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Research Article

Generalizations of $(\in, \in \vee q)$-Fuzzy Filters in R_0-Algebras

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Generalizations of a part of the paper (Ma et al., 2009) are considered. As a generalization of an $(\in, \in \vee q)$-fuzzy filter, the notion of an $(\in, \in \vee q_k)$-fuzzy filter is introduced, and its characterizations are provided. The implication-based fuzzy filters of an R_0-algebra are discussed.

1. Introduction

One important task of artificial intelligence is to make the computers simulate beings in dealing with certainty and uncertainty in information. Logic appears in a “sacred” (resp., a “profane”) form which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold—as a tool for applications in both areas, and a technique for laying the foundations. Nonclassical logic including many-valued logic and fuzzy logic takes the advantage of classical logic to handle information with various facets of uncertainty (see [1] for generalized theory of uncertainty), such as fuzziness and randomness. Nonclassical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life. The concept of R_0-algebras was first introduced by Wang in [2] by providing an algebraic proof of the completeness theorem of a formal deductive system [3]. Obviously, R_0-algebras are different from the BL-algebras. Jun and Lianzhen [4] studied filters of R_0-algebras. Lianzhen and Kaitai [5] discussed the fuzzy set theory of filters in R_0-algebras. As a generalization of the notion of fuzzy filters, Ma et al. [6] dealt with the notion of $(\in, \in \vee q)$-fuzzy filters in R_0-algebras.

In this article, we try to get more general form of the notion of $(\in, \in \vee q)$-fuzzy filters. We introduce the notion of $(\in, \in \vee q_k)$-fuzzy filters and investigate related properties. We establish characterizations of an $(\in, \in \vee q_k)$-fuzzy filter and finally consider the implication-based fuzzy filters of an R_0-algebra. The important achievement of the study with an $(\in, \in \vee q_k)$-fuzzy filter is that the notion of an $(\in, \in \vee q)$-fuzzy filter is a special case of an $(\in, \in \vee q_k)$-fuzzy filter, and thus the related results obtained in the paper [6] are a corollary of our results obtained in this paper.

2. Preliminaries

Definition 2.1 (see [2]). Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\to$. Then $(L, \wedge, \vee, \neg, \to)$ is called an R_0-algebra if it satisfies the following axioms:

(R1) $x \to y = \neg y \to \neg x$,

(R2) $1 \to x = x$,

(R3) $(y \to z) \wedge ((x \to y) \to (x \to z)) = y \to z$,

(R4) $x \to (y \to z) = y \to (x \to z)$,

(R5) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(R6) $(x \to y) \vee ((x \to y) \to (\neg x \vee y)) = 1$.

Let L be an R_0-algebra. For any $x, y \in L$, we define $x \odot y = \neg(x \to \neg y)$ and $x \oplus y = \neg x \to y$. It is proved that $\odot$ and $\oplus$ are commutative, associative, and $x \oplus y = \neg(\neg x \odot \neg y)$, and $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice.

Example 2.2 (see [5]). Let $L = [0,1]$. For any $x, y \in L$, we define $x \wedge y = \min\{x, y\}, x \vee y = \max\{x, y\}, \neg x = 1 - x$, and

$$x \longrightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ \neg x \vee y & \text{if } x > y. \end{cases} \tag{2.1}$$

Then $(L, \wedge, \vee, \neg, \to)$ is an R_0-algebra which is neither a BL-algebra nor a lattice implication algebra.

An R_0-algebra has the following useful properties.

Proposition 2.3 (see [7]). *For any elements x, y, and z of an R_0-algebra L, one has the following properties:*

(a1) $x \leq y$ *if and only if* $x \to y = 1$,

(a2) $x \leq y \to x$,

(a3) $\neg x = x \to 0$,

(a4) $(x \to y) \vee (y \to x) = 1$,

(a5) $x \leq y$ *implies* $y \to z \leq x \to z$,

(a6) $x \leq y$ *implies* $z \to x \leq z \to y$, (a7) $((x \to y) \to y) \to y = x \to y$,

(a8) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(a9) $x \odot \neg x = 0$ *and* $x \oplus \neg x = 1$,

(a10) $x \odot y \leq x \wedge y$ *and* $x \odot (x \to y) \leq x \wedge y$,

(a11) $(x \odot y) \to z = x \to (y \to z)$,

(a12) $x \leq y \to (x \odot y)$,

(a13) $x \odot y \leq z$ *if and only if* $x \leq y \to z$,

(a14) $x \leq y$ *implies* $x \odot z \leq y \odot z$,

(a15) $x \to y \leq (y \to z) \to (x \to z)$,

(a16) $(x \to y) \odot (y \to z) \leq x \to z$.

A nonempty subset A of an R_0-algebra L is called a *filter* of L if it satisfies the following two conditions:

(b1) $1 \in A$,

(b2) (for all $x \in A$)(for all $y \in L$)$(x \to y \in A \Rightarrow y \in A)$.

It can be easily verified that a nonempty subset A of an R_0-algebra L is a filter of L if and only if it satisfies the following conditions:

(b3) (for all $x, y \in A$)$(x \odot y \in A)$,

(b4) (for all $y \in L$)$(\exists x \in A)(x \leq y \Rightarrow y \in A)$.

Definition 2.4. A fuzzy set μ in an R_0-algebra L is called a *fuzzy filter* of L if it satisfies the following:

(c1) (for all $x, y \in L$)$(\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})$,

(c2) μ is order-preserving, that is, (for all $x, y \in L$)$(x \leq y \Rightarrow \mu(x) \leq \mu(y))$.

Theorem 2.5. *A fuzzy set μ in an R_0-algebra L is a fuzzy filter of L if and only if it satisfies the following:*

(c3) (*for all* $x \in L$)$(\mu(1) \geq \mu(x))$,

(c4) (*for all* $x, y \in L$)$(\mu(y) \geq \min\{\mu(x \to y), \mu(x)\})$.

For any fuzzy set μ in L and $t \in (0, 1]$, the set

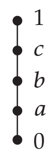
$$U(\mu; t) = \{x \in L \mid \mu(x) \geq t\} \tag{2.2}$$

is called a *level subset* of L. A fuzzy set μ in a set L of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases} \tag{2.3}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x, t).

