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Research Article

Existence of Multiple Solutions of a Second-Order Difference Boundary Value Problem

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This paper studies the existence of multiple solutions of the second-order difference boundary value problem $\Delta^2 u(n-1) + V'(u(n)) = 0$, $n \in \mathbb{Z}(1,T)$, $u(0) = 0 = u(T+1)$. By applying Morse theory, critical groups, and the mountain pass theorem, we prove that the previous equation has at least three nontrivial solutions when the problem is resonant at the eigenvalue λ_k $(k \geq 2)$ of linear difference problem $\Delta^2 u(n-1) + \lambda u(n) = 0$, $n \in \mathbb{Z}(1,T)$, $u(0) = 0 = u(T+1)$ near infinity and the trivial solution of the first equation is a local minimizer under some assumptions on V.

1. Introduction

Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}$ be the sets of real numbers, natural numbers, and integers, respectively. For any $a, b \in \mathbb{Z}$, $a \leq b$, define $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$.

Consider the second-order difference boundary value problem (BVP)

$$\begin{gathered} \Delta^2 u(n-1) + V'(u(n)) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1), \end{gathered} \tag{1.1}$$

where $V \in C^2(\mathbb{R}, \mathbb{R})$ and Δ denotes the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, $\Delta^2 u(n) = \Delta(\Delta u(n))$.

By a solution u of the BVP (1.1), we mean a real sequence $\{u(n)\}_{n=0}^{T+1}$ $(= (u(0), u(1), \dots, u(T+1)))$ satisfying the BVP (1.1). For $u = \{u(n)\}_{n=0}^{T+1}$ with $u(0) = 0 = u(T+1)$, we say that $u \neq 0$ if there exists at least one $n \in \mathbb{Z}(1,T)$ such that $u(n) \neq 0$. We say that u is *positive* (and write $u > 0$) if for all $n \in \mathbb{Z}(1,T)$, $u(n) > 0$, and similarly, u is *negative* $(u < 0)$

if for all $n \in \mathbb{Z}(1,T)$, $u(n) < 0$. The aim of this paper is to obtain the existence of multiple solutions of the BVP (1.1) and analyse the sign of solutions.

Recently, a few authors applied the minimax methods to examine the difference boundary value problems. For example, in [1], Agarwal et al. employed the Mountain Pass Lemma to study the following BVP:

$$\begin{gathered}\Delta^2 u(n-1) + f(n,u(n)) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1)\end{gathered} \tag{1.2}$$

and obtained the existence of multiple positive solutions, where f may be singular at $u = 0$. In [2], Jiang and Zhou employed the Mountain Pass Lemma together with strongly monotone operator principle, to study the following difference BVP:

$$\begin{gathered}\Delta^2 u(n-1) + f(n,u(n)) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = \Delta u(T)\end{gathered} \tag{1.3}$$

and obtained existence and uniqueness results, where $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ is continuous. In [3], Cai and Yu employed the Linking Theorem and the Mountain Pass Lemma to study the following difference BVP:

$$\begin{gathered}\Delta\Big(p(n)(\Delta u(n-1))^{\delta}\Big) + q(n)u^{\delta}(n) = f(n,u(n)), \quad n \in \mathbb{Z}(1,T), \\ \Delta u(0) = A, \quad u(T+1) = B\end{gathered} \tag{1.4}$$

and obtained the existence of multiple solutions, where $\delta > 0$ is the ratio of odd positive integers, $\{p(n)\}_{n=1}^{T+1}$ and $\{q(n)\}_{n=1}^{T}$ are real sequences, $p(n) \neq 0$ for all $n \in \mathbb{Z}(1,T+1)$, and A, B are two given constants, $f : \mathbb{Z}(1,T) \times \mathbb{R} \to \mathbb{R}$ is continuous.

Although applications of the minimax methods in the field of the difference BVP have attracted some scholarly attention in the recent years, efforts in applying Morse theory to the difference BVP are scarce. The main purpose of this paper is to develop a new approach to the BVP (1.1) by using Morse theory. To this end, we first consider the following linear difference eigenvalue problem:

$$\begin{gathered}\Delta^2 u(n-1) + \lambda u(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ u(0) = 0 = u(T+1).\end{gathered} \tag{1.5}$$

On the above eigenvalue problem, the following results hold; see [4].

Proposition 1.1. *The eigenvalues of* (1.5) *are*

$$\lambda = \lambda_l = 4\sin^2\frac{l\pi}{2(T+1)}, \quad l = 1,2,\dots,T, \tag{1.6}$$

and the corresponding eigenfunction with λ_l *is* $\phi_l(n) = \sin(l\pi n/(T+1))$, $l = 1,2,\dots,T$.

Remark 1.2. (1) The set of functions $\{\phi_l(n),\ l = 1,2,\ldots,T\}$ is orthogonal on $\mathbb{Z}(1,T)$ with respect to the weight function $r(n) \equiv 1$, that is,

$$\sum_{n=1}^{T}(\phi_l(n),\phi_j(n)) = 0 \quad \forall l \neq j. \tag{1.7}$$

Moreover, for each $l \in \mathbb{Z}(1,T)$, $\sum_{n=1}^{T}\sin^2(l\pi n/(T+1)) = (T+1)/2$.

(2) It is easy to see that ϕ_1 is positive and ϕ_l changes sign for each $l \in \mathbb{Z}(2,T)$, that is, $\{n : \phi_l(n) > 0\} \neq \emptyset$ and $\{n : \phi_l(n) < 0\} \neq \emptyset$.

For (1.1), we assume that

$$V(0) = V'(0) = 0, \tag{1.8}$$

$$V''(\infty) := \lim_{|t|\to\infty}\frac{V'(t)}{t} = \lambda_k, \tag{1.9}$$

where λ_k is an eigenvalue of (1.5). Hence the BVP (1.1) has a trivial solution $u \equiv 0$. And we say that BVP (1.1) is resonant at infinity if (1.9) holds.

