Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 943565, 9 pages
doi:10.1155/2010/943565

Research Article

$\mathcal{N}$-Structures Applied to Closed Ideals in BCH-Algebras

Young Bae Jun,[1] Mehmet Ali Öztürk,[2] and Eun Hwan Roh[3]

[1] *Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, South Korea*
[2] *Department of Mathematics, Faculty of Arts and Sciences, Adiyaman University, 02040 Adiyaman, Turkey*
[3] *Department of Mathematics Education (and IME), Chinju National University of Education, Chinju 660-756, South Korea*

Correspondence should be addressed to Eun Hwan Roh, ehroh@cue.ac.kr

Received 28 December 2009; Revised 2 February 2010; Accepted 9 February 2010

Academic Editor: Ilya M. Spitkovsky

The notions of $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals in BCH-algebras are introduced, and the relation between $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals is considered. Characterizations of $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals are provided. Using special subsets, $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals are constructed. A condition for an $\mathcal{N}$-subalgebra to be an $\mathcal{N}$-closed ideal is discussed. Given an $\mathcal{N}$-structure, the greatest $\mathcal{N}$-closed ideal which is contained in the $\mathcal{N}$-structure is established.

1. Introduction

In [1, 2], Hu and Li introduced the notion of BCH-algebras which are a generalization of BCK/BCI-algebras. Ahmad [3] classified BCH-algebras, and decompositions of BCH-algebras are considered by Dudek and Thomys [4]. Jun et al. [5] discussed the notion of $\mathcal{N}$-structures and applied it to BCK/BCI-algebras. In [6], Chaudhry et al. studied closed ideals and filters in BCH-algebras. In this paper, we apply the $\mathcal{N}$-structures to the closed ideal theory in BCH-algebras. We introduced the notion of $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals in BCH-algebras, and investigate the relation between $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals. We provide characterizations of $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals. Using special subsets, we construct $\mathcal{N}$-subalgebras and $\mathcal{N}$-closed ideals. We provide a condition for an $\mathcal{N}$-subalgebra to be an $\mathcal{N}$-closed ideal. Given an $\mathcal{N}$-structure (X, μ), we make the greatest $\mathcal{N}$-closed ideal which is contained in (X, μ).

2. Preliminaries

By a *BCH-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(H1) $x * x = 0$,

(H2) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(H3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

In a BCH-algebra X, the following conditions are valid (see [1, 4]).

(a1) $x * 0 = x$,

(a2) $x * 0 = 0$ implies $x = 0$,

(a3) $0 * (x * y) = (0 * x) * (0 * y)$,

(a4) $0 * (0 * (0 * x)) = 0 * x$.

A nonempty subset S of a BCH-algebra X is called a *subalgerba* of X if $x * y \in S$ for all $x, y \in S$. A nonempty subset A of a BCH-algebra X is called a *closed ideal* of X (see [7]) if it satisfies:

(1) (for all $x \in X$)($x \in A \Rightarrow 0 * x \in A$),

(2) (for all $y \in X$)(for all $x \in A$)($y * x \in A \Rightarrow y \in A$).

Note that every closed ideal is a subalgebra, but the converse is not true (see [7]). Since every closed ideal is a subalgebra, we know that any closed ideal contains the element 0. Denote by $\mathcal{S}(X)$ and $\mathcal{I}(X)$ the set of all subalgebras and closed ideals of X, respectively.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases} \tag{2.1}$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases} \tag{2.2}$$

3. $\mathcal{N}$-Closed Ideals of BCH-Algebras

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, μ) of X and an $\mathcal{N}$-function μ on X. In what follows, let X denote a BCH-algebra and μ an $\mathcal{N}$-function on X unless otherwise specified.

For any $\mathcal{N}$-structure (X, μ) and $t \in [-1, 0]$, the set

$$C(\mu; t) := \{x \in X \mid \mu(x) \le t\} \tag{3.1}$$

is called a *closed (μ, t)-cut* of (X, μ).

Using the similar method to the transfer principle in fuzzy theory (see [8, 9]), we can consider transfer principle in $\mathcal{N}$-structures. Let A be a subset of X and satisfy the following property $\wp$ expressed by a first-order formula:

$$\wp : \frac{t_1(x,\ldots,y) \in A,\ldots,t_n(x,\ldots,y) \in A}{t(x,\ldots,y) \in A}, \tag{3.2}$$

where $t_1(x,\ldots,y),\ldots,t_n(x,\ldots,y)$ and $t(x,\ldots,y)$ are terms of X constructed by variables $x,\ldots,y$. We note that the subset A satisfies the property $\wp$ if, for all elements $a,\ldots,b \in X, t(a,\ldots,b) \in A$ whenever $t_1(a,\ldots,b),\ldots,c_n(a,\ldots,b) \in A$. For the subset A we define an $\mathcal{N}$-structure (X,μ_A) which satisfies the following property

$$\overline{\wp} : \mu_A(t(x,\ldots,y)) \leq \vee\{\mu_A(t_1(x,\ldots,y)),\ldots \mu_A(t_n(x,\ldots,y))\}. \tag{3.3}$$

We establish a statement without proof, and we call it $\mathcal{N}$-transfer principle in $\mathcal{N}$-structures.