Table 1: Hasse diagram and Cayley tables.



x	$\neg x$	$\to$	0	a	b	c	1
0	1	0	1	1	1	1	1
a	c	a	c	1	1	1	1
b	b	b	b	b	1	1	1
c	a	c	a	a	b	1	1
1	0	1	0	a	b	c	1

For a fuzzy point (x,t) and a fuzzy set μ in a set L, Pu and Liu [8] introduced the symbol $(x,t)\alpha\mu$, where $\alpha \in \{\in, \mathrm{q}, \in\vee\mathrm{q}, \in\wedge\mathrm{q}\}$. To say that $(x,t)\in\mu$ (resp. $(x,t)\mathrm{q}\mu$), we mean $\mu(x)\geq t$ (resp. $\mu(x)+t>1$), and in this case, (x,t) is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $(x,t)\in\vee\mathrm{q}\,\mu$ (resp. $(x,t)\in\wedge\mathrm{q}\,\mu$), we mean that $(x,t)\in\mu$ or $(x,t)\mathrm{q}\mu$ (resp. $(x,t)\in\mu$ and $(x,t)\mathrm{q}\mu$).

3. Generalizations of $(\in, \in\vee\mathrm{q})$-Fuzzy Filters

In what follows, L is an R_0-algebra and let k denote an arbitrary element of $[0,1)$ unless otherwise specified. To say that $(x,t)\mathrm{q}_k\,\mu$, we mean $\mu(x)+t+k>1$. To say that $(x,t)\in\vee\mathrm{q}_k\mu$, we mean $(x,t)\in\mu$ or $(x,t)\mathrm{q}_k\,\mu$. For $\alpha\in\{\in,\in\vee\mathrm{q}_k\}$, to say that $(x,t)\overline{\alpha}\,\mu$, we mean $(x,t)\alpha\,\mu$ does not hold.

Definition 3.1. A fuzzy set μ in L is said to be an $(\in,\in\vee\mathrm{q}_k)$*-fuzzy filter* of L if it satisfies the following:

(d1) $(x,t)\in\mu \& (y,r)\in\mu \Rightarrow (x\odot y, \min\{t,r\})\in\vee\mathrm{q}_k\mu$,

(d2) $(x,t)\in\mu \& x\leq y \Rightarrow (y,t)\in\vee\mathrm{q}_k\mu$

for all $x,y\in L$ and $t,r\in(0,1]$.

An $(\in,\in\vee\mathrm{q}_k)$-fuzzy filter of L with $k=0$ is called an $(\in,\in\vee\mathrm{q})$*-fuzzy filter* of L.

Example 3.2. Let $L=\{0,a,b,c,1\}$ be a set with Hasse diagram and Cayley tables which are given in Table 1. Then $(L,\wedge,\vee,\neg,\to,0,1)$ is an R_0-algebra (see [5]), where $x\wedge y=\min\{x,y\}$ and $x\vee y=\max\{x,y\}$. Define a fuzzy set μ in L by

$$\mu = \begin{pmatrix} 0 & a & b & c & 1 \\ 0.3 & 0.3 & 0.3 & 0.8 & 0.45 \end{pmatrix}. \tag{3.1}$$

It is routine to verify that μ is an $(\in,\in\vee\mathrm{q}_{0.2})$-fuzzy filter of L. But it is neither a fuzzy filter nor an $(\in,\in\vee\mathrm{q})$-fuzzy filter of L since $\mu(1)=0.45\ngeq 0.8=\mu(c)$, and $c\leq 1$ and $(c,0.5)\in\mu$ but $(1,0.5)\,\overline{\in\vee\mathrm{q}}\mu$.

Theorem 3.3. *Every fuzzy filter is an* $(\in,\in\vee\mathrm{q}_k)$*-fuzzy filter.*

Proof. It is straightforward. □

Example 3.2 shows that the converse of Theorem 3.3 may not be true and shows that an $(\in, \in \vee q_k)$-fuzzy filter may not be an $(\in, \in \vee q)$-fuzzy filter in general.

We establish characterizations of an $(\in, \in \vee q_k)$-fuzzy filter.

Theorem 3.4. *A fuzzy set μ in L is an $(\in, \in \vee q_k)$-fuzzy filter of L if and only if it satisfies the following:*

(d3) (*for all* $x, y \in L)(\mu(x \odot y) \geq \min\{\mu(x), \mu(y), (1-k)/2\})$,

(d4) (*for all* $x, y \in L)(x \leq y \Rightarrow \mu(y) \geq \min\{\mu(x), (1-k)/2\})$.

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy filter of L. Assume that (d3) is not valid. Then there exist $a, b \in L$ such that

$$\mu(a \odot b) < \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\}. \tag{3.2}$$

If $\min\{\mu(a), \mu(b)\} < (1-k)/2$, then $\mu(a \odot b) < \min\{\mu(a), \mu(b)\}$. Hence

$$\mu(a \odot b) < t \leq \min\{\mu(a), \mu(b)\} \tag{3.3}$$

for some $t \in (0, (1-k)/2]$. It follows that $(a, t) \in \mu$ and $(b, t) \in \mu$, but $(a \odot b, t)\, \overline{\in} \mu$. Moreover, $\mu(a \odot b) + t < 2t < 1-k$, and so $(a \odot b, t)\, \overline{q_k} \mu$. Consequently $(a \odot b, t)\, \overline{\in \vee q_k} \mu$, a contradiction. If $\min\{\mu(a), \mu(b)\} \geq (1-k)/2$, then $\mu(a) \geq (1-k)/2, \mu(b) \geq (1-k)/2$ and $\mu(a \odot b) < (1-k)/2$. Thus $(a, (1-k)/2) \in \mu$ and $(b, (1-k)/2) \in \mu$, but $(a \odot b, (1-k)/2)\, \overline{\in} \mu$. Also,

$$\mu(a \odot b) + \frac{1-k}{2} < \frac{1-k}{2} + \frac{1-k}{2} = 1-k, \tag{3.4}$$

that is, $(a \odot b, (1-k)/2)\, \overline{q_k} \mu$. Hence $(a \odot b, (1-k)/2)\, \overline{\in \vee q_k} \mu$, again, a contradiction. Therefore (d3) is valid. Let $x, y \in L$ be such that $x \leq y$. Assume that $\mu(y) < \min\{\mu(x), (1-k)/2\}$. Then

$$\mu(y) < r \leq \min\left\{\mu(x), \frac{1-k}{2}\right\} \tag{3.5}$$

for some $r \in (0, (1-k)/2]$. If $\mu(x) < (1-k)/2$, then $\mu(y) < r \leq \mu(x)$ by (3.5). Hence $(x, r) \in \mu$ and $(y, r)\, \overline{\in} \mu$. Furthermore, $\mu(y) + r < 2r \leq 1-k$, that is, $(y, r)\, \overline{q_k} \mu$. Thus $(y, r)\, \overline{\in \vee q_k} \mu$, a contradiction. If $\mu(x) \geq (1-k)/2$, then $\mu(y) < r \leq (1-k)/2$ by (3.5). Hence $(x, (1-k)/2) \in \mu$ and $(y, (1-k)/2)\, \overline{\in} \mu$. Also, $\mu(y) + (1-k)/2 \leq 1-k$, that is, $(y, (1-k)/2)\, \overline{q_k} \mu$. Thus $(y, (1-k)/2)\, \overline{\in \vee q_k} \mu$ which is also a contradiction. Therefore $\mu(y) \geq \min\{\mu(x), (1-k)/2\}$ for all $x, y \in L$ with $x \leq y$; that is, (d4) is valid.