Let

$$W^- = \operatorname{span}\{\phi_1,\phi_2,\ldots,\phi_{k-1}\}, \qquad W^0 = \operatorname{span}\{\phi_k\}, \qquad W^+ = \operatorname{span}\{\phi_{k+1},\phi_{k+2},\ldots,\phi_T\}. \tag{1.10}$$

Let $G(t) = \int_0^t G'(s)ds = V(t) - (\lambda_k/2)t^2$. By (1.9) we have

$$\lim_{|t|\to\infty}\frac{G'(t)}{t} = 0. \tag{1.11}$$

Assume that the following conditions on $G'(t)$ hold.

($G^{\pm}$) If $\|u_m\| \to \infty$ such that $\|v_m\|/\|u_m\| \to 1$, then there exist $\delta > 0$ and $M \in \mathbb{N}$ such that

$$\pm\sum_{n=1}^{T}(G'(u_m(n)),v_m(n)) \geq \delta, \quad \forall m \geq M, \tag{1.12}$$

where $u_m = v_m + w_m$, $v_m \in W^0$, $w_m \in W := W^+ \oplus W^-$.

The main result of this paper is as follows.

Theorem 1.3. *Let* (1.8), (1.9) *hold and*

($V1$) $V''(t) > 0$ *for all* $t \in \mathbb{R}$,

($V2$) $V''(0) < \lambda_1$

hold. Then the BVP (1.1) *has at least three nontrivial solutions, with one positive solution and one negative solution, in each of the following cases:*

(i) *(G^+) and $k \geq 2$;*

(ii) *(G^-) and $k \geq 3$.*

To the author's best knowledge, only Bin et al. [5] deal with the existence and multiplicity of nontrivial periodic solutions for asymptotically linear resonant difference problem by the aid of Su [6]. In [5], G satisfies

$$\left|G'(z)\right| \leq c_1|z|^s + c_2, \tag{1.13}$$

$$\lim_{\|v\|\to\infty} \inf_{v\in W^0} \frac{1}{\|v\|^{2s}} G(z) \geq \frac{4\beta^2}{\delta T}, \tag{1.14}$$

where $c_1 > 0$, $c_2 > 0$, $s \in (0,1)$, $\beta = c_1 T^{(1-s)/2}$, $\delta > 0$. In [5], the authors obtained the existence of one nontrivial periodic solution. Notice that (1.13) implies that (1.11) holds; however, $(G^{\pm})$ is not covered by (1.14). In fact, conditions (1.13) and (1.14) are borrowed from [6]. The conditions in Theorem 1.3 coincide with the assumptions of Theorem 1 in [7]. The aim of this paper is to develop a new approach to study the discrete systems by using Morse theory, minimax theorems, and some analysis technique. We wish to have some breakthrough points with the aid of the method of discretization.

The remaining part of this paper proceeds as follows. In the next section, we establish the variational framework of the BVP (1.1) and collect some results which will be used in the proof of Theorem 1.3. In Section 3, we give the proof of Theorem 1.3. Finally, in Section 4, we give an example to illustrate our main result and summarize conclusions and future directions.

2. Variational Framework and Auxiliary Results

Let

$$E = \left\{u : u = \{u(n)\}_{n=0}^{T+1} \text{ with } u(0) = 0 = u(T+1) \in \mathbb{R}\right\}. \tag{2.1}$$

E can be equipped with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$ as follows:

$$\begin{aligned} \|u\| &= \left(\sum_{n=0}^{T}|\Delta u(n)|^2\right)^{1/2}, \quad \forall u \in E, \\ \langle u,v\rangle &= \sum_{n=0}^{T}(\Delta u(n), \Delta v(n)), \quad \forall u,v \in E, \end{aligned} \tag{2.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$ and $(\cdot,\cdot)$ denotes the usual scalar product in $\mathbb{R}$. It is easy to see that $(E, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Consider the functional defined on E by

$$J(u) = \frac{1}{2}\sum_{n=0}^{T}|\Delta u(n)|^2 - \sum_{n=1}^{T} V(u(n)). \tag{2.3}$$

We claim that if $u \in E$ is a critical point of J, then u is precisely a solution of the BVP (1.1). Indeed, for every $u, v \in E$, we have

$$\langle J'(u), v\rangle = \sum_{n=0}^{T}(\Delta u(n), \Delta v(n)) - \sum_{n=1}^{T}(V'(u(n)), v(n)) = -\sum_{n=1}^{T}\Big(\Delta^2 u(n-1) + V'(u(n)), v(n)\Big). \tag{2.4}$$

So, if $J'(u) = 0$, then we have

$$\sum_{n=1}^{T}\Big(\Delta^2 u(n-1) + V'(u(n)), v(n)\Big) = 0. \tag{2.5}$$

Since $v \in E$ is arbitrary, we obtain

$$\Delta^2 u(n-1) + V'(u(n)) = 0, \quad n \in \mathbb{Z}(1,T). \tag{2.6}$$

Therefore, we reduce the problem of finding solutions of the BVP (1.1) to that of seeking critical points of the functional J in E.

According to Proposition 1.1 and Remark 1.2, E can be decomposed as $E = W^- \oplus W^0 \oplus W^+$. For all $u \in E$, denote $u = w^0 + w^+ + w^-$ with $w^0 \in W^0$, $w^+ \in W^+$, and $w^- \in W^-$, then we have the following Wirtinger type inequalities:

$$\lambda_1\sum_{n=1}^{T}(u(n), u(n)) \le \|u\|^2 \le \lambda_T\sum_{n=1}^{T}(u(n), u(n)), \quad \forall u \in E, \tag{2.7}$$

$$\lambda_1\sum_{n=1}^{T}(w^-(n), w^-(n)) \le \|w^-\|^2 \le \lambda_{k-1}\sum_{n=1}^{T}(w^-(n), w^-(n)), \quad \forall w^- \in W^-, \tag{2.8}$$

$$\lambda_{k+1}\sum_{n=1}^{T}(w^+(n), w^+(n)) \le \|w^+\|^2 \le \lambda_T\sum_{n=1}^{T}(w^+(n), w^+(n)), \quad \forall w^+ \in W^+, \tag{2.9}$$

see [4] for details.

Now we collect some results on Morse theory and the minimax methods.

Let E be a real Hilbert space and $J \in C^1(E,\mathbb{R})$. Denote

$$J^c = \{u \in E : J(u) \le c\}, \qquad \mathcal{K}_c = \{u \in E : J'(u) = 0, J(u) = c\} \tag{2.10}$$

for $c \in \mathbb{R}$. The following is the definition of the Palais-Smale condition ((PS) condition).