Theorem 3.1. *($\mathcal{N}$-transfer principle) An $\mathcal{N}$-structure (X,μ) satisfies the property $\overline{\wp}$ if and only if for all $\alpha \in [-1,0]$,*

$$C(\mu;\alpha) \neq \emptyset \Longrightarrow C(\mu;\alpha) \text{ satisfies the property } \wp. \tag{3.4}$$

Definition 3.2. By an *$\mathcal{N}$-subalgebra* of X we mean an $\mathcal{N}$-structure (X,μ) in which μ satisfies:

$$(\forall x,y \in X) \quad (\mu(x * y) \leq \vee\{\mu(x),\mu(y)\}). \tag{3.5}$$

Theorem 3.3. *For an $\mathcal{N}$-structure (X,μ), the following are equivalent:*

(1) *(X,μ) is an $\mathcal{N}$-subalgerba of X;*

(2) *(for all $t \in [-1,0]$)$(C(\mu;t) \in \mathcal{S}(X) \cup \{\emptyset\})$.*

Proof. It follows from the $\mathcal{N}$-transfer principle. □

Definition 3.4. By an *$\mathcal{N}$-closed ideal* of X we mean an $\mathcal{N}$-structure (X,μ) in which μ satisfies:

$$(\forall x,y \in X) \quad (\mu(0 * x) \leq \mu(x) \leq \vee\{\mu(x * y),\mu(y)\}). \tag{3.6}$$

It is clear that if (X,μ) is an $\mathcal{N}$-closed ideal or an $\mathcal{N}$-subalgebra, then $\mu(0) \leq \mu(x)$ for all $x \in X$.

Table 1: Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Theorem 3.5. *Every $\mathcal{N}$-closed ideal is an $\mathcal{N}$-subalgebra.*

Proof. Let (X, μ) be an $\mathcal{N}$-closed ideal of X. For any $x, y \in X$, we have

$$\begin{aligned} \mu(x * y) &\leq \vee\{\mu((x * y) * x), \mu(x)\} \\ &= \vee\{\mu((x * x) * y), \mu(x)\} \\ &= \vee\{\mu(0 * y), \mu(x)\} \\ &\leq \vee\{\mu(x), \mu(y)\}. \end{aligned} \tag{3.7}$$

Hence (X, μ) is an $\mathcal{N}$-subalgebra of X. □

The converse of Theorem 3.5 may not be true as seen in the following example.

Example 3.6. Consider a BCH-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1 (see [7]). Let (X, μ) be an $\mathcal{N}$-structure in which μ is given by

$$\mu = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -0.8 & -0.3 & -0.3 & -0.3 & -0.8 \end{pmatrix}. \tag{3.8}$$

It is easy to check that (X, μ) is an $\mathcal{N}$-subalgebra of Xbut it is not an $\mathcal{N}$-closed ideal of X since $\mu(3) = -0.3 > -0.8 = \vee\{\mu(3 * 4), \mu(4)\}$.

In order to discuss the converse of Theorem 3.5 we need to strengthen some conditions. We first consider the following lemma.

Lemma 3.7. *Every $\mathcal{N}$-subalgebra (X, μ) of X satisfies the following inequality:*

$$(\forall x \in X) \quad (\mu(x) \geq \mu(0 * x)). \tag{3.9}$$

Proof. For any $x \in X$, we get

$$\begin{aligned} \mu(0 * x) &\leq \vee\{\mu(0), \mu(x)\} = \vee\{\mu(x * x), \mu(x)\} \\ &= \vee\{\vee\{\mu(x), \mu(x)\}, \mu(x)\} = \mu(x), \end{aligned} \tag{3.10}$$

which is the desired result. □

Theorem 3.8. *If an $\mathcal{N}$-subalgerba (X, μ) satisfies*

$$(\forall x, y \in X) \quad (\mu(y) \leq \vee\{\mu(y * x), \mu(x)\}), \tag{3.11}$$

then (X, μ) is an $\mathcal{N}$-closed ideal of X.

Proof. It is straightforward by Lemma 3.7. □

Proposition 3.9. *Let (X, μ) be an $\mathcal{N}$-closed ideal of X that satisfies the following inequality*

$$(\forall x \in X) \quad (\mu(x) \leq \mu(0 * x)). \tag{3.12}$$

Then (X, μ) satisfies the inequality

$$(\forall x, y \in X) \quad (\mu(y * x) \leq \mu(x * y)). \tag{3.13}$$

Proof. Using (3.12), (3.6), (a3), (H1), and (H3), we have

$$\begin{aligned}
\mu(y * x) &\leq \mu(0 * (y * x)) \\
&\leq \vee\{\mu((0 * (y * x)) * (x * y)), \mu(x * y)\} \\
&= \vee\{\mu(((0 * y) * (0 * x)) * (x * y)), \mu(x * y)\} \\
&= \vee\