Conversely, let μ be a fuzzy set in L satisfying two conditions (d3) and (d4). Let $x, y \in L$ and $t, r \in (0, 1]$ be such that $(x, t) \in \mu$ and $(y, r) \in \mu$. Then $\mu(x) \geq t$ and $\mu(y) \geq r$. It follows from (d3) that

$$\begin{aligned}\mu(x \odot y) &\geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \\ &\geq \min\left\{t, r, \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\{t, r\} & \text{if } t \leq \dfrac{1-k}{2} \text{ or } r \leq \dfrac{1-k}{2}, \\ \dfrac{1-k}{2} & \text{if } t > \dfrac{1-k}{2} \text{ and } r > \dfrac{1-k}{2}. \end{cases}\end{aligned} \tag{3.6}$$

The case $\mu(x \odot y) \geq \min\{t, r\}$ implies that $(x \odot y, \min\{t, r\}) \in \mu$. From the case $\mu(x \odot y) \geq (1-k)/2$, we have

$$\mu(x \odot y) + \min\{t, r\} > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k, \tag{3.7}$$

that is, $(x \odot y, \min\{t, r\})\, q_k\, \mu$. Hence $(x \odot y, \min\{t, r\}) \in \vee q_k \mu$. Finally let $x, y \in L$ and $t \in (0, 1]$ be such that $x \leq y$ and $(x, t) \in \mu$. Then $\mu(x) \geq t$, and so

$$\mu(y) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} \tag{3.8}$$

by (d4). If $t \leq (1-k)/2$, then $\mu(y) \geq t$, and thus $(y, t) \in \mu$. If $t > (1-k)/2$, then $\mu(y) \geq (1-k)/2$ which implies that $\mu(y) + t > (1-k)/2 + (1-k)/2 = 1 - k$, that is, $(y, t)\, q_k \mu$. Thus $(y, t) \in \vee q_k \mu$. Consequently, μ is an $(\in, \in \vee q_k)$-fuzzy filter of L. □

If we take $k = 0$ in Theorem 3.4, then we have the following corollary.

Corollary 3.5 (see [6]). *A fuzzy set μ in L is an $(\in, \in \vee q)$-fuzzy filter of L if and only if it satisfies the following:*

(1) *(for all $x, y \in L$)$(\mu(x \odot y) \geq \min\{\mu(x), \mu(y), 0.5\})$,*

(2) *(for all $x, y \in L$)$(x \leq y \Rightarrow \mu(y) \geq \min\{\mu(x), 0.5\})$.*

Theorem 3.6. *A fuzzy set μ in L is an $(\in, \in \vee q_k)$-fuzzy filter of L if and only if it satisfies the following:*

(d5) *(for all $x \in L$)$(\mu(1) \geq \min\{\mu(x), (1-k)/2\})$,*

(d6) *(for all $x, y \in L$)$(\mu(y) \geq \min\{\mu(x), \mu(x \to y), (1-k)/2\})$.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy filter of L. Since $x \leq 1$ for all $x \in L$, it follows from (d4) that $\mu(1) \geq \min\{\mu(x), (1-k)/2\}$ for all $x \in L$. Let $x, y \in L$. Since $x \leq (x \to y) \to y$, we have $x \odot (x \to y) \leq y$ by (a13). Using (d4) and (d3), we obtain

$$\mu(y) \geq \min\left\{\mu(x \odot (x \longrightarrow y)), \frac{1-k}{2}\right\} \geq \min\left\{\mu(x), \mu(x \longrightarrow y), \frac{1-k}{2}\right\}. \tag{3.9}$$

Conversely, let μ be a fuzzy set in L satisfying two conditions (d5) and (d6). Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$, and so

$$\begin{aligned}
\mu(y) &\geq \min\left\{\mu(x), \mu(x \longrightarrow y), \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(x), \mu(1), \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(x), \frac{1-k}{2}\right\}
\end{aligned} \tag{3.10}$$

by (d6) and (d5). Note from (a11) that

$$x \longrightarrow (y \longrightarrow (x \odot y)) = (x \odot y) \longrightarrow (x \odot y) = 1 \tag{3.11}$$

for all $x, y \in L$. It follows from (d5) and (d6) that

$$\begin{aligned}
\mu(x \odot y) &\geq \min\left\{\mu(y), \mu(y \longrightarrow (x \odot y)), \frac{1-k}{2}\right\} \\
&\geq \min\left\{\mu(y), \min\left\{\mu(x), \mu(x \longrightarrow (y \longrightarrow (x \odot y))), \frac{1-k}{2}\right\}, \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(y), \min\left\{\mu(x), \mu(1), \frac{1-k}{2}\right\}, \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\}.
\end{aligned} \tag{3.12}$$

Using Theorem 3.4, we conclude that μ is an $(\in, \in \vee q_k)$-fuzzy filter of L. □

Corollary 3.7 (see [6]). *A fuzzy set μ in L is an $(\in, \in \vee q)$-fuzzy filter of L if and only if it satisfies the following:*

(1) (*for all* $x \in L$)$(\mu(1) \geq \min\{\mu(x), 0.5\})$,

(2) (*for all* $x, y \in L$)$(\mu(y) \geq \min\{\mu(x), \mu(x \to y), 0.5\})$.

Proof. It is straightforward by taking $k = 0$ in Theorem 3.6. □

Corollary 3.8. *If μ is an $(\in, \in \vee q_k)$-fuzzy filter of L with $\mu(1) < (1-k)/2$, then μ is a fuzzy filter.*

If we take $k = 0$ in Corollary 3.8, then we obtain the following corollary.

Corollary 3.9 (see [6]). *If μ is an $(\in, \in \vee \mathrm{q})$-fuzzy filter of L with $\mu(1) < 0.5$, then μ is a fuzzy filter.*

Theorem 3.10. *A fuzzy set μ in L is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L if and only if it satisfies the following:*

(d7) (*for all* $x, y, z \in L)(x \leq y \to z \Rightarrow \mu(z) \geq \min\{\mu(x), \mu(y), (1-k)/2\})$.