Definition 2.1. The functional J satisfies the (PS) condition if any sequence $\{u_m\} \subset E$ such that $\{J(u_m)\}$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$ has a convergent subsequence.

In [8], Cerami introduced a weak version of the (PS) condition as follows.

Definition 2.2. The functional J satisfies the Cerami condition ((C) condition) if any sequence $\{u_m\} \subset E$ such that $\{J(u_m)\}$ is bounded and $(1 + \|u_m\|)\|J'(u_m)\| \to 0$ as $m \to \infty$ has a convergent subsequence.

If J satisfies the (PS) condition or the (C) condition, then J satisfies the following deformation condition which is essential in Morse theory (cf. [9, 10]).

Definition 2.3. The functional J satisfies the (D_c) condition at the level $c \in \mathbb{R}$ if for any $\overline{\epsilon} > 0$ and any neighborhood $\mathcal{N}$ of $\mathcal{K}_c$, there are $\epsilon > 0$ and a continuous deformation $\eta : [0,1] \times E \to E$ such that

(1) $\eta(0,u) = u$ for all $u \in E$;

(2) $\eta(t,u) = u$ for all $u \notin J^{-1}([c - \overline{\epsilon}, c + \overline{\epsilon}])$;

(3) $J(\eta(t,u))$ is nonincreasing in t for any $u \in E$;

(4) $\eta(1, J^{c+\epsilon} \setminus \mathcal{N}) \subset J^{c-\epsilon}$.

J satisfies the (D) condition if J satisfies the (D_c) condition for all $c \in \mathbb{R}$.

Let u_0 be an isolated critical point of J with $J(u_0) = c \in \mathbb{R}$, and let U be a neighborhood of u_0, the group

$$C_q(J, u_0) := H_q(J^c \cap U, J^c \cap U \setminus \{u_0\}), \quad q \in \mathbb{Z}, \tag{2.11}$$

is called the qth critical group of J at u_0, where $H_q(A,B)$ denotes the qth singular relative homology group of the pair (A,B) over a field F, which is defined to be quotient $H_q(A,B) = Z_q(A,B)/B_q(A,B)$, where $Z_q(A,B)$ is the qth singular relative closed chain group and $B_q(A,B)$ is the qth singular relative boundary chain group.

Let $\mathcal{K} = \{u \in E : J'(u) = 0\}$. If $J(\mathcal{K})$ is bounded from below by $a \in \mathbb{R}$ and J satisfies the (D_c) condition for all $c \leq a$, then the group

$$C_q(J, \infty) := H_q(E, J^a), \quad q \in \mathbb{Z}, \tag{2.12}$$

is called the qth critical group of J at infinity [11].

Assume that $\#\mathcal{K} < \infty$ and J satisfies the (D) condition. The Morse-type numbers of the pair (E, J^a) are defined by

$$M_q = M_q(E, J^a) = \sum_{u \in \mathcal{K}} \dim C_q(J, u), \tag{2.13}$$

and the Betti numbers of the pair (E, J^a) are

$$\beta_q := \dim C_q(J, \infty). \tag{2.14}$$

By Morse theory [12, 13], the following relations hold:

$$\sum_{j=0}^{q}(-1)^{q-j}M_j \geq \sum_{j=0}^{q}(-1)^{q-j}\beta_j, \quad q \in \mathbb{Z},$$
$$\sum_{q=0}^{\infty}M_q = \sum_{q=0}^{\infty}\beta_q. \tag{2.15}$$

Thus, if $C_q(J,\infty) \ncong 0$, for some $k \in \mathbb{Z}$, then there must exist a critical point u of J with $C_q(J,u) \ncong 0$, which can be rephrased as follows.

Proposition 2.4. *Let E be a real Hilbert space and $J \in C^2(E,\mathbb{R})$. Assume that $\#\mathcal{K} < \infty$ and that J satisfies the (D) condition. If there exists some $q \in \mathbb{Z}$ such that $C_q(J,\infty) \ncong 0$, then J must have a critical point u with $C_q(J,u) \ncong 0$.*

In order to prove our main result, we need the following result about the critical group on $C_q(J,\infty)$.

Proposition 2.5. *Let the functional $J : E \to \mathbb{R}$ be of the form*

$$J(u) = \frac{1}{2}\langle Au, u\rangle + Q(u), \tag{2.16}$$

where $A : E \to E$ is a self-adjoint linear operator such that 0 is isolated in $\sigma(A)$, the spectrum of A. Assume that $Q \in C^1(E,\mathbb{R})$ satisfies

$$\lim_{\|u\|\to\infty}\frac{\|Q'(u)\|}{\|u\|} = 0. \tag{2.17}$$

Denote $V := \ker A$, $W := V^{\perp} = W^{+} \oplus W^{-}$, where W^{+} (W^{-}) is the subspace of E on which A is positive (negative) definite. Assume that $\mu := \dim W^{-}$, $\nu := \dim V \neq 0$ are finite and that J satisfies the (D) condition. Then

$$C_q(J,\infty) \cong \delta_{q,k^{\pm}}F, \quad q \in \mathbb{Z}, \tag{2.18}$$

provided that J satisfies the angle conditions at infinity.

$(AC_{\infty}^{\pm})$: *there exist $M > 0$ and $\alpha \in (0,1)$ such that*

$$\pm\langle J'(u), v\rangle \geq 0 \quad \text{for } u = v + w,\ \|u\| \geq M,\ \|w\| \leq \alpha\|u\|, \tag{2.19}$$

where $k^{+} = \mu$, $k^{-} = \mu + \nu$, $v \in V$, and $w \in W$.

Remark 2.6. Conditions (2.16) and (2.17) imply that J is asymptotically quadratic. Bartsch and Li [11] introduced the notion of critical groups at infinity and proved that if J satisfied some angle properties at infinity, the critical groups can be completely figured out. Proposition 2.5

is a slight improvement of [11, Proposition 3.10] by Su and Zhao [7]. There are many other papers considering concrete problems by computing the critical groups at infinity with different methods, for example, see [14–17].

We will use the Mountain Pass Lemma (cf. [12, 18]) in our proof.

