i,j,x_h)\right\}\right)\right). \tag{7.10}$$

Thus one obtains a sequence of the interaction graphs as follows:

$$\begin{gathered}(G_0, G_1, \ldots),\\ G_i = G\left(\{f_i\}_i^k; \{x_j\}_j^l; \Phi^i\left(\left\{\overline{k}(i,j,x_h)\right\}_{i,j,h=0}^{i,j=k,h=l}\right)\right).\end{gathered} \tag{7.11}$$

This gives a dynamics of the interaction graphs. Below we will formulate several geometric spaces arising from dynamics of the interaction graphs which we call *the spaces from the interaction graphs*.

7.1. Veronese Map

Here we have an easy example of spaces from the interaction graphs. Let $(\{f_i\}, \{x_j\}, \{\overline{k}(i,j,x_h)\})$ be an interaction system, and denote the corresponding interaction graphs by $(G_0, G_1, \ldots)$. Passing through the forgetful map, one obtains a sequence of finite graphs $(G'_0, G'_1, \ldots)$.

For each vertex $v \in G'_i$, let $e(v)$ be the number of the edges with a common vertex v. Let us fix $m \geq 0$ and put

$$P(m,G) = \left\{(i_0,\ldots,i_k) : \Sigma_{a=0}^{k} e(i_a) = m,\ k \leq N\right\}. \tag{7.12}$$

Let $N+1$ be the number of the edges in the interaction graphs. The *Veronese map* with respect to $\{G'_i\}_i$ is a family of embeddings:

$$\begin{gathered} I_i : \mathbf{CP}^N \hookrightarrow \mathbf{CP}^M, \\ [x_0,\ldots,x_N] \longrightarrow \left[\left\{x_0^{i_0}\ldots,x_k^{i_k} : (i_0,\ldots,i_k) \in P(m,G_i)\right\}\right], \end{gathered} \tag{7.13}$$

where $N_i = \#P(m_i, G_i) + 1$ and $M = (N+1)N/2$ is the maximum number of the edges in the graphs.

7.1.1. Reduction to Dynamics of Toric Ideals

Let $\mathfrak{G}$ be the set of finite graphs, and let

$$F : \mathfrak{G}\left(\{f_i\}_{i=0}^{k}; \{x_j\}_{j=0}^{l}\right) \longrightarrow \mathfrak{G} \tag{7.14}$$

be the forgetful map. Then, one obtains a family of finite graphs

$$G_1, G_2, \ldots \subset \mathfrak{G} \tag{7.15}$$

consisted by the images of F of the interaction graphs. We call them just the *associated graphs*.

Let $\{G_0, G_1, \ldots\}$ be a family of finite graphs. We say that the family is *strongly regular* if the numbers of edges of G_i are all the same. In Section 7.1, we will always assume that sequences consisted by the images of F of the interaction graphs are strongly regular.

For each $G \in \mathfrak{G}$, let us associate a *configuration*

$$\mathfrak{A} = \{\overline{a}_1, \ldots, \overline{a}_m\} \subset \mathbf{Z}^N \tag{7.16}$$

as follows, where $N = (k+1)(l+1)$ is the number of the vertices. Let us make a numbering of the set of vertices, $v_1, \ldots, v_N$, and let $e_i \in \mathbf{Z}^N$ be the unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 appears only at the ith. Then, $e_i + e_j \in \mathfrak{A}$ if and only if v_i and v_j are mutually connected by an edge.

Thus one obtains a reduction from dynamics of the interaction graphs to the one of the following configurations:

$$\{\mathfrak{A}_0, \mathfrak{A}_1, \ldots\}, \tag{7.17}$$

which we call the *transcripted configurations*.

For a configuration $\mathfrak{A} = \{\overline{a}_1, \ldots, \overline{a}_m\} \subset \mathbf{Z}^N$, we associate a Laurent polynomial:

$$\mathbf{C}[\mathfrak{A}] \equiv \mathbf{C}\left[t^{\overline{a}_1}, \ldots, t^{\overline{a}_m}\right] \subset \mathbf{C}\left[t_1^{\pm 1}, \ldots, t_N^{\pm 1}\right], \tag{7.18}$$

where $t^{\overline{a}_k} = t_1^{a_k^1} \cdots t_N^{a_k^N}$, $\overline{a}_k = (a_k^1, \ldots, a_k^N)$. We call $\mathbf{C}[\mathfrak{A}]$ as the *associated toric ring*.