Proof. Assume that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L. Let $x, y, z \in L$ be such that $x \leq y \to z$. It follows from (d4) that $\mu(y \to z) \geq \min\{\mu(x), (1-k)/2\}$ and so from (d6) that

$$\mu(z) \geq \min\left\{\mu(y), \mu(y \longrightarrow z), \frac{1-k}{2}\right\} \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\}. \tag{3.13}$$

Conversely, let μ be a fuzzy set in L satisfying (d7). Since $x \leq x \to 1$ for all $x \in L$, we have $\mu(1) \geq \min\{\mu(x), (1-k)/2\}$ by (d7). Note that $x \to y \leq x \to y$ for all $x, y \in L$. It follows from (d7) that

$$\mu(y) \geq \min\left\{\mu(x), \mu(x \longrightarrow y), \frac{1-k}{2}\right\}. \tag{3.14}$$

Using Theorem 3.6, we conclude that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L. □

Corollary 3.11 (see [6]). *A fuzzy set μ in L is an $(\in, \in \vee \mathrm{q})$-fuzzy filter of L if and only if it satisfies the following:*

$$(\textit{for all } x, y, z \in L) \quad (x \leq y \longrightarrow z \Longrightarrow \mu(z) \geq \min\{\mu(x), \mu(y), 0.5\}). \tag{3.15}$$

Proof. It is obvious by taking $k = 0$ in Theorem 3.10. □

Theorem 3.12. *For an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter μ of L, the followings are equivalent:*

(1) (*for all* $x, y, z \in L)(\mu(x \to z) \geq \min\{\mu(x \to (y \to z)), \mu(x \to y), (1-k)/2\})$,

(2) (*for all* $x, y \in L)(\mu(x \to y) \geq \min\{\mu(x \to (x \to y)), (1-k)/2\})$,

(3) (*for all* $x, y, z \in L)(\mu((x \to y) \to (x \to z)) \geq \min\{\mu(x \to (y \to z)), (1-k)/2\})$.

Proof. (1)⇒(2): Suppose that μ satisfies the condition (1). If we take $z = y$ and $y = x$ in (1), then

$$\begin{aligned}\mu(x \longrightarrow y) &\geq \min\left\{\mu(x \longrightarrow (x \longrightarrow y)), \mu(x \longrightarrow x), \frac{1-k}{2}\right\} \\ &= \min\left\{\mu(x \longrightarrow (x \longrightarrow y)), \mu(1), \frac{1-k}{2}\right\} \\ &= \min\left\{\mu(x \longrightarrow (x \longrightarrow y)), \frac{1-k}{2}\right\}\end{aligned} \tag{3.16}$$

for all $x, y \in L$ by (d5).

(2)⇒(3): Assume that μ satisfies the condition (2). Note that

$$x \longrightarrow (y \longrightarrow z) \leq x \longrightarrow ((x \longrightarrow y) \longrightarrow (x \longrightarrow z)) \tag{3.17}$$

for all $x, y, z \in L$. It follows from (R4), (2), and (d4) that

$$\begin{aligned}
\mu((x \longrightarrow y) \longrightarrow (x \longrightarrow z)) &= \mu(x \longrightarrow ((x \longrightarrow y) \longrightarrow z)) \\
&\geq \min\left\{\mu(x \longrightarrow (x \longrightarrow ((x \longrightarrow y) \longrightarrow z))), \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(x \longrightarrow ((x \longrightarrow y) \longrightarrow (x \longrightarrow z))), \frac{1-k}{2}\right\} \\
&\geq \min\left\{\min\left\{\mu(x \longrightarrow (y \longrightarrow z)), \frac{1-k}{2}\right\}, \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(x \longrightarrow (y \longrightarrow z)), \frac{1-k}{2}\right\}
\end{aligned} \tag{3.18}$$

for all $x, y, z \in L$.

(3)⇒(1): Suppose that μ satisfies the condition (3). Using (d6), we have

$$\begin{aligned}
\mu(x \longrightarrow z) &\geq \min\left\{\mu(x \longrightarrow y), \mu((x \longrightarrow y) \longrightarrow (x \longrightarrow z)), \frac{1-k}{2}\right\} \\
&\geq \min\left\{\mu(x \longrightarrow y), \min\left\{\mu(x \longrightarrow (y \longrightarrow z)), \frac{1-k}{2}\right\}, \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(x \longrightarrow y), \mu(x \longrightarrow (y \longrightarrow z)), \frac{1-k}{2}\right\}
\end{aligned} \tag{3.19}$$

for all $x, y, z \in L$. □

Corollary 3.13 (see [6]). *For an* $(\in, \in \vee \mathrm{q})$*-fuzzy filter* μ *of* L*, the followings are equivalent:*

(1) *(for all* $x, y, z \in L)(\mu(x \to z) \geq \min\{\mu(x \to (y \to z)), \mu(x \to y), 0.5\})$,

(2) *(for all* $x, y \in L)(\mu(x \to y) \geq \min\{\mu(x \to (x \to y)), 0.5\})$,

(3) *(for all* $x, y, z \in L)(\mu((x \to y) \to (x \to z)) \geq \min\{\mu(x \to (y \to z)), 0.5\})$.

Theorem 3.14. *A fuzzy set* μ *in* L *is an* $(\in, \in \vee \mathrm{q}_k)$*-fuzzy filter of* L *if and only if it satisfies the following:*

$$\left(\forall t \in \left(0, \frac{1-k}{2}\right]\right)(U(\mu; t) \neq \emptyset \Longrightarrow U(\mu; t) \text{ is a filter of } L). \tag{3.20}$$

Proof. Assume that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L. Let $t \in (0, (1-k)/2]$ be such that $U(\mu; t) \neq \emptyset$. Let $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Hence $\mu(x \odot y) \geq \min\{\mu(x), \mu(y), (1-k)/2\} \geq \min\{t, (1-k)/2\} = t$ by (d3), and so $x \odot y \in U(\mu; t)$. Let $x, y \in L$ be such that $x \leq y$.

If $x \in U(\mu; t)$, then

$$\mu(y) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t \tag{3.21}$$

by (d4). This shows that $y \in U(\mu; t)$. Therefore $U(\mu; t)$ is a filter of L.

