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Research Article

On q-Operators and Summation of Some q-Series

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Using Jackson's q-derivative and the q-Stirling numbers, we establish some transformation theorems leading to the values of some convergent q-series.

1. Introduction

The operator $(x(d/dx))^n$ has many assets and plays a central role in arithmetic fields and in computation of some finite or infinite sums. For example, when we try to compute the sum $\sum_{k=0}^{+\infty} k^n x^k$, we use the operators $(x(d/dx))^n$, which give

$$\sum_{k=0}^{+\infty} k^n x^k = \left(x\frac{d}{dx}\right)^n \left(\frac{1}{1-x}\right), \quad |x| < 1,\ n = 0, 1, 2 \ldots. \tag{1.1}$$

These operators are intimately related to the Stirling numbers of second kind $\left\{ {n \atop k} \right\}$ by the formula (see [1])

$$\left(x\frac{d}{dx}\right)^n f(x) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^k \frac{d^k f}{dx^k}, \tag{1.2}$$

where f is a suitable function. We note that the q-analogue of formula (1.2) has been studied by many authors (see [2, 3] and references therein) and has found applications in many fields such as arithmetic partitions and asymptotic expansions.

This paper deals with the analogues of the operators $(x(d/dx))^n$ in *Quantum Calculus* and some q-transformation theorems that will be used to establish the sums of some q-series.

This paper is organized as follows. In Section 2, we present some preliminary notions and notations useful in the sequel. Section 3 gives three applications of a result proved in [2], states a transformation theorem using the q-Stirling numbers, and presents some related applications. Section 4 attempts to give a new q-analogue of formula (1.2) by studying the transformation theorem related to a q-derivative operator.

2. Notations and Preliminaries

To make this paper self-containing and easily decipherable, we recall some useful preliminaries about the *Quantum Calculus* and we select Gasper-Rahman's book [4], for the notations and for a deep study in this way. Throughout this paper, we fix $q \in]0,1[$.

2.1. q-Shifted Factorials

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \qquad (a;q)_\infty = \prod_{k=0}^{+\infty}\left(1-aq^k\right). \tag{2.1}$$

We also write

$$(a_1,\ldots,a_k;q)_n = \prod_{j=1}^{k}(a_j;q)_n, \quad n = 0,1,\ldots,\infty. \tag{2.2}$$

We put

$$\begin{aligned} [x]_q &= \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}, \\ [n]_q! &= \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}. \end{aligned} \tag{2.3}$$

For $a, x \in \mathbb{C}$ and $n \in \mathbb{N}$, we adopt the following notation [5]:

$$(x-a)_q^n = \begin{cases} 1 & \text{if } n = 0, \\ (x-a)(x-aq)\cdots\left(x-aq^{n-1}\right) & \text{if } n \geqslant 1. \end{cases} \tag{2.4}$$

The q-analogue of the Jordan factorial is given by

$$\begin{aligned} [x]_{k,q} &= [x]_q[x-1]_q\cdots[x-k+1]_q \\ &= \frac{(1-q^x)\left(1-q^{x-1}\right)\cdots\left(1-q^{x-k+1}\right)}{(1-q)^k}, \end{aligned} \tag{2.5}$$

and the q-binomial coefficient is defined by

$$\begin{bmatrix} x \\ k \end{bmatrix}_q = \frac{[x]_{k,q}}{[k]_q!}. \tag{2.6}$$

2.2. The Jackson's q-Derivative

The q-derivative $D_q f$ of a function f is defined by (see [4])

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad \text{if } x \neq 0, \tag{2.7}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. Note that when f is differentiable, at x, then $(D_q) f(x)$ tends to $f'(x)$ as q tends to 1^-.