Let B_ρ denote the open ball in E about 0 of radius r and let ∂B_ρ denote its boundary.

Theorem 2.7 (mountain pass lemma). *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfying the (PS) condition. Suppose $J(0) = 0$ and that*

($J1$) *there are constants $\rho > 0, a > 0$ such that $J|_{\partial B_\rho} \geq a > 0$,*

($J2$) *there is a $u_0 \in E \setminus B_\rho$ such that $J(u_0) \leq 0$,*

then J possesses a critical value $c \geq a$. Moreover c can be characterized as

$$c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)), \tag{2.20}$$

where

$$\Gamma = \{h \in C([0,1], E) \mid h(0) = 0, h(1) = u_0\}. \tag{2.21}$$

Definition 2.8 (mountain pass point). An isolated critical point u of J is called a mountain pass point, if $C_1(J, u) \not\cong 0$.

The following result is useful in computing the critical group of a mountain pass point; see [13, 19] for details.

Theorem 2.9. *Let E be a real Hilbert space. Suppose that $J \in C^2(E, \mathbb{R})$ has a mountain pass point u, and that $J''(u)$ is a Fredholm operator with finite Morse index, satisfying*

$$J''(u_0) \geq 0,\ 0 \in \sigma(J''(u_0)) \Longrightarrow \dim\ker(J''(u_0)) = 1, \tag{2.22}$$

then

$$C_q(J, u_0) \cong \delta_{q,1} F, \quad q \in \mathbb{Z}. \tag{2.23}$$

3. Proof of Theorem 1.3

We give the proof of Theorem 1.3 in this section. Firstly, we prove that the functional J satisfies the (C) condition (Lemma 3.1) and compute the critical group $C_q(J, \infty)$ (Lemma 3.2). Then, we employ the cut-off technique and the Mountain Pass Lemma to obtain two critical points u^+, u^- of J and compute the critical groups $C_q(J, u^+)$ and $C_q(J, u^-)$ (Lemmas 3.3 and 3.4). Finally, we prove Theorem 1.3.

Rewrite the functional J as

$$J(u) = \frac{1}{2}\sum_{n=0}^{T} |\Delta u(n)|^2 - \frac{\lambda_k}{2}\sum_{n=1}^{T} |u(n)|^2 - \sum_{n=1}^{T} G(u(n)), \quad \forall u \in E. \tag{3.1}$$

Lemma 3.1. *Let* (1.8) *and* (1.9) *hold. If* G *satisfies* $(G^{\pm})$*, then the functional* J *satisfies the* (C) *condition.*

Proof. We only prove the case where (G^{+}) holds. Let $\{u_m\} \subset E$ such that

$$J(u_m) \longrightarrow c \in \mathbb{R}, \quad (1 + \|u_m\|)\left\|J'(u_m)\right\| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{3.2}$$

Then for all $\varphi \in E$, we have

$$\langle J'(u_m), \varphi\rangle = \langle u_m, \varphi\rangle - \lambda_k \sum_{n=1}^{T} (u_m(n), \varphi(n)) - \sum_{n=1}^{T} (G'(u_m(n)), \varphi(n)). \tag{3.3}$$

Denote $u_m = v_m + w_m^{+} + w_m^{-}$ with $v_m \in W^0$, $w_m^{+} \in W^{+}$ and $w_m^{-} \in W^{-}$. Since E is a finite-dimensional Hilbert space, it suffices to show that $\{u_m\}$ is bounded. Suppose that $\{u_m\}$ is unbounded. Passing to a subsequence we may assume that $\|u_m\| \to \infty$ as $m \to \infty$.

By (1.11), for any $\epsilon > 0$, there exists $b \in \mathbb{R}$ such that

$$\left|G'(t)\right| \le \epsilon|t| + b, \quad \forall t \in \mathbb{R}. \tag{3.4}$$

Let $\varphi = w_m^{+}$ in (3.3). Then by (2.7), (2.9), and (3.4), we have

$$\begin{aligned}
c_1\|w_m^{+}\|^2 &:= \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|w_m^{+}\|^2 \\
&\le \|w_m^{+}\|^2 - \lambda_k \sum_{n=1}^{T} (w_m^{+}(n), w_m^{+}(n)) \\
&= \sum_{n=1}^{T} (G'(u_m(n)), w_m^{+}(n)) + \langle J'(u_m), w_m^{+}\rangle \\
&\le \|w_m^{+}\| + \sum_{n=1}^{T} (\epsilon|u_m(n)| + b)|w_m^{+}(n)| \\
&\le \|w_m^{+}\| + \frac{\epsilon}{\sqrt{\lambda_1\lambda_{k+1}}}\|u_m\|\|w_m^{+}\| + \frac{b\sqrt{T}}{\sqrt{\lambda_{k+1}}}\|w_m^{+}\| \\
&:= c_2\|w_m^{+}\| + c_3\|u_m\|\|w_m^{+}\|,
\end{aligned} \tag{3.5}$$

where

$$c_1 = 1 - \frac{\lambda_k}{\lambda_{k+1}} > 0, \qquad c_2 = 1 + \frac{b\sqrt{T}}{\sqrt{\lambda_{k+1}}}, \qquad c_3 = \frac{\epsilon}{\sqrt{\lambda_1\lambda_{k+1}}}. \tag{3.6}$$

And since $\epsilon > 0$ is arbitrary, we have

$$\frac{\|w_m^+\|}{\|u_m\|} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{3.7}$$

Similarly, let $\varphi = w_m^-$ in (3.3), by (2.8) and (3.4) we get

$$\begin{aligned}
-c_4\left\|w_m^-\right\|^2 =: \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\left\|w_m^-\right\|^2 \\
&\geq \left\|w_m^-\right\|^2 - \lambda_k \sum_{n=1}^{T}\left(w_m^-(n), w_m^-(n)\right) \\
&= \sum_{n=1}^{T}\left(G'(u_m(n)), w_m^-(n)\right) + \langle J'(u_m), w_m^- \rangle \\
&\geq -\left\|w_m^-\right\| - \sum_{n=1}^{T}(\epsilon|u_m(n)| + b)\left|w_m^-(n)\right| \\
&\geq -\left\|w_m^-\right\| - \frac{\epsilon}{\lambda_1}\|u_m\|\left\|w_m^-\right\| - \frac{b\sqrt{T}}{\sqrt{\lambda_1}}\left\|w_m^-\right\| \\
&:= -c_5\left\|w_m^-\right\| - c_6\|u_m\|\left\|w_m^-\right\|,
\end{aligned} \tag{3.8}$$

where

$$c_4 = \frac{\lambda_k}{\lambda_{k-1}} - 1 > 0, \qquad c_5 = 1 + \frac{b\sqrt{T}}{\sqrt{\lambda_1}}, \qquad c_6 = \frac{\epsilon}{\lambda_1}. \tag{3.9}$$