By assigning $y_i \rightarrow t^{\overline{a}_i}$, one obtains a ring homomorphism

$$\pi : \mathbf{C}[y_1, \ldots, y_m] \longrightarrow \mathbf{C}[\mathfrak{A}] \tag{7.19}$$

and its kernel $I_{\mathfrak{A}}$ is called the *toric ideal*.

For each interaction graph $G_i \in \mathfrak{G}(\{f_i\}_{i=0}^k, \{x_j\}_{j=0}^l)$, one forgets markings on edges and orientation and then obtains a finite graph $F(G_i)$. Then, one can assign a configuration $\mathfrak{A}_i \subset \mathbf{Z}^N$. Now correspondingly one has the associated toric ideal $I_i \subset \mathbf{C}[y_1, \ldots, y_m]$. Thus one has obtained a sequence of toric ideals as

$$I_0, I_1, \ldots, I_k, I_{k+1}, \ldots \subset \mathbf{C}[y_1, \ldots, y_m] \tag{7.20}$$

which we will call the *associated ideals*. Notice that there are a finite number of ideals $\{J_1, \ldots, J_d\}$ such that each $I_{\rm i}$ coincides with one of $\{J_j\}$. We say that the associated ideals are *regular* if all I_i has the same dimension.

Let us fix a total ordering $<$ on the set of monomials of the polynomial ring $\mathbf{C}[y_1, \ldots, y_m]$, for example, lexicographic or its reverse ones.

Let $I \subset \mathbf{C}[y_1, \ldots, y_m]$ be an ideal, and let $\overline{z} = \{z_1, \ldots, z_k\}$ be a generating set. We denote by $\mathrm{int} z_i$ to imply the leading term of the polynomial with respect to the ordering. Let $\mathrm{int}\, I \subset \mathbf{C}[y_1, \ldots, y_N]$ be another ideal generated by $\mathrm{int}\, z$ for all $z \in I$. It is called the *initial ideal*. We say that $\overline{z}$ is a *Gröbner basis* if

$$\mathrm{int}\, I = \mathrm{gen}\{\mathrm{int}\, z_1, \ldots, \mathrm{int}\, z_k\}. \tag{7.21}$$

Notice that the sequence of the toric ideals $\{I_i\}_i$ is obtained originally from the data $(\{f_i\}_i, \{x_j\}_j, \{\overline{k}(i, j, x_h)\})$. The following problem seems natural, since, in some cases, these interactions come from some combinatoric structures, like cell automata [3].

Question 4. Can one find an algorithm to find out Gröbner basis successively for $I_0, I_1, \ldots$ from the interaction data?

7.1.2. Gröbner Fans and Translated Varieties

Let $w = (w_1, \ldots, w_m) \in \mathbf{R}^m$ be a vector which is called the *weight vector*. Then, the weight of a monomial $y_1^{a_1} \cdots y_m^{a_m}$ is equal to $\langle w, \overline{a} \rangle = w_1 a_1 + w_2 a_2 + \cdots + w_m a_m$. For each polynomial $f = \Sigma c_{\overline{a}} y^{\overline{a}} \in \mathbf{C}[y_1, \ldots, y_m]$, let $\mathrm{in}_w(f) = \Sigma_{\overline{b}} c_{\overline{b}} y^{\overline{b}}$, where any $\overline{b}$ satisfies $\langle w, \overline{b} \rangle = \max_{c_{\overline{a}} \neq 0} \langle w, \overline{a} \rangle$.

Let I be an ideal. Then, we denote the corresponding initial ideal $\mathrm{in}_w(I)$ generated by all elements of the form $\mathrm{in}_w(f)$. It is known that, for every ideal I and ordering $<$, a weight vector w is associated which satisfies $\mathrm{in}_<(I) = \mathrm{in}_w(I)$.

For an ideal I, we say that w and w' are equivalent if they give the same initial ideal $\mathrm{int}_w I = \mathrm{int}_{w'} I$. The equivalence class

$$\mathfrak{C}(I, w) = \left\{ w' \in \mathbf{R}^m : \mathrm{int}_w I = \mathrm{int}_{w'} I \right\} \tag{7.22}$$

is an open polyhedral cone in $\mathbf{R}^m$.

The set of the cones $\{\mathfrak{C}(I, w)\}_w$ is finite and defines a polyhedral fan $\mathfrak{F}(I)$. We say that it is the *Gröbner fan* of I (see [18]).