It is easy to see that for suitable functions f and g, we have

$$D_q(fg)(x) = f(qx) D_q g(x) + g(x) D_q f(x), \tag{2.8}$$

$$D_q\left(\frac{f}{g}\right)(x) = \frac{g(qx) D_q f(x) - f(qx) D_q g(x)}{g(x) g(qx)}. \tag{2.9}$$

2.3. Elementary q-Special Functions

Two q-analogues of the exponential function are given by (see [4])

$$\begin{aligned} e_q(z) &= \sum_{n=0}^{+\infty} \frac{z^n}{[n]_q!} = \frac{1}{((1-q)z; q)_\infty}, \quad |z| < (1-q)^{-1}, \\ E_q(z) &= \sum_{n=0}^{+\infty} q^{n(n-1)/2} \frac{z^n}{[n]_q!} = (-(1-q)z; q)_\infty, \quad z \in \mathbb{C}. \end{aligned} \tag{2.10}$$

They satisfy the relations

$$\begin{gathered} D_q e_q(z) = e_q(z), \qquad D_q E_q(z) = E_q(qz), \\ e_q(z) E_q(-z) = E_q(z) e_q(-z) = 1, \qquad E_q(z) = e_{1/q}(z). \end{gathered} \tag{2.11}$$

In 1910, F. H. Jackson defined a q-analogue of the Gamma function by (see [4, 6])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \quad x \neq 0, -1, -2, \ldots. \tag{2.12}$$

It satisfies the following functional equations:

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \qquad \Gamma_q(1) = 1, \qquad \Gamma_q(n+1) = [n]_q!, \quad n \in \mathbb{N}. \tag{2.13}$$

2.4. *q*-Stirling Numbers of Noncentral Type

In [7], Charalambides introduced the so-called noncentral q-Stirling numbers, which are q-analogues of the Stirling numbers and classified into two kinds.

The noncentral q-Stirling numbers of the first kind $s_q(n,k;r)$ are defined by the following generating relation:

$$[t-r]_{n,q} = q^{-\binom{n}{2}-rn} \sum_{k=0}^{n} s_q(n,k;r)[t]_q^k, \quad n = 0,1,\ldots, \tag{2.14}$$

and they are given by

$$s_q(n,k;r) = \frac{1}{(1-q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{\binom{n-j}{2}+r(n-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \binom{j}{k}. \tag{2.15}$$

The noncentral q-Stirling numbers of the second kind $S_q(n,k;r)$ are defined by the following generating relation:

$$[t]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}-rk} S_q(n,k;r)[t-r]_{k,q}, \quad n = 0,1,\ldots, \tag{2.16}$$

and they are given by

$$\begin{aligned} S_q(n,k;r) &= \frac{1}{[k]_q!} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{j+1}{2}-(r+j)k} \begin{bmatrix} k \\ j \end{bmatrix}_q [r+j]_q^n \\ &= \frac{1}{(1-q)^{n-k}} \sum_{j=k}^{n} (-1)^{j-k} q^{r(j-k)} \binom{n}{j} \begin{bmatrix} j \\ k \end{bmatrix}_q. \end{aligned} \tag{2.17}$$

Remark 2.1. Note that when $r = 0$, then $s_q(n,k;r)$ and $S_q(n,k;r)$ reduce to the q-Stirling numbers, respectively, of the first and the second kind studied by Gould, Carlitz, and Kim (see [8–11]).

Properties

The noncentral q-Stirling numbers satisfy the following properties.

(i) For $n = 1,2,\ldots$ and $k = 1,2,\ldots,n$,

$$s_q(n,k;r) = s_q(n-1,k-1;r) - [n+r-1]_q s_q(n-1,k;r), \tag{2.18}$$

under the following conditions:

$$s_q(0,0;r)=1,\qquad s_q(n,0;r)=q^{\binom{n}{2}+rn}[-r]_{n,q},\quad n>0,$$
$$s_q(0,k;r)=0,\quad k>0,\qquad s_q(n,k;r)=0,\quad k>n. \tag{2.19}$$

(ii) For $n=1,2,\ldots$ and $k=1,2,\ldots,n$,

$$S_q(n,k;r)=S_q(n-1,k-1;r)+[r+k]_q S_q(n-1,k;r), \tag{2.20}$$

under the following conditions:

$$S_q(0,0;r)=1,\qquad S_q(n,0;r)=[r]_q^n,\quad n>0,$$
$$S_q(0,k;r)=0,\quad k>0,\qquad S_q(n,k;r)=0,\quad k>n. \tag{2.21}$$

3. The Operator $(xD_q)^m$ and Some Related Transformations Theorems

As in the classical case (see [1]), the iterate $(xD_q)^m$, $m\in\mathbb{N}$, can be expanded in finite terms involving the q-Stirling numbers. This is the purpose of the following result.