And hence we also have

$$\frac{\|w_m^-\|}{\|u_m\|} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{3.10}$$

By (3.7) and (3.10), we have

$$\frac{\|w_m\|}{\|u_m\|} \longrightarrow 0, \quad \frac{\|v_m\|}{\|u_m\|} \longrightarrow 1 \quad \text{as} \quad m \longrightarrow \infty. \tag{3.11}$$

By (G^+), there exist $\delta > 0$ and $M \in \mathbb{N}$ such that

$$\sum_{n=1}^{T}\left(G'(u_m(n)), v_m(n)\right) \geq \delta, \quad \forall m \geq M. \tag{3.12}$$

This implies that

$$\langle J'(u_m), v_m\rangle = -\sum_{n=1}^{T}\left(G'(u_m(n)), v_m(n)\right) \le -\delta, \quad \forall m \ge M, \tag{3.13}$$

and hence

$$\left\|J'(u_m)\right\|\|u_m\| \ge \left\|J'(u_m)\right\|\|v_m\| \ge \left|\langle J'(u_m), v_m\rangle\right| \ge \delta, \quad \forall m \ge M, \tag{3.14}$$

which is a contradiction to (3.2). Thus $\{u_m\}$ is bounded. The proof is complete. □

Lemma 3.2. *Let* (1.8) *and* (1.9) *hold. Then*

(1) $C_q(J,\infty) \cong \delta_{q,k}F$ *provided that* (G^+) *holds;*

(2) $C_q(J,\infty) \cong \delta_{q,k-1}F$ *provided that* (G^-) *holds.*

Proof. We only prove the case (1). Define a bilinear function

$$a(u,v) = \lambda_k \sum_{n=1}^{T}(u(n), v(n)), \quad \forall u, v \in E. \tag{3.15}$$

Then by (2.7) we have

$$|a(u,v)| \le \frac{\lambda_k}{\lambda_1}\|u\|\|v\|. \tag{3.16}$$

And hence there exists a unique continuous bounded linear operator $K : E \to E$ such that

$$\langle Ku, v\rangle = \lambda_k \sum_{n=1}^{T}(u(n), v(n)). \tag{3.17}$$

Since $\langle Ku, u\rangle \in \mathbb{R}$ for all $u \in E$, we can conclude that K is a self-adjoint operator and

$$J(u) = \frac{1}{2}\langle (I-K)u, u\rangle - \sum_{n=1}^{T}G(u(n)). \tag{3.18}$$

Then J has the form (2.16) with

$$Q(u) = -\sum_{n=1}^{T}G(u(n)), \tag{3.19}$$

and (1.11) implies that (2.17) holds. Let $A = I - K$. Then $\ker A = W^0 = \operatorname{span}\{\phi_k\}$. Next we show that (G^+) implies that the angle condition (AC_∞^-) at infinity holds.

If not, then for any $m \in \mathbb{N}$ and each $\alpha_m = 1/m$, there exists $u_m = v_m + w_m \in W^0 \oplus (W^+ \oplus W^-)$ with $v_m \in W^0$, $w_m \in W^+ \oplus W^-$ such that

$$\|u_m\| \geq m, \quad \|w_m\| \leq \frac{1}{m}\|u_m\|, \tag{3.20}$$

$$\langle J'(u_m), v_m \rangle > 0. \tag{3.21}$$

On the other hand, (3.20) implies

$$\|u_m\| \longrightarrow \infty, \quad \frac{\|v_m\|}{\|u_m\|} \longrightarrow 1 \quad \text{as} \quad m \longrightarrow \infty. \tag{3.22}$$

Thus, by (G^+) there exist $\delta > 0$ and $M \in \mathbb{N}$ such that

$$\sum_{n=1}^{T} (G'(u_m(n)), v_m(n)) \geq \delta, \quad \forall m \geq M. \tag{3.23}$$

Therefore,

$$\langle J'(u_m), v_m \rangle = -\sum_{n=1}^{T} (G'(u_m(n)), v_m(n)) \leq -\delta, \quad \forall m \geq M, \tag{3.24}$$

which is a contradiction to (3.21). Consequently (AC_∞^-) holds and by Lemma 3.1 and Proposition 2.5, $C_q(J, \infty) = \delta_{q,k} F$. Similarly, we can prove that (2) holds. □

In order to obtain a mountain pass point, we need the following lemmas.

Lemma 3.3. *Let*

$$V'^{+}(t) = \begin{cases} V'(t), & t \geq 0, \\ 0, & t \leq 0, \end{cases} \qquad V'^{-}(t) = \begin{cases} V'(t), & t \leq 0, \\ 0, & t \geq 0, \end{cases} \tag{3.25}$$

and $V^{\pm}(t) = \int_0^t V'^{\pm}(s) ds$. *If*

$$\lim_{|t| \to \infty} \frac{V'(t)}{t} = \alpha \neq \lambda_1, \tag{3.26}$$

then the functional

$$J^{\pm}(u) = \frac{1}{2} \sum_{n=0}^{T} |\Delta u(n)|^2 - \sum_{n=1}^{T} V^{\pm}(u(n)) \tag{3.27}$$

satisfies the (PS) condition.

Proof. We only prove the case (J^+). Let $\{u_m\} \subset E$ such that

$$J(u_m) \longrightarrow c \in \mathbb{R}, \qquad J'^{+}(u_m) \longrightarrow 0 \tag{3.28}$$

as $m \to \infty$. Since E is a finite-dimensional space, it suffices to show that $\{u_m\}$ is bounded in E. Suppose that $\{u_m\}$ is unbounded. Passing to a subsequence we may assume that $\|u_m\| \to \infty$ and for each n, either $|u_m(n)| \to \infty$ or $\{u_m(n)\}$ is bounded.