Conversely, let μ be a fuzzy set in L satisfying (3.20). If there exist $a, b \in L$ such that $\mu(a \odot b) < \min\{\mu(a), \mu(b), (1-k)/2\}$, then

$$\mu(a \odot b) < t \leq \min\left\{\mu(a), \mu(b), \frac{1-k}{2}\right\} \tag{3.22}$$

for some $t \in (0, (1-k)/2]$. Thus $a, b \in U(\mu; t)$ and $a \odot b \notin U(\mu; t)$, which is a contradiction. Hence $\mu(x \odot y) \geq \min\{\mu(x), \mu(y), (1-k)/2\}$ for all $x, y \in L$. Let $x, y \in L$ be such that $x \leq y$. If $\mu(y) < \min\{\mu(x), (1-k)/2\}$, then

$$\mu(y) < r \leq \min\left\{\mu(x), \frac{1-k}{2}\right\} \tag{3.23}$$

for some $r \in (0, (1-k)/2]$. It follows that $x \in U(\mu; r)$ and $y \notin U(\mu; r)$, a contradiction. Therefore $\mu(y) \geq \min\{\mu(x), (1-k)/2\}$ for all $x, y \in L$ with $x \leq y$. Using Theorem 3.4, we conclude that μ is an $(\in, \in \vee q_k)$-fuzzy filter of L. □

Corollary 3.15 (see [6]). *A fuzzy set μ in L is an $(\in, \in \vee q)$-fuzzy filter of L if and only if it satisfies the following:*

$$(\forall t \in (0, 0.5])\left(U(\mu; t) \neq \emptyset \Longrightarrow U(\mu; t) \text{ is a filter of } L\right). \tag{3.24}$$

Theorem 3.16. *If A is a filter of L, then a fuzzy set μ in L such that $\mu(x) = t_1$ for $x \in A$ and $\mu(x) = t_2$ otherwise, where $t_1 \in [(1-k)/2, 1]$ and $t_2 \in (0, (1-k)/2)$, is an $(\in, \in \vee q_k)$-fuzzy filter of L.*

Proof. Note that

$$U(\mu; r) = \begin{cases} A & \text{if } r \in \left(t_2, \dfrac{1-k}{2}\right], \\ L & \text{if } r \in (0, t_2], \end{cases} \tag{3.25}$$

which is a filter of L. It follows from Theorem 3.14 that μ is an $(\in, \in \vee q_k)$-fuzzy filter of L. □

Corollary 3.17. *If A is a filter of L, then a fuzzy set μ in L such that $\mu(x) = t_1$ for $x \in A$ and $\mu(x) = t_2$ otherwise, where $t_1 \in [0.5, 1]$ and $t_2 \in (0, 0.5)$, is an $(\in, \in \vee q)$-fuzzy filter of L.*

For any fuzzy set μ in L and any $t \in (0,1]$, we consider four subsets:

$$Q(\mu;t) := \{x \in L \mid (x,t)\mathrm{q}\mu\},\ [\mu]_t := \{x \in L \mid (x,t) \in \vee\mathrm{q}\mu\},$$
$$Q^k(\mu;t) := \{x \in L \mid (x,t)\mathrm{q}_k\mu\},\ [\mu]_t^k := \{x \in L \mid (x,t) \in \vee\mathrm{q}_k\mu\}.$$

It is clear that $[\mu]_t = U(\mu;t) \cup Q(\mu;t)$ and $[\mu]_t^k = U(\mu;t) \cup Q^k(\mu;t)$.

Theorem 3.18. *If μ is an $(\in, \in \vee\mathrm{q}_k)$-fuzzy filter of L, then*

$$\left(\forall t \in \left(\frac{1-k}{2}, 1\right]\right)\left(Q^k(\mu;t) \neq \emptyset \Longrightarrow Q^k(\mu;t) \text{ is a filter of } L\right). \tag{3.26}$$

Proof. Assume that μ is an $(\in, \in \vee\mathrm{q}_k)$-fuzzy filter of L and let $t \in ((1-k)/2, 1]$ be such that $Q^k(\mu;t) \neq \emptyset$. Let $x, y \in Q^k(\mu;t)$. Then $(x,t)\mathrm{q}_k\mu$ and $(y,t)\mathrm{q}_k\mu$, that is, $\mu(x) + t + k > 1$ and $\mu(y) + t + k > 1$. Using (d3), we have

$$\begin{aligned} \mu(x \odot y) &\geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\{\mu(x), \mu(y)\} & \text{if } \min\{\mu(x), \mu(y)\} < \dfrac{1-k}{2}, \\ \dfrac{1-k}{2} & \text{if } \min\{\mu(x), \mu(y)\} \geq \dfrac{1-k}{2} \end{cases} \\ &> 1 - t - k, \end{aligned} \tag{3.27}$$

that is, $(x \odot y, t)\mathrm{q}_k\mu$. Hence $x \odot y \in Q^k(\mu;t)$. Let $x, y \in L$ be such that $x \leq y$. If $x \in Q^k(\mu;t)$, then $(x,t)\mathrm{q}_k\mu$, that is, $\mu(x) + t + k > 1$. It follows from (d4) that

$$\begin{aligned} \mu(y) &\geq \min\left\{\mu(x), \frac{1-k}{2}\right\} \\ &= \begin{cases} \dfrac{1-k}{2} & \text{if } \mu(x) \geq \dfrac{1-k}{2}, \\ \mu(x) & \text{if } \mu(x) < \dfrac{1-k}{2}, \end{cases} \\ &> 1 - t - k, \end{aligned} \tag{3.28}$$

that is, $(y,t)\mathrm{q}_k\mu$. Hence $y \in Q^k(\mu;t)$. Therefore $Q^k(\mu;t)$ is a filter of L. □

Corollary 3.19. *If μ is an $(\in, \in \vee\mathrm{q})$-fuzzy filter of L, then*

$$(\forall t \in (0.5, 1])(Q(\mu;t) \neq \emptyset \Longrightarrow Q(\mu;t) \text{ is a filter of } L). \tag{3.29}$$

Proof. It is clear by taking $k = 0$ in Theorem 3.18. □

Theorem 3.20. *For any fuzzy set μ in L, the followings are equivalent:*

(1) *μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L,*

(2) *(for all $t \in (0,1]$)($[\mu]_t^k \neq \emptyset \Rightarrow [\mu]_t^k$ is a filter of L).*