For each fan, there associated with a toric variety. Thus for each interaction graph G, one associates with a toric variety X_G which we call the *translated variety*.

Thus corresponding to a sequence of the associated ideals $I_0, I_1, \ldots$, one obtains a sequence of fans over $\mathbf{R}^m$ and the associated translated varieties

$$\overline{X} = (X_0, X_1, \ldots, X_i, \ldots). \tag{7.23}$$

Remark 7.2. In order to study structure of $\overline{X}$, one may use resultants for the defining polynomials of these ideals.

Now we have started from a finite data

$$\mathbf{D} = \left(\{f_i\}_{i=0}^{k}, \{x_j\}_{j=0}^{l}, \{\overline{a}(i,j,h)\}_{i,j,h=0}^{i,j=k,h=l} \right). \tag{7.24}$$

We will call such data an *interaction data*.

Then, we have obtained a sequence of the interaction graphs:

$$\begin{gathered} G_0, G_1, \ldots, \\ G_i = G\left(\{f_i\}_{i=0}^{k}, \{x_j\}_{j=0}^{l}, \Phi_*^i(\{\overline{a}(i,j,h)\}) \right). \end{gathered} \tag{7.25}$$

By forgetting extra data, one obtains a sequence of the transcripted configurations:

$$\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \tag{7.26}$$

where $\mathfrak{A}_i \subset \mathbf{Z}^N$. Then, one has obtained a sequence of the toric ideals $\{I_0, I_1, \ldots\}$ and the translated toric varieties:

$$\{X_0, X_1, \ldots\} \subset \mathbf{CP}^{m-1}, \tag{7.27}$$

where each X_i coincides with one in a finite set of toric varieties $\{Y_1, \ldots, Y_l\} \subset \mathbf{CP}^{m-1}$.

7.1.3. Correspondence on Polytopes

Let $(\{f_i\}^k, \{x_j\}^l, \{\overline{a}(i,j,h)\})$ be a triple, and consider the corresponding interaction graphs $\{G_i\}_{i=0}^{\infty}$. Then, there are finite configurations $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_l\} \subset \mathbf{Z}^N$ such that each G_i associates one of $\mathfrak{A}_j$ for some $j = j(i)$.

Let $\mathfrak{A} \subset \mathbf{Z}^N$ be a configuration, and let m be the number of the elements in $\mathfrak{A}$.

Let Δ be a triangulation of $\mathfrak{A}$. Then, every $\psi \in \mathbf{R}^m$ defines the corresponding piecewise linear function $g_{\psi,\Delta}$ on $\mathbf{R}^m$ satisfying $g_{\psi,\Delta}(a_i) = \psi(a_i)$ for each vertex a_i of Δ which is affine on each simplex of Δ.

Then, we put

$$\mathfrak{C}(\mathfrak{A}, \Delta) \equiv \left\{\psi \in R^m : g_{\psi,\Delta} \text{ is concave and } g_{\psi,\Delta} \geq \psi(a_i) \text{ whenever } a_i \text{ is not a vertex of } \Delta\right\}. \tag{7.28}$$

Moreover, $\mathfrak{C}(\mathfrak{A}, \Delta)$ is a closed polyhedral cone, and

$$\mathfrak{F}(\mathfrak{A}) \equiv \left\{\mathfrak{C}(\mathfrak{A}, \Delta) : \Delta \text{ is a triangulation}\right\} \tag{7.29}$$

forms a complete fan on $\mathbf{R}^m$ and is called the *secondary fan* of $\mathfrak{A}$ [17, page 219]. It is known that the Gröbner fan is a refinement of the secondary fan [18].

Let $Q \subset \mathbf{R}^m$ be a polytope. Then, for each $p \in Q$, let us define the *normal cone*

$$N(Q,p) \equiv \left\{v \in \mathbf{R}^m : \langle v, p\rangle \geq \langle v, y\rangle \, \forall y \in Q\right\}. \tag{7.30}$$

The set of the normal cones

$$N(Q) = \bigcup_{p \in \operatorname{Vert} Q} N(Q,p) \tag{7.31}$$

is called the *normal fan*.

Theorem 7.3 (see [17]). *For a configuration $\mathfrak{A}$, there is a polytope $Q \equiv \Sigma(\mathfrak{A})$ and an assignment of a vertex $\varphi_\Delta \in Q$ for each triangulation Δ such that the normal cone $N(\Sigma(\mathfrak{A}), \varphi_\Delta)$ coincides with $\mathfrak{C}(\mathfrak{A}, \Delta)$.*

In particular there is a natural correspondence from the secondary fan $\mathfrak{F}(\mathfrak{A})$ to the corresponding normal fan $N(Q)$.