Noticing that for all $\varphi \in E$,

$$\langle J'^{+}(u_m), \varphi\rangle = \langle u_m, \varphi\rangle - \sum_{n=1}^{T}\Big(V'^{+}(u_m(n)), \varphi(n)\Big). \tag{3.29}$$

Denote $w_m := u_m/\|u_m\|$, for a subsequence, w_m converges to some w with $\|w\| = 1$. By (3.29), we have

$$\frac{\langle J'^{+}(u_m), \varphi\rangle}{\|u_m\|} = \langle w_m, \varphi\rangle - \sum_{n=1}^{T}\left(\frac{V'^{+}(u_m(n))}{\|u_m\|}, \varphi(n)\right). \tag{3.30}$$

If $|u_m(n)| \to \infty$, then

$$\lim_{m\to\infty} \frac{V'^{+}(u_m(n))}{u_m(n)} w_m(n) = \alpha w^{+}(n), \tag{3.31}$$

where $w^{+}(n) = \max\{w(n), 0\}$ with $n \in \mathbb{Z}(1, T)$. If $\{u_m(n)\}$ is bounded, then

$$\lim_{m\to\infty} \frac{V'^{+}(u_m(n))}{\|u_m\|} = 0, \qquad w(n) = 0. \tag{3.32}$$

Since $w \neq 0$, there is an n for which $|u_m(n)| \to \infty$. So passing to the limit in (3.30), we have

$$\sum_{n=0}^{T}(\Delta w(n), \Delta\varphi(n)) - \alpha\sum_{n=1}^{T}(w^{+}(n), \varphi(n)) = 0. \tag{3.33}$$

This implies that $w \neq 0$ satisfies

$$\begin{gathered}\Delta^2 w(n-1) + \alpha w^{+}(n) = 0, \quad n \in \mathbb{Z}(1, T), \\ w(0) = 0 = w(T+1).\end{gathered} \tag{3.34}$$

Now, we claim that

$$w(n) > 0, \quad \forall n \in \mathbb{Z}(1, T). \tag{3.35}$$

In fact, let

$$w(n_0) = \min\{w(n) : n \in \mathbb{Z}(1,T)\}. \tag{3.36}$$

We only need to prove $w(n_0) > 0$. If not, assume that $w(n_0) \le 0$. Then by (3.34), we have $\Delta^2 w(n_0 - 1) = 0$ and hence $w(n_0 - 1) = w(n_0) = w(n_0 + 1)$. By induction, it is easy to get $w(n) = 0$ for all $n \in \mathbb{Z}(1,T)$ which is a contradiction to $w \neq 0$ and hence (3.35) holds.

On the other hand, by Proposition 1.1 and Remark 1.2, we see that only the eigenfunction corresponding to the eigenvalue λ_1 is positive, which is a contradiction to $\alpha \neq \lambda_1$. The proof is complete. □

Lemma 3.4. *Under the conditions of Theorem 1.3, the functional J^+ has a critical point $u^+ > 0$ and $C_q(J^+, u^+) \cong \delta_{q,1}F$; the functional J^- has a critical point $u^- < 0$ and $C_q(J^-, u^-) \cong \delta_{q,1}F$.*

Proof. We only prove the case of J^+. Firstly, we prove that J^+ satisfies the Mountain Pass Lemma and hence J^+ has a nonzero critical point u^+. In fact, $J^+ \in C^1(E, \mathbb{R})$ and by Lemma 3.3 we see that J^+ satisfies the (PS) condition. Clearly $J^+(0) = 0$. Thus we need to show that J^+ satisfies $(J1)$ and $(J2)$. To verify $(J1)$, by (1.8) and $(V2)$, there exist $\rho_1 > 0$ and $\rho_2 > 0$ with $V''(0) < \rho_2 < \lambda_1$ such that

$$V(t) \le \frac{1}{2}\rho_2 t^2 \tag{3.37}$$

for $|t| \le \rho_1$. So, for all $u \in E$, if $\|u\| \le \sqrt{\lambda_1}\rho_1$, then for each $n \in \mathbb{Z}(1,T)$, $|u(n)| \le \rho_1$ and

$$\begin{aligned} J^+(u) &= \frac{1}{2}\|u\|^2 - \sum_{n=1}^{T} V^+(u(n)) \\ &= \frac{1}{2}\|u\|^2 - \sum_{n\in N_1} V(u(n)) \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\rho_2 \sum_{n\in N_1} (u(n), u(n)) \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\rho_2 \sum_{n=1}^{T} (u(n), u(n)) \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\frac{\rho_2}{\lambda_1}\|u\|^2, \end{aligned} \tag{3.38}$$

where $N_1 = \{n \in \mathbb{Z}(1,T) \mid u(n) \ge 0\}$. Let

$$\rho = \sqrt{\lambda_1}\rho_1, \qquad a = \frac{1}{2}\left(1 - \frac{\rho_2}{\lambda_1}\right)\rho^2. \tag{3.39}$$

Then $J^+(u)|_{\partial B_\rho} \geq a > 0$ and hence $(J1)$ holds. For $(J2)$, by $V''(\infty) = \lambda_k \in (\lambda_{k-1}, \lambda_{k+1})$, we claim that there exist $\overline{\gamma} > \lambda_{k-1}\ (\geq \lambda_1)$, $b \in \mathbb{R}$ such that

$$V(t) \geq \frac{\overline{\gamma}}{2}t^2 + b, \quad \forall t \in \mathbb{R}. \tag{3.40}$$

In fact, by assumption (1.9), there exist $M > 0$ and $b_1 \in \mathbb{R}$ such that $V(t) \geq (\overline{\gamma}/2)t^2 + b_1$ for $|t| \geq M$. Meanwhile, there exists $b_2 \in \mathbb{R}$ such that $V(t) - (\overline{\gamma}/2)t^2 \geq b_2$ for $|t| \leq M$ by virtue of the continuity of V. Let $b = \min\{b_1, b_2\}$, we get the conclusion.