Lemma 3.1 (see [2, 3]). *Letting f be a differentiable function, then one has*

$$(xD_q)^m f(x)=\sum_{k=1}^{m}\left\{ {m \atop k} \right\}_{q,1} x^k D_q^k f(x),\quad m=1,2,\ldots, \tag{3.1}$$

where

$$\left\{ {m \atop k} \right\}_{q,1}=q^{k(k-1)/2}S_q(m-1,k-1;1)=\frac{1}{[k-1]_q!}\sum_{j=0}^{k-1}(-1)^j q^{\binom{j}{2}}\begin{bmatrix} k-1 \\ j \end{bmatrix}_q [k-j]_q^{m-1}. \tag{3.2}$$

Now, let us give three applications of the previous lemma.

Example 3.2 (q-binomial series). The q-binomial theorem asserts that

$${}_1\Phi_0(q^a;-;q,x)=\sum_{n=0}^{+\infty}\frac{(q^a;q)_n}{(q;q)_n}x^n=\frac{(q^a x;q)_\infty}{(x;q)_\infty},\quad |x|<1. \tag{3.3}$$

Using the fact that, for all $m\in\mathbb{N}$,

$$(xD_q)^m x^n=[n]_q^m x^n \tag{3.4}$$

and the previous lemma, we deduce that

$$\sum_{n=0}^{+\infty}\frac{(q^a;q)_n}{(q;q)_n}[n]_q^m x^n=\sum_{k=1}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_{q,1}x^kD_q^k\left(\frac{(q^ax;q)_\infty}{(x;q)_\infty}\right),\quad |x|<1. \tag{3.5}$$

On the other hand, the definition of q-derivative (2.9) gives

$$D_q\left(\frac{(q^ax;q)_\infty}{(x;q)_\infty}\right)=[a]_q\frac{(q^{a+1}x;q)_\infty}{(x;q)_\infty}, \tag{3.6}$$

and by iteration we have

$$D_q^k\left(\frac{(q^ax;q)_\infty}{(x;q)_\infty}\right)=\frac{\Gamma_q(a+k)}{\Gamma_q(a)}\frac{(q^{a+k}x;q)_\infty}{(x;q)_\infty}. \tag{3.7}$$

Thus,

$$\sum_{n=0}^{+\infty}\frac{(q^a;q)_n}{(q;q)_n}[n]_q^m x^n=\frac{1}{\Gamma_q(a)(x;q)_\infty}\sum_{k=1}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_{q,1}x^k\Gamma_q(a+k)\left(q^{a+k}x;q\right)_\infty. \tag{3.8}$$

So, taking $a=1$, we obtain

$$\sum_{n=0}^{+\infty}[n]_q^m x^n=\frac{1}{1-x}\sum_{k=1}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_{q,1}x^k\frac{[k]_q!}{(xq;q)_k}. \tag{3.9}$$

Remark that if q tends to 1^-, we obtain the formula given in [13, page 366].

Example 3.3 (q-Bessel function). We consider the function

$$C_p(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k x^k}{[k]_q![k+p]_q!}=\left(\frac{x}{2}\right)^{-p}J_p^{(1)}\left((1-q)\sqrt{2}x,q\right), \tag{3.10}$$

where $J_p^{(1)}(\cdot,q)$ is the first Jackson's q-Bessel function of order p (see [12, 13]).