$\Sigma(\mathfrak{A})$ is called the *secondary polytope*.

Let us denote $\mathfrak{A} = \{a_1, \ldots, a_m\}$. Then, one can describe Q explicitly as follows. Let us put

$$\begin{aligned} \phi_\Delta &= \left(\phi_\delta^1, \ldots, \phi_\Delta^m\right) \in \mathbf{R}^m, \\ \phi_\Delta^i &= \Sigma\{\operatorname{vol}(\tau) : \tau \in \Delta,\ a_i \in \tau\}. \end{aligned} \tag{7.32}$$

Then, $Q = \Sigma(\mathfrak{A})$ is given by

$$Q \equiv \operatorname{Conv}\{\phi_\Delta : \Delta \text{ is a triangulation of } \mathfrak{A}\}, \tag{7.33}$$

where $\operatorname{vol}(\tau)$ is the volume of τ.

Now let $\{\mathfrak{A}_0, \dots, \mathfrak{A}_l, \dots\} \subset \mathbf{Z}^N$ be the transcripted configurations. Then, correspondingly one obtains other sequences of polytopes:

$$\{\Sigma(\mathfrak{A}_0), \Sigma(\mathfrak{A}_1), \dots\}, \tag{7.34}$$

which is called the *secondary transcripted polytopes*.

7.2. Dynamics over Local Charts

Let us fix two sets $\{f_i\}_{i=0}^k$ and $\{x_j\}_{j=0}^l$. Then, there associates with finite numbers of toric ideals $\{J_1, \dots, J_m\} \subset \mathbf{C}[y_1, \dots, y_N]$.

If we choose a finite subset $\{\overline{a}(i,j,h)\} \subset X_2$, then one obtains an infinite sequence of ideals

$$I(\{\overline{a}(i,j,h)\}) = (I_0, I_1, \dots, I_i, \dots) \tag{7.35}$$

among a finite set $\{J_j\}_j$ above.

Let us put all of the set of sequences

$$\mathbf{I} = \{I(\{\overline{a}(i,j,h)\}) : \{\overline{a}(i,j,h)\} \subset X_2\}. \tag{7.36}$$

We call it the *sequence of local charts*.

One may consider this as though it might be a symbolic dynamics of some "Markov partition" over some algebraic variety V, by regarding each I_k as a defining ideal of a local chart of V.

7.2.1. Algebraic Markov Partition

Let V be an algebraic variety with the affine coordinates V_i defined by an ideal J_i. Let us take an automorphism A on V. We say that an affine coordinate $\{(V_i, J_i)\}_{i=1}^m$ is an (stable) *algebraic Markov partition* for A if, for each i, there is some j so that

$$A(V_i) \subset V_j \tag{7.37}$$

holds.

Example 7.4. Let A be an automorphism on $\mathbf{CP}^N$ by $[z_0, z_1, \dots, z_N] \to [z_1, z_0, z_2, z_3, \dots, z_N]$. Then, for $V_i = \{[z_0, z_1, \dots, z_N] : z_i \neq 0\}$, $\mathbf{CP}^N$ admits an affine covering by $V_0 \cup V_1$. Moreover $A(V_i) = V_{i+1} \bmod 2$.

Let $A : V \cong V$ be an automorphism, and consider its iteration $A^t : V \cong V$. Let us have an algebraic Markov partition by the set $\{V_i\}_i$. Then, one obtains a symbolic dynamics of the algebraic Markov partition:

$$\Sigma(A; \{V_l\}_l) = \{(a_0, a_1, \ldots) : A(V_{a_i}) \subset V_{a_{i+1}})\} \subset X_m, \tag{7.38}$$

where m is the number of the local charts. It can be expressed by a set of sequences of the defining ideals $\{(I_0, I_1, \ldots)\}$.

One can consider its converse. Let us put the set of all sequences coming from the algebraic Markov partition:

$$\mathbf{A} = \{(I_0, I_1, \ldots)\}. \tag{7.39}$$

When one is given an interaction data $(\{f_i\}_i, \{x_j\}_j, \{\overline{a}(i,j,h)\})$, then one obtains a sequence of ideals. Thus a fundamental question in symbolic dynamics of ideals will be to construct correspondence from interaction data to algebraic Markov partitions.