Thus, if we choose $e \in \operatorname{span}\{\phi_1\}$ with $e > 0$ and $\|e\| = 1$, then

$$\begin{aligned} J^+(te) &= \frac{t^2}{2} - \sum_{n=1}^{T} V(te(n)) \\ &\leq \frac{t^2}{2} - \frac{\overline{\gamma}t^2}{2}(e,e) - bT \\ &= \frac{t^2}{2} - \frac{\overline{\gamma}t^2}{2\lambda_1} - bT \longrightarrow -\infty \end{aligned} \tag{3.41}$$

as $0 < t \to +\infty$. Thus, we can choose a constant t large enough with $t > \rho$ and $u_0 = te \in E$ such that $J^+(u_0) \leq 0$. $(J2)$ holds.

Therefore, by Theorem 2.7, J^+ has a critical point $u^+ \neq 0$ and similar to the proof of Lemma 3.3, we can prove that $u^+ > 0$. So u^+ is also a critical point of J. In the following we compute the critical group $C_q(J^+, u^+)$ by using Theorem 2.9.

Assume that

$$\langle J''(u^+)v, v\rangle = \langle v, v\rangle - \sum_{n=1}^{T}\left(V''(u^+(n))v(n), v(n)\right) \geq 0, \quad \forall v \in E, \tag{3.42}$$

and that there exists $v_0 \not\equiv 0$ such that

$$\langle J''(u^+)v_0, v\rangle = 0, \quad \forall v \in E. \tag{3.43}$$

This implies that v_0 satisfies

$$\begin{gathered} \Delta^2 v_0(n-1) + V''(u^+(n))v_0(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ v_0(0) = v_0(T+1) = 0. \end{gathered} \tag{3.44}$$

Hence the eigenvalue problem

$$\begin{gathered} \Delta^2 v(n-1) + \lambda V''(u^+(n))v(n) = 0, \quad n \in \mathbb{Z}(1,T), \\ v(0) = v(T+1) = 0 \end{gathered} \tag{3.45}$$

has an eigenvalue $\lambda = 1$. $(V1)$ implies that 1 must be a simple eigenvalue; see [4]. So, $\dim \ker(J''(u_0)) = 1$. Since E is a finite-dimensional Hilbert space, the Morse index of u^+ must be finite and $J''(u^+)$ must be a Fredholm operator. By Theorem 2.9, $C_q(J^+, u^+) \cong \delta_{q,1}F$. The proof is complete. □

Remark 3.5. We can choose the neighborhood U of u^+ such that $u > 0$ for all $u \in U$. Therefore,

$$C_q(J, u^+) \cong C_q(J^+, u^+) \cong \delta_{q,1}F. \tag{3.46}$$

Similarly,

$$C_q(J, u^-) \cong C_q(J^-, u^-) \cong \delta_{q,1}F. \tag{3.47}$$

Now, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. We only prove the case (i). By Lemma 3.2,

$$C_q(J, \infty) \cong \delta_{q,k}F. \tag{3.48}$$

Hence by Proposition 2.4 the functional J has a critical point u_1 satisfying

$$C_k(J, u_1) \ncong 0. \tag{3.49}$$

Since

$$\langle J''(0)u, u\rangle \geq \left(1 - \frac{V''(0)}{\lambda_1}\right)\|u\|^2, \tag{3.50}$$

by $(V2)$ and $J(0) = J'(0) = 0$, we see that 0 is a local minimum of J. Hence

$$C_q(J, 0) \cong \delta_{q,0}F. \tag{3.51}$$

By Remark 3.5, (3.49), (3.51), and $k \geq 2$ we get that u^+, u^-, and u_1 are three nonzero critical points of J with $u^+ > 0$ and $u^- < 0$. The proof is complete. □

4. An Example and Future Directions

To illustrate the use of Theorem 1.3, we offer the following example.

Example 4.1. Consider the BVP

$$\begin{gathered} \Delta^2 u(n-1) + V'(u(n)) = 0, \quad n \in \mathbb{Z}(1,5), \\ u(0) = 0 = u(6), \end{gathered} \tag{4.1}$$

where $V \in C^2(\mathbb{R},\mathbb{R})$ is defined as follows:

$$V(t) = \begin{cases} \dfrac{1}{10}t^2, & |t| \le 1, \\ \dfrac{1}{2}t^2 + \dfrac{3}{4}t^{4/3}, & |t| \ge 10, \\ \text{a strictly convex function}, & \text{otherwise}. \end{cases} \tag{4.2}$$

It is easy to verify that V satisfies (1.8), (1.9), (1.11), $(V1)$, and $(V2)$ with $k = 2$. To verify the condition (G^+), note that $G'(t) = t^{1/3}$ for $|t| \ge 10$, we claim that

$$\sum_{n=1}^{5}\left(G'(u_m(n)), v_m(n)\right) \longrightarrow +\infty, \quad \text{as} \quad m \longrightarrow \infty \tag{4.3}$$

which implies that (G^+) holds.

To this end, for any constant $r > 1$, we introduce another norm in $E(T = 5)$ as follows:

$$\|u\|_r = \left(\sum_{n=1}^{5}|u(n)|^r\right)^{1/r}, \quad \forall u \in E. \tag{4.4}$$

Since E is finite dimensional, there exist two constants $C_2 \ge C_1 > 0$ such that

$$C_1\|u\| \le \|u\|_r \le C_2\|u\|, \quad \forall u \in E. \tag{4.5}$$

Now, by (G^+), for any ϵ small enough, it is easy to see that

$$\|w_m\| \le \epsilon\|u_m\| \tag{4.6}$$

holds for m large enough.