Proof. Assume that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L and let $t \in (0,1]$ be such that $[\mu]_t^k \neq \emptyset$. Let $x, y \in [\mu]_t^k$. Then $\mu(x) \geq t$ or $\mu(x) + t + k > 1$, and $\mu(y) \geq t$ or $\mu(y) + t + k > 1$. We can consider four cases:

$$\mu(x) \geq t, \qquad \mu(y) \geq t, \tag{3.30}$$

$$\mu(x) \geq t, \qquad \mu(y) + t + k > 1, \tag{3.31}$$

$$\mu(x) + t + k > 1, \qquad \mu(y) \geq t, \tag{3.32}$$

$$\mu(x) + t + k > 1, \qquad \mu(y) + t + k > 1. \tag{3.33}$$

For the first case, (d3) implies that

$$\mu(x \odot y) \geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = \begin{cases} \dfrac{1-k}{2} & \text{if } t > \dfrac{1-k}{2}, \\ t & \text{if } t \leq \dfrac{1-k}{2}, \end{cases} \tag{3.34}$$

and so $\mu(x \odot y) + t + k > (1-k)/2 + (1-k)/2 + k = 1$, that is, $(x \odot y)_t \mathrm{q}_k \mu$, or $x \odot y \in U(\mu; t)$. Therefore $x \odot y \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. For the case (3.31), assume that $t > (1-k)/2$. Then $1 - t - k \leq 1 - t < (1-k)/2$, and so

$$\begin{aligned} \mu(x \odot y) &\geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\left\{\mu(y), \dfrac{1-k}{2}\right\} > 1 - t - k & \text{if } \min\left\{\mu(y), \dfrac{1-k}{2}\right\} \leq \mu(x), \\ \mu(x) \geq t & \text{if } \min\left\{\mu(y), \dfrac{1-k}{2}\right\} > \mu(x). \end{cases} \end{aligned} \tag{3.35}$$

Thus $x \odot y \in U(\mu; t) \cup Q^k(\mu; t) = [\mu]_t^k$. Suppose that $t \leq (1-k)/2$. Then $1 - t \geq (1-k)/2$, which implies that

$$\begin{aligned} \mu(x \odot y) &\geq \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} \\ &= \begin{cases} \min\left\{\mu(x), \dfrac{1-k}{2}\right\} \geq t & \text{if } \min\left\{\mu(x), \dfrac{1-k}{2}\right\} \leq \mu(y), \\ \mu(y) > 1 - t - k & \text{if } \min\left\{\mu(x), \dfrac{1-k}{2}\right\} > \mu(y), \end{cases} \end{aligned} \tag{3.36}$$

and thus $x \odot y \in U(\mu;t) \cup Q^k(\mu;t) = [\mu]_t^k$. We have similar result for the case (3.32). For the final case, if $t > (1-k)/2$, then $1-t-k \le 1-t < (1-k)/2$. Hence

$$\mu(x \odot y) \ge \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} = \begin{cases} \dfrac{1-k}{2} > 1-t-k & \text{if } \min\{\mu(x), \mu(y)\} \ge \dfrac{1-k}{2}, \\ \min\{\mu(x), \mu(y)\} > 1-t-k & \text{if } \min\{\mu(x), \mu(y)\} < \dfrac{1-k}{2}, \end{cases} \tag{3.37}$$

and so $x \odot y \in Q^k(\mu;t) \subseteq [\mu]_t^k$. If $t \le (1-k)/2$, then

$$\mu(x \odot y) \ge \min\left\{\mu(x), \mu(y), \frac{1-k}{2}\right\} = \begin{cases} \dfrac{1-k}{2} \ge t & \text{if } \min\{\mu(x), \mu(y)\} \ge \dfrac{1-k}{2}, \\ \min\{\mu(x), \mu(y)\} > 1-t-k & \text{if } \min\{\mu(x), \mu(y)\} < \dfrac{1-k}{2}, \end{cases} \tag{3.38}$$

which implies that $x \odot y \in U(\mu;t) \cup Q^k(\mu;t) = [\mu]_t^k$. Let $x, y \in L$ be such that $x \le y$. If $x \in [\mu]_t^k$, then $\mu(x) \ge t$ or $\mu(x) + t + k > 1$. Assume that $\mu(x) \ge t$. (d4) implies that

$$\mu(y) \ge \min\left\{\mu(x), \frac{1-k}{2}\right\} \ge \min\left\{t, \frac{1-k}{2}\right\} = \begin{cases} t & \text{if } t \le \dfrac{1-k}{2}, \\ \dfrac{1-k}{2} > 1-t-k & \text{if } t > \dfrac{1-k}{2} \end{cases} \tag{3.39}$$

so that $y \in U(\mu;t) \cup Q^k(\mu;t) = [\mu]_t^k$. Suppose that $\mu(x) + t + k > 1$. If $t > (1-k)/2$, then

$$\mu(y) \ge \min\left\{\mu(x), \frac{1-k}{2}\right\} = \begin{cases} \dfrac{1-k}{2} > 1-t-k & \text{if } \mu(x) \ge \dfrac{1-k}{2}, \\ \mu(x) > 1-t-k & \text{if } \mu(x) < \dfrac{1-k}{2}, \end{cases} \tag{3.40}$$

and thus $y \in Q^k(\mu;t) \subseteq [\mu]_t^k$. If $t \le (1-k)/2$, then

$$\mu(y) \ge \min\left\{\mu(x), \frac{1-k}{2}\right\} = \begin{cases} \dfrac{1-k}{2} \ge t & \text{if } \mu(x) \ge \dfrac{1-k}{2}, \\ \mu(x) > 1-t-k & \text{if } \mu(x) < \dfrac{1-k}{2}, \end{cases} \tag{3.41}$$

which implies that $y \in U(\mu;t) \cup Q^k(\mu;t) = [\mu]_t^k$. Therefore $[\mu]_t^k$ is a filter of L.

Conversely, let μ be a fuzzy set in L such that $[\mu]_t^k$ is a filter of L whenever it is nonempty for all $t \in (0,1]$. If there exists $a \in L$ such that $\mu(1) < \min\{\mu(a),(1-k)/2\}$, then $\mu(1) < t_a \leq \min\{\mu(a),(1-k)/2\}$ for some $t_a \in (0,(1-k)/2]$. It follows that $a \in U(\mu;t_a) \subseteq [\mu]_{t_a}^k$ but $1 \notin U(\mu;t_a)$. Also, $\mu(1)+t_a < 2t_a \leq 1-k$, and so $(1,t_a)\overline{\mathrm{q}_k}\mu$, that is, $1 \notin Q^k(\mu;t_a)$. Therefore $1 \notin [\mu]_{t_a}^k$, a contradiction. Hence $\mu(1) \geq \min\{\mu(x),(1-k)/2\}$ for all $x \in L$. Suppose that there exist $a,b \in L$ such that $\mu(b) < \min\{\mu(a),\mu(a \to b),(1-k)/2\}$. Then

$$\mu(b) < t_b \leq \min\left\{\mu(a),\mu(a \longrightarrow b),\frac{1-k}{2}\right\} \tag{3.42}$$

for some $t_b \in (0,(1-k)/2]$, which implies that $a, a \to b \in U(\mu;t_b) \subseteq [\mu]_{t_b}^k$ and so from (b2) that $b \in [\mu]_{t_b}^k = U(\mu;t_b) \cup Q^k(\mu;t_b)$ since $[\mu]_{t_b}^k$ is a filter of L. But, (3.42) implies that $b \notin U(\mu;t_b)$ and $\mu(b)+t_b < 2t_b \leq 1-k$, that is, $b \notin Q^k(\mu;t_b)$. This is a contradiction, and therefore $\mu(y) \geq \min\{\mu(x),\mu(x \to y),(1-k)/2\}$ for all $x,y \in L$. Using Theorem 3.6, we conclude that μ is an $(\in,\in\vee\mathrm{q}_k)$-fuzzy filter of L. □

If we take $k=0$ in Theorem 3.20, then we have the following corollary.