Question 5. Let $\mathbf{I}$ be a set of sequences of ideals among a finite set of ideals. Then, can one construct an algebraic Markov partition $\{J_1, \ldots, J_k\}$ for an algebraic variety V and an automorphism A, so that $\mathbf{I} \subset \mathbf{A}$ might hold? Namely, when are a set of sequences of ideals symbolic dynamics of algebraic Markov partitions?

Conversely, given $\mathbf{A}$, can one find some $\{f_i\}_{i=0}^k$ and $\{x_j\}_{j=0}^l$ so that the corresponding $\mathbf{I}$ might satisfy $\mathbf{I} \subset \mathbf{A}$?

We say that associated ideals $I = (I_0, I_1, \ldots)$ are *regular* if they have the same dimension as the others. Let us put $\bigcup_i I_i = \{J_1, \ldots, J_k\}$.

Definition 7.5. Let $(\{f_i\}_i, \{x_j\}_j, \{\overline{a}(i,j,h)\})$ be an interaction data, and suppose that the associated ideals by $I = (I_0, I_1, \ldots)$ are regular. The sequence is called a symbolic flow of an automorphism if there is an algebraic Markov partition for (V, A) with an affine coordinate $\{(V_i, J_i\}_{i=1}^k$ and some $x \in V$ so that its orbit $\{A^n(x)\}_{n=0,1,\ldots}$ corresponds to the sequence.

We call such pair (V, A) corresponding to I a *prohedron.* (One may imagine as though it represents some state of a protein.)

Let us consider the simplest case. Let us consider the partition of $\mathbf{CP}^N = V_0 \cup V_1$ and the involution A in Example 7.4.

Lemma 7.6. *Let* $(\{f_i\}_{i=0}^k, \{x_j\}_{j=0}^l, \{\overline{a}(i,j,h)\})$ *be an interaction data such that the corresponding sequence of the interaction graphs* $G_0, G_1, \ldots$ *satisfies that (1) the numbers of edges are all constant* m*, that (2)* $F(G_{2i})$ *and* $F(G_{2i+1})$ *are mutually the same finite graphs* G *and* G'*, respectively for all* i*, and (3) there are no primitive loops of even length for any graph.*

Then, the corresponding sequence of ideals gives an algebraic Markov partition for (V_0, V_1, A) *above.*

This follows from the following general facts.

Sublemma 6.1

The toric ideal is generated by the set of primitive loops of the even length.

It follows from this that the corresponding ideals are all zero, which defines the affine plane $\mathbf{C}^N$. This completes the proof.

This simple case suggests that, in general, possibility of construction of algebraic Markov partitions will be reflected by combinatorics of the transcripted configurations.

7.2.2. Automorphism Groups

Let us choose a family of interval maps $\{f_i\}^k$, and let V be an algebraic variety. We denote

$$D\Big(\{f_i\}^k, V\Big) \equiv \Big\{\Big(\{x_j\}^l, \{\overline{a}(i,j,h)\}\Big) : \Big(\{f_i\}^k, \{x_j\}^l, \Big\{\overline{k}(i,j,h)\Big\}\Big) \\ \text{give algebraic Markov partitions for some } A \text{ on } V\Big\} \tag{7.40}$$

in $(\cup_l([0,1]^{l+1} \times X_2^{k,l})$.

Let us put the set of the associated automorphisms E and the automorphism groups G generated by E as

$$E\Big(\{f_i\}^k, V\Big) \equiv \Big\{A = A\Big(\{f_i\}^k, \{x_j\}^k, \Big\{\overline{k}(i,j,h)\Big\}\Big) : \Big(\{x_j\}^l, \Big\{\overline{k}(i,j,h)\Big\}\Big) \in D\Big(\{x_j\}^l, V\Big)\Big\}, \\ G\Big(\{f_i\}^k, V\Big) = \operatorname{gen} E\Big(\{f_i\}^k, V\Big). \tag{7.41}$$

We also put the closure of $G(\{f_i\}^k, V)$ by $\overline{G}(\{f_i\}^k, V) \subset \mathrm{Aut}V$.