By application of the operator $(xD_q)^m$ to $C_0(x)$ and the use of relation (3.4), we obtain

$$(xD_q)^m C_0(x)=\sum_{k=0}^{+\infty}(-1)^k\frac{[k]_q^m}{\left([k]_q!\right)^2}x^k. \tag{3.11}$$

Then, using Lemma 3.1 and the fact that

$$D_qC_p(x)=-C_{p+1}(x), \tag{3.12}$$

we get

$$\sum_{k=0}^{+\infty}(-1)^k\frac{[k]_q^m}{\left([k]_q!\right)^2}x^k=\sum_{k=1}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_{q,1}(-1)^k x^k C_k(x). \tag{3.13}$$

Example 3.4 (q-polynomial exponential). Take $f(x) = e_q(x)$. From relation (3.4) and Lemma 3.1, we obtain

$$(xD_q)^m e_q(x)=e_q(x)\Phi_{m,q}(x)=\sum_{n=1}^{+\infty}\frac{[n]_q^m}{[n]_q!}x^n, \tag{3.14}$$

where

$$\Phi_{m,q}(x)=\sum_{k=1}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_{q,1}x^k, \tag{3.15}$$

which is called the q-polynomial exponential. So,

$$\sum_{k=1}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_{q,1}x^k=E_q(-x)\sum_{n=1}^{+\infty}\frac{[n]_q^m}{[n]_q!}x^n=\sum_{n=1}^{+\infty}\sum_{k=1}^{n}(-1)^k q^{\binom{k}{2}}\frac{[n-k]_q^m}{[k]_q![n-k]_q!}x^n. \tag{3.16}$$

In many mathematical fields there are some transformation theorems using the Stirling numbers leading one to compute certain sums (see [14]). The purpose of the following result is to give a q-analogue context.

Theorem 3.5. *Let $f(x)$ and $g(x)$ be two functions satisfying*

$$f(x)=\sum_{n=0}^{+\infty}a_n[x]_q^n,\qquad g(x)=\sum_{n=0}^{+\infty}c_n x^n. \tag{3.17}$$

Then

$$\sum_{n=0}^{+\infty}c_n f(n)x^n=\sum_{m=1}^{+\infty}a_m\sum_{k=1}^{m}\left\{\begin{matrix}m\\k\end{matrix}\right\}_{q,1}x^k {D_q}^k g(x) \tag{3.18}$$

provided the series

$$\sum_{n=0}^{+\infty}c_n f(n)x^n \tag{3.19}$$

converges absolutely.

Proof. From Lemma 3.1 and the properties of the q-Stirling numbers of the second kind (2.21), we obtain

$$\begin{aligned}\sum_{m=1}^{+\infty} a_m \sum_{k=1}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k D_q^k g(x) &= \sum_{m=1}^{+\infty} a_m \left(xD_q\right)^m g(x) \\ &= \sum_{m=1}^{+\infty} a_m \left(xD_q\right)^m \left(\sum_{n=0}^{+\infty} c_n x^n \right).\end{aligned} \tag{3.20}$$

The result follows, then, from relations (3.4) and (3.19). □

Corollary 3.6. *Let* $f(x) = \sum_{n=0}^{+\infty} a_n [x]_q^n$. *Then*

$$\sum_{n=0}^{+\infty} f(n) x^n = \sum_{m=1}^{+\infty} a_m \sum_{k=1}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \frac{[k]_q!}{(x;q)_{k+1}} \tag{3.21}$$

provided the series $\sum_{n=0}^{+\infty} f(n) x^n$ *converges absolutely.*

Proof. By taking $g(x) = 1/(1-x) = \sum_{n=0}^{+\infty} x^n$, $|x| < 1$, in the previous theorem, and by application of relation (3.7), we obtain

$$\begin{aligned}\sum_{n=0}^{+\infty} f(n) x^n &= \sum_{m=1}^{+\infty} a_m \sum_{k=1}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k D_q^k \left(\frac{1}{1-x} \right) \\ &= \sum_{m=1}^{+\infty} a_m \sum_{k=1}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \frac{\Gamma_q(k+1)}{\Gamma_q(1)} \frac{(q^{k+1}x;q)_\infty}{(x;q)_\infty} \\ &= \sum_{m=1}^{+\infty} a_m \sum_{k=1}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\}_{q,1} x^k \frac{[k]_q!}{(x;q)_{k+1}}.\end{aligned} \tag{3.22}$$

□

Example 3.7. Let $f(x) = [x]_{n,q}$. Then, the fact that

$$[x]_{n,q} = q^{-\binom{n}{2}} \sum_{k=0}^{n} s_q(n,k;0) [x]_q^k, \quad n = 0, 1, \ldots \tag{3.23}$$

and Corollary 3.6 give

$$\sum_{k=0}^{+\infty} [k]_{n,q} x^k = \frac{q^{-\binom{n}{2}}}{1-x} \sum_{m=1}^{n} s_q(n,m;0) \sum_{l=1}^{m} \left\{ \begin{matrix} m \\ l \end{matrix} \right\}_{q,1} x^l \frac{[l]_q!}{(qx;q)_l}. \tag{3.24}$$

Remark that when q tends to 1^-, we obtain the formula given in [1, (6.4), page 3863].