Corollary 3.21. *For any fuzzy set μ in L, the followings are equivalent:*

(1) *μ is an $(\in,\in\vee\mathrm{q})$-fuzzy filter of L,*

(2) *(for all $t \in (0,1]$)($[\mu]_t \neq \emptyset \Rightarrow [\mu]_t$ is a filter of L).*

4. Implication-Based Fuzzy Filters

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example, $\wedge,\vee,\neg,\to$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper:

$$[x \in \mu] = \mu(x), \tag{4.1}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi],[\Psi]\}, \tag{4.2}$$

$$[\Phi \longrightarrow \Psi] = \min\{1, 1-[\Phi]+[\Psi]\}, \tag{4.3}$$

$$[\forall x \Phi(x)] = \inf_{x \in U}[\Phi(x)], \tag{4.4}$$

$$\models \Phi \quad \text{iff } [\Phi] = 1 \text{ for all valuations.} \tag{4.5}$$

The truth valuation rules given in (4.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}) is

$$I_{\mathrm{GR}}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 0 & \text{otherwise}. \end{cases} \tag{4.6}$$

(b) Gödel implication operator (I_{G}) is

$$I_{\mathrm{G}}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ b & \text{otherwise}. \end{cases} \tag{4.7}$$

(c) The contraposition of Gödel implication operator (I_{cG}) is

$$I_{\mathrm{cG}}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 1-a & \text{otherwise}. \end{cases} \tag{4.8}$$

Ying [9] introduced the concept of fuzzifying topology. We can expand his/her idea to R_0-algebras, and we define a fuzzifying filter as follows.

Definition 4.1. A fuzzy subset μ of L is called a *fuzzifying filter* of L if it satisfies the following conditions:

(1) for all $x \in L$, we have

$$\vDash [x \in \mu] \longrightarrow [1 \in \mu]; \tag{4.9}$$

(2) for all $x, y \in R$, we get

$$\vDash [x \in \mu] \wedge [x \longrightarrow y \in \mu] \longrightarrow [y \in \mu]. \tag{4.10}$$

Obviously, conditions (4.9) and (4.10) are equivalent to (c3) and (c4), respectively. Therefore a fuzzifying filter is an ordinary fuzzy filter.

In [10], the concept of t-tautology is introduced, that is,

$$\vDash_t \Phi \quad \text{iff } [\Phi] \ge t \text{ for all valuations}. \tag{4.11}$$

Definition 4.2. Let μ be a fuzzy set in L and $t \in (0,1]$. Then μ is called a *t-implication-based fuzzy filter* of L if it satisfies the following conditions:

(1) for all $x \in L$, we have

$$\vDash_t [x \in \mu] \longrightarrow [1 \in \mu], \tag{4.12}$$

(2) for all $x, y \in R$, we get

$$\vDash_t [x \in \mu] \wedge [x \longrightarrow y \in \mu] \longrightarrow [y \in \mu]. \tag{4.13}$$

Let I be an implication operator. Clearly, μ is a t-implication-based fuzzy filter of L if and only if it satisfies the following:

(1) (for all $x \in L)(I(\mu(x), \mu(1)) \geq t)$,

(2) (for all $x, y \in L)(I(\min\{\mu(x), \mu(x \to y)\}, \mu(y)) \geq t)$.

Theorem 4.3. *For any fuzzy set μ in L, one has the following.*

(1) *If $I = I_{\mathrm{GR}}$, then μ is a* 0.5*-implication-based fuzzy filter of L if and only if μ is a fuzzy filter of L.*

(2) *If $I = I_{\mathrm{G}}$, then μ is a $((1-k)/2)$-implication-based fuzzy filter of L if and only if μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L.*

(3) *If $I = I_{\mathrm{cG}}$, then μ is a $((1-k)/2)$-implication-based fuzzy filter of L if and only if μ satisfies the following conditions:*

(3.1) (*for all* $x \in L)(\max\{\mu(1), (1-k)/2\} \geq \min\{\mu(x), 1\})$.

(3.2) (*for all* $x, y \in L)(\max\{\mu(y), (1-k)/2\} \geq \min\{\mu(x), \mu(x \to y), 1\})$.

Proof. (1) It is straightforward.

(2) Assume that μ is a $((1-k)/2)$-implication-based fuzzy filter of L. Then

(i) (for all $x \in L)(I_{\mathrm{G}}(\mu(x), \mu(1)) \geq (1-k)/2)$,

(ii) (for all $x, y \in L)(I_{\mathrm{G}}(\min\{\mu(x), \mu(x \to y)\}, \mu(y)) \geq (1-k)/2)$.

From (i), we have $\mu(1) \geq \mu(x)$ or $\mu(x) \geq \mu(1) \geq (1-k)/2$, and so $\mu(1) \geq \min\{\mu(x), (1-k)/2\}$. The second case implies that $\mu(y) \geq \min\{\mu(x), \mu(x \to y)\}$ or $\min\{\mu(x), \mu(x \to y)\} > \mu(y) \geq (1-k)/2$. It follows that

$$\mu(y) \geq \min\left\{\mu(x), \mu(x \longrightarrow y), \frac{1-k}{2}\right\}. \tag{4.14}$$

Using Theorem 3.6, we know that μ is an $(\in, \in \vee \mathrm{q}_k)$-fuzzy filter of L.