Thus for each algebraic variety V, one has obtained a map from a set of interval maps and a Lie subgroup of Aut V:

$$\Big(\{f_i\}^k, V\Big) \longrightarrow \overline{G}\Big(\{f_i\}^k, V\Big) \subset \mathrm{Aut}V. \tag{7.42}$$

It is known that, when V is compact and nonsingular toric variety, then Aut V is linear, and its root system can be written explicitly from the fan of V. Thus in this case $\overline{G} \subset$ Aut V is a closed subgroup of a linear algebraic group. Thus it will be natural to ask the following.

Question 6. Whether one might write down the above subgroups from the information of the associated family of Gröbner fans.

7.3. Zariski Subsets on the Moduli of Interaction Graphs

Let us choose an interaction data: (1) a set of interval maps $\{f_1, \ldots, f_k\}$, (2) an index $\{1, \ldots, l\}$, and (3) a set of $\{0,1\}$ sequences $\{\overline{a}(i,j,h)\}_{i,j,h} \subset X_2$.

For each assignment from $\{1,\ldots,l\}$ to $\{x_1,\ldots,x_l\}$, one obtains the associated interaction graph G. Each edge $e = ((i,h),(k,v)) \in G$ is assigned with some $j \in \{1,\ldots,k\}$ so that $(f_i, x_h) \xrightarrow{(f_j,\overline{a}(i,j,h))} (f_k, x_v)$ hold. So an interaction graph G_k is a weighted finite graph such that each edge is assigned with an element in $\{1,\ldots,k\}$.

Let us fix k and l as above and denote the set of interaction graphs as follows:

$$\mathbf{G}(k,l) = \{G(k,l) : \text{interaction graphs}\}. \tag{7.43}$$

Then $\mathbf{G}(k,l)$ can be parameterized as

$$\mathbf{G}_N \cong [0,1]^l \times \mathrm{Map}\Big[(\{1,\ldots,k\} \times \{1,\ldots,l\})^2 \longrightarrow \{0,1,\ldots,k\}\Big] \cong [0,1]^M, \tag{7.44}$$

where 0 in the last term implies no edges and $M = M(k,l)$.

So once one gives an interaction data

$$(\{f_1,\ldots,f_k\}, \{\overline{a}(i,j,h)\}), \tag{7.45}$$

then one obtains the family of the associated interaction graphs as follows:

$$\mathbf{G}(\{f_i\}_i, \{\overline{a}(i,j,h)\}) = \bigcup_{\{x_j\}_j \in [0,1]^l} G\Big(\{f_i\}_i, \{x_j\}_j, \{\overline{a}(i,j,h)\}\Big) \tag{7.46}$$

in $\mathbf{G}(k,l) \cong [0,1]^M$.

Definition 7.7. A Zariski subset $X \subset \mathbf{G}(k,l) \cong [0,1]^M$ is a subset of the form

$$X = \mathbf{G}(\{f_i\}_i, \{\overline{a}(i,j,h)\}) \subset [0,1]^M. \tag{7.47}$$

Let $\mathfrak{A}$ be all of the set of the interval maps. Then, we have obtained a map

$$\mathbf{J} : \mathfrak{A}^k \times [0,1]^l \times X_2^{k^2+l} \longmapsto \mathbf{G}(k,l). \tag{7.48}$$

Let $\{f_i\}$ and $\{g_i\}$ be two k interval maps. Then consider the following.

Question 7. (1) In order to guarantee that $\mathbf{J}(\{f_i\},) = \mathbf{J}(\{g_i\},)$ implies that $\{f_i\} = \{g_i\}$, how should k and l be large?

(2) Can one find some continuous properties for $\mathbf{J}$?

Let X be a Zariski subset, and consider a proper decreasing Zariski subsets

$$X = X_0 \supset X_1 \supset \cdots \supset X_n. \tag{7.49}$$

We define the *dimension* of X to be the largest number n with the above property.

7.3.1. Dynamics on Zariski Subsets

Let us choose a set $\{x_j\}^l \subset [0,1]^l$. Then, one obtains sequences $\{\overline{a}(i,j,h)^n\}_{n=0,1,\dots}$ by using the interaction map. Thus one obtains a sequence of Zariski subsets as follows:

$$X_0, X_1, \dots, X_n, X_{n+1}, \dots \subset [0,1]^M,$$
$$X_n = \mathbf{G}(\{f_i\}, \{\overline{a}(i,j,h)^n\}). \tag{7.50}$$

We say that $\bigcap_n X_n$ is an *invariant* subset. If $\lim_n X_n \subset \mathbf{G}(k,l)$ exists, then we say that the associated dynamics of the Zariski subsets converges.

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