Some others summation formulas are presented in the following statements.

Corollary 3.8. *For $f(x)=\sum_{n=0}^{+\infty}a_n[x]_q^n$, the transformation formulas lead to the following:*

(1) $\sum_{n=0}^{+\infty}q^{n(n-1)/2}\left[{\alpha \atop n}\right]_q f(n)x^n=\sum_{n=1}^{+\infty}a_n\sum_{k=1}^{n}q^{k(k-1)/2}\left\{{n \atop k}\right\}_{q,1}[k]_q!\left[{\alpha \atop k}\right]_q x^k(1+q^kx)_q^{\alpha-k}$;

(2) $\sum_{n=0}^{+\infty}((1-q^\alpha)_q^n/(1-q)_q^n)f(n)x^n=\sum_{n=1}^{+\infty}a_n\sum_{k=1}^{n}\left\{{n \atop k}\right\}_{q,1}\left[{\alpha+k-1 \atop k-1}\right]_q(x^k/(1-x)_q^{\alpha+k})$;

(3) $\sum_{n=0}^{+\infty}(f(n)/[n]_q!)x^n=e_q(x)\sum_{n=1}^{+\infty}a_n\sum_{k=1}^{n}\left\{{n \atop k}\right\}_{q,1}x^k=e_q(x)\sum_{n=1}^{+\infty}a_n\Phi_{n,q}(x)$;

(4) $\sum_{n=0}^{+\infty}q^{n(n-1)}(f(n)/[n]_q!)x^n=\sum_{n=1}^{+\infty}a_n\sum_{k=1}^{n}\left\{{n \atop k}\right\}_{q,1}q^{k(k-1)/2}(x^k/(-(1-q)q^kx;q)_\infty)$

provided the series converge absolutely.

Proof. The results are direct consequences of Theorem 3.5 by putting the following:

(1) $g(x)=(1+x)_q^\alpha=\sum_{n=0}^{+\infty}q^{n(n-1)/2}\left[{\alpha \atop n}\right]_q x^n$ and remark that $D_q^k(1+x)_q^\alpha=q^{k(k-1)/2}[\alpha]_q[\alpha-1]_q\cdots[\alpha-k+1]_q(1+q^kx)_q^{\alpha-k}$;

(2) $g(x)=1/(1-x)_q^\alpha=\sum_{n=0}^{+\infty}((1-q^\alpha)_q^n/(1-q)_q^n)x^n$ and remark that $D_q^k(1/(1-x)_q^\alpha)=[\alpha]_q[\alpha+1]_q\cdots[\alpha+k-1]_q/(1-x)_q^{\alpha+k}$;

(3) $g(x)=e_q(x)$;

(4) $g(x)=E_q(x)$ and remark that $D_q^kE_q(x)=q^{k(k-1)/2}E_q(q^kx)$. □

Remark 3.9. The last formulas coincide with some of the ones given in [15] when q tends to 1^-.

4. The Operator $((x;q)_1D_q)^m$ and Related Transformation Theorem

Lemma 4.1. *For a suitable function f, one has for $m=1,2,\dots$*

$$\left[(x;q)_1D_q\right]^m f(x)=\sum_{k=1}^{m}(-1)^{m-k}S_q(m-1,k-1;1)(x;q)_kD_q^kf(x). \tag{4.1}$$

Proof. The formula can be obtained by induction with respect to m. Indeed, for $m=1$, we have

$$\left[(x;q)_1D_q\right]f(x)=(x;q)_1D_qf(x)=S_q(0,0;1)(x;q)_1D_qf(x). \tag{4.2}$$