Conversely, suppose that μ is an $(\in, \in \vee q_k)$-fuzzy filter of L. From (d5), if $\min\{\mu(x), (1-k)/2\} = \mu(x)$, then $I_G(\mu(x), \mu(1)) = 1 \geq (1-k)/2$. Otherwise, $I_G(\mu(x), \mu(1)) \geq (1-k)/2$. From (d6), if

$$\min\left\{\mu(x), \mu(x \longrightarrow y), \frac{1-k}{2}\right\} = \min\{\mu(x), \mu(x \longrightarrow y)\}, \tag{4.15}$$

then $\mu(y) \geq \min\{\mu(x), \mu(x \to y)\}$ and so

$$I_G(\min\{\mu(x), \mu(x \longrightarrow y)\}, \mu(y)) = 1 \geq \frac{1-k}{2}. \tag{4.16}$$

If $\min\{\mu(x), \mu(x \to y), (1-k)/2\} = (1-k)/2$, then $\mu(y) \geq (1-k)/2$ and thus

$$I_G(\min\{\mu(x), \mu(x \longrightarrow y)\}, \mu(y)) \geq \frac{1-k}{2}. \tag{4.17}$$

Consequently, μ is a $((1-k)/2)$-implication-based fuzzy filter of L.

(3) Suppose that μ satisfies (3.5) and (3.20). In (3.5), if $\mu(x) = 1$, then $\max\{\mu(1), (1-k)/2\} = 1$ and hence $I_{cG}(\mu(x), \mu(1)) = 1 \geq (1-k)/2$. If $\mu(x) < 1$, then

$$\max\left\{\mu(1), \frac{1-k}{2}\right\} \geq \mu(x). \tag{4.18}$$

If $\max\{\mu(1), (1-k)/2\} = \mu(1)$ in (4.18), then $\mu(1) \geq \mu(x)$. Hence

$$I_{cG}(\mu(x), \mu(1)) = 1 \geq \frac{1-k}{2}. \tag{4.19}$$

If $\max\{\mu(1), (1-k)/2\} = (1-k)/2$ in (4.18), then $\mu(x) \leq (1-k)/2$ which implies that

$$I_{cG}(\mu(x), \mu(1)) = \begin{cases} 1 \geq \dfrac{1-k}{2} & \text{if } \mu(1) \geq \mu(x), \\ 1 - \mu(x) \geq \dfrac{1-k}{2} & \text{otherwise.} \end{cases} \tag{4.20}$$

In (3.20), if $\min\{\mu(x), \mu(x \to y), 1\} = 1$, then $\max\{\mu(y), (1-k)/2\} = 1$ and so $\mu(y) = 1 \geq \min\{\mu(x), \mu(x \to y)\}$. Therefore

$$I_{cG}(\min\{\mu(x), \mu(x \longrightarrow y)\}, \mu(y)) = 1 \geq \frac{1-k}{2}. \tag{4.21}$$

If $\min\{\mu(x), \mu(x \to y), 1\} = \min\{\mu(x), \mu(x \to y)\}$, then

$$\max\left\{\mu(y), \frac{1-k}{2}\right\} \geq \min\{\mu(x), \mu(x \longrightarrow y)\}. \tag{4.22}$$

Thus, if $\max\{\mu(y),(1-k)/2\}=(1-k)/2$ in (4.22), then $\mu(y)\leq(1-k)/2$ and

$$\min\{\mu(x),\mu(x\longrightarrow y)\}\leq\frac{1-k}{2}. \tag{4.23}$$

Therefore

$$I_{cG}(\min\{\mu(x),\mu(x\longrightarrow y)\},\mu(y)) = \begin{cases} 1\geq\dfrac{1-k}{2} & \text{if } \mu(y)\geq\min\{\mu(x),\mu(x\longrightarrow y)\}, \\ 1-\min\{\mu(x),\mu(x\longrightarrow y)\}\geq\dfrac{1-k}{2} & \text{otherwise.} \end{cases} \tag{4.24}$$

If $\max\{\mu(y),(1-k)/2\}=\mu(y)$ in (4.22), then $\mu(y)\geq\min\{\mu(x),\mu(x\to y)\}$ and so

$$I_{cG}(\min\{\mu(x),\mu(x\longrightarrow y)\},\mu(y))=1\geq\frac{1-k}{2}. \tag{4.25}$$

Consequently, μ is a $((1-k)/2)$-implication-based fuzzy filter of L.

Conversely assume that μ is a $((1-k)/2)$-implication-based fuzzy filter of L. Then

(iii) (for all $x\in L$) $(I_{cG}(\mu(x),\mu(1))\geq(1-k)/2)$,

(iv) (for all $x,y\in L$) $(I_{cG}(\min\{\mu(x),\mu(x\to y)\},\mu(y))\geq(1-k)/2)$.

The case (iii) implies that $I_{cG}(\mu(x),\mu(1))=1$, that is, $\mu(x)\leq\mu(1)$, or $1-\mu(x)\geq(1-k)/2$ and so $\mu(x)\leq(1-k)/2$. It follows that

$$\max\left\{\mu(1),\frac{1-k}{2}\right\}\geq\mu(x)=\min\{\mu(x),1\}. \tag{4.26}$$

From (iv), we have

$$I_{cG}(\min\{\mu(x),\mu(x\longrightarrow y)\},\mu(y))=1, \tag{4.27}$$

that is, $\min\{\mu(x),\mu(x\to y)\}\leq\mu(y)$, or $1-\min\{\mu(x),\mu(x\to y)\}\geq(1-k)/2$. Hence

$$\max\left\{\mu(y),\frac{1-k}{2}\right\}\geq\min\{\mu(x),\mu(x\longrightarrow y)\}=\min\{\mu(x),\mu(x\longrightarrow y),1\}. \tag{4.28}$$

This completes the proof. □

Corollary 4.4. *For any fuzzy set μ in L, one has the following.*

(1) *If $I = I_G$, then μ is a* 0.5*-implication-based fuzzy filter of L if and only if μ is an* $(\in, \in \vee q_k)$*-fuzzy filter of L.*

(2) *If $I = I_{cG}$, then μ is a* 0.5*-implication-based fuzzy filter of L if and only if μ satisfies the following conditions:*

(2.1) (*for all* $x \in L$) $(\max\{\mu(1), 0.5\} \geq \min\{\mu(x), 1\})$.

(2.2) (*for all* $x, y \in L$) $(\max\{\mu(y), 0.5\} \geq \min\{\mu(x), \mu(x \to y), 1\})$.

5. Conclusions

We introduced the notion of an $(\in, \in \vee q_k)$-fuzzy filter which is a generalization of an $(\in, \in \vee q)$-fuzzy filter in R_0-algebras. We investigated related properties and provided characterizations of an $(\in, \in \vee q_k)$-fuzzy filter in R_0-algebras. We discussed the implication-based fuzzy filters of an R_0-algebra. Using our notions/results, we know that the related results in the paper [6] are induced from the notions/results in this paper.

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