Set

$$\Omega_1 = \{n \in \mathbb{Z}(1,5) : |u_m(n)| \ge 10\}, \qquad \Omega_2 = \mathbb{Z}(1,5) \setminus \Omega_1. \tag{4.7}$$

Since $\|u_m\| \to \infty$, $\Omega_1 \ne \emptyset$, for m large enough. And for m large enough, we have

$$\begin{aligned} \sum_{n=1}^{5}\left(G'(u_m(n)), v_m(n)\right) &= \sum_{n\in\Omega_1}\left(u_m^{1/3}(n), v_m(n)\right) + \sum_{n\in\Omega_2}\left(G'(u_m(n)), u_m(n)\right) \\ &\quad - \sum_{n\in\Omega_2}\left(G'(u_m(n)), w_m(n)\right) \\ &\ge \sum_{n\in\Omega_1}\left(u_m^{1/3}(n), v_m(n)\right) - c - c\epsilon\|u_m\| \\ &= \sum_{n\in\Omega_1}\left(u_m^{1/3}(n), u_m(n)\right) - \sum_{n\in\Omega_1}\left(u_m^{1/3}(n), w_m(n)\right) - c - c\epsilon\|u_m\|. \end{aligned} \tag{4.8}$$

Here and below we denote by c various positive constants. Since

$$\begin{aligned}
\sum_{n\in\Omega_1}\left(u_m^{1/3}(n),u_m(n)\right) &= \sum_{n=1}^{5}\left(u_m^{1/3}(n),u_m(n)\right) - \sum_{n\in\Omega_2}\left(u_m^{1/3}(n),u_m(n)\right)\\
&= \|u_m\|_{4/3}^{4/3}-c,\\
\sum_{n\in\Omega_1}\left(u_m^{1/3}(n),w_m(n)\right) &\le \sum_{n=1}^{5}|u_m(n)|^{1/3}|w_m(n)|\\
&\le \left(\sum_{n=1}^{5}|u_m(n)|^{4/3}\right)^{1/4}\left(\sum_{n=1}^{5}|w_m(n)|^{4/3}\right)^{3/4}\\
&= \|u_m\|_{4/3}^{1/3}\|w_m\|_{4/3}\le c\epsilon\|u_m\|_{4/3}^{4/3}.
\end{aligned} \tag{4.9}$$

Hence

$$\sum_{n=1}^{5}\left(G'(u_m(n)),v_m(n)\right)\ge(1-c\epsilon)\|u_m\|_{4/3}^{4/3}-c\epsilon\|u_m\|-c. \tag{4.10}$$

Since ϵ is small enough, we get (4.3) holds by the above and (4.5). Hence, by Theorem 1.3, BVP (4.1) has at least three nontrivial solutions.

Morse theory has been proved very useful in proving the existence and multiplicity of solutions of operator equations with variational frameworks. However, it is well known that the minimax methods is also a useful tool for the same purpose. The advantage of the minimax methods is that it provides an estimate of the critical value. But it is hard to distinguish critical points obtained by this methods with those by other methods, if the local behavior of the critical points is not very well known. However, critical groups serve as a topological tool in distinguishing isolated critical points. Hence, in order to obtain multiple solutions by using Morse theory, it is crucial to describe critical groups clearly.

A natural question is: can we use the same methods in this paper to other BVPs? Noticing that the key conditions which guarantee the multiplicity of solutions of the BVP (1.1) are as follows:

(1) the BVP has a variational framework;

(2) the eigenvalues of the corresponding linear BVP are nonzero and there is a one-sign eigenfunction,

hence, if the difference equation

$$\Delta^2 u(n-1)+V'(u(n))=0,\quad n\in\mathbb{Z}(1,T) \tag{4.11}$$

subject to some other boundary value conditions satisfying (1) and (2), then we can obtain similar results to Theorem 1.3.

Example 4.2. Consider the BVP

$$\Delta^2 u(n-1) + V'(u(n)) = 0, \quad n \in \mathbb{Z}(1,T),$$
$$u(0) = 0 = \Delta u(T). \tag{4.12}$$

Let

$$E = \left\{ u : u = \{u(n)\}_{n=0}^{T+1} \text{ with } u(0) = 0 = \Delta u(T) \right\}. \tag{4.13}$$

Then E is a T-dimensional Hilbert space with the inner product

$$\langle u, v \rangle = \sum_{n=0}^{T} (\Delta u(n), \Delta v(n)). \tag{4.14}$$

Define the functional J on E by

$$J(u) = \sum_{n=0}^{T} \frac{1}{2} |\Delta u(n)|^2 - \sum_{n=1}^{T} V(u(n)). \tag{4.15}$$

It is easy to see that u is a critical point of J in E if and only if u is a solution of the BVP (4.12).

The eigenvalues of the linear BVP

$$\Delta^2 u(n-1) + \lambda u(n) = 0, \quad n \in \mathbb{Z}(1,T),$$
$$u(0) = 0 = \Delta u(T) \tag{4.16}$$

are

$$\lambda = \lambda_l = 4 \sin^2 \frac{l\pi}{2(2T+1)}, \quad l = 1, 2, \ldots, T, \tag{4.17}$$

and the corresponding eigenfunctions are

$$\phi_l(n) = \sin \frac{l\pi n}{2T+1}, \quad l = 1, 2, \ldots, T. \tag{4.18}$$

Hence, $\lambda_l \neq 0$ for all $l \in \mathbb{Z}(1,T)$ and $\phi_1(n) > 0$ for all $n \in \mathbb{Z}(1,T)$. Therefore, the BVP (4.12) satisfies (1) and (2) and hence we can obtain similar results as in Theorem 1.3.

However, consider the following difference BVP:

$$\Delta^2 u(n-1) + V'(u(n)) = 0, \quad n \in \mathbb{Z}(1,T),$$
$$u(0) = u(T), \qquad \Delta u(0) = \Delta u(T). \tag{4.19}$$

It is easy to verify that the variational functional of the BVP (4.19) is

$$J(u) = \sum_{n=1}^{T} \left[\frac{1}{2}|\Delta u(n)|^2 - V(u(n))\right], \quad \forall u \in E_1, \tag{4.20}$$

where

$$E_1 = \left\{u : u = \{u(n)\}_{n=0}^{T+1} \text{ with } u(0) = u(T), \Delta u(0) = \Delta u(T+1)\right\}. \tag{4.21}$$

But, $\lambda = 0$ is an eigenvalue of the linear BVP:

$$\begin{gathered} \Delta^2 u(n-1) + \lambda u(n) = 0, \quad n \in \mathbb{Z}(1, T), \\ u(0) = u(T), \qquad \Delta u(0) = \Delta u(T). \end{gathered} \tag{4.22}$$

So, for the BVP (4.19), we need to find other techniques (e.g., dual variational methods if possible) to study the BVP (4.19).

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