Assuming that formula (4.1) is true for m, then

$$\begin{aligned}
\left[(x;q)_1 D_q\right]^{m+1} f(x) &= (x;q)_1 D_q\left[\sum_{k=1}^{m}(-1)^{m-k} S_q(m-1,k-1;1)(x;q)_k D_q^k f(x)\right] \\
&= (x;q)_1 \sum_{k=1}^{m}(-1)^{m-k} S_q(m-1,k-1;1)(qx;q)_k D_q^{k+1} f(x) \\
&\quad - (x;q)_1 \sum_{k=1}^{m}(-1)^{m-k} S_q(m-1,k-1;1)[k]_q (qx;q)_{k-1} D_q^k f(x) \\
&= \sum_{k=2}^{m+1}(-1)^{m-k+1} S_q(m-1,k-2;1)(x;q)_k D_q^k f(x) \\
&\quad - \sum_{k=1}^{m}(-1)^{m-k}[k]_q S_q(m-1,k-1;1)(x;q)_k D_q^k f(x) \\
&= \sum_{k=2}^{m}(-1)^{m-k+1}\left[S_q(m-1,k-2;1)-[k]_q S_q(m-1,k-1;1)\right](x;q)_k D_q^k f(x) \\
&\quad + S_q(m-1,m-1;1)(x;q)_{m+1} D_q^{m+1} f(x) \\
&\quad - (-1)^{m-1} S_q(m-1,0;1)(x;q)_1 D_q f(x).
\end{aligned} \tag{4.3}$$

The result is easily deduced by formulas (2.20), and (2.21). □

Theorem 4.2. *Let $f(x)$ and $g(x)$ be two functions defined by*

$$f(x)=\sum_{n=0}^{+\infty}(-1)^n \alpha_n [x]_q^n, \qquad g(x)=\sum_{n=0}^{+\infty} c_n (x;q)_n. \tag{4.4}$$

If the series

$$\sum_{n=0}^{+\infty} c_n f(n)(x;q)_n \tag{4.5}$$

converges absolutely, then

$$\sum_{n=0}^{+\infty} c_n f(n)(x;q)_n = \sum_{m=1}^{+\infty}\alpha_m \sum_{k=1}^{m}(-1)^{m-k} S_q(m-1,k-1;1)(x;q)_k D_q^k g(x). \tag{4.6}$$

Proof. From the previous lemma, we obtain for $m=1,2,\ldots,$

$$\sum_{m=0}^{+\infty}\alpha_m\left[(x;q)_1 D_q\right]^m g(x) = \sum_{m=1}^{+\infty}\alpha_m \sum_{k=1}^{m}(-1)^{m-k} S_q(m-1,k-1;1)(x;q)_k D_q^k g(x). \tag{4.7}$$

So, the absolute convergence of the series (4.5) and the fact that

$$\left[(x;q)_1 D_q\right]^m (x;q)_n = (-1)^m [n]_q^m (x;q)_n, \quad m \in \mathbb{N} \tag{4.8}$$

achieve the proof. □

Corollary 4.3. *Let* $f(x) = \sum_{m=0}^{+\infty} (-1)^m \alpha_m [x]_q^m$. *Then*

$$\sum_{m=1}^{+\infty} \alpha_m \sum_{k=1}^{m} (-1)^{m-k} S_q(m-1,k-1;1)(x;q)_k [n]_{k,q} x^{n-k} = \sum_{k=0}^{n} (-q)^k s_q(k,0,-n) f(k)(x;q)_k. \tag{4.9}$$

Proof. Put $g(x) = x^n$.

Using the representation (see [5])

$$x^n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-(n-1)k} q^{\binom{k}{2}} (x;q)_k = \sum_{k=0}^{n} (-q)^k s_q(k,0,-n)(x;q)_k, \tag{4.10}$$

relation (4.6) and the fact that

$$D_q^k(x^n) = [n]_{k,q} x^{n-k} \tag{4.11}$$

give the desired result. □

Remark 4.4. Note that recently Liu in his paper (see [16]) has obtained some interesting q-identities in showing that the solutions of two difference equations involve some series of q-operators D_q^n of q-Cauchy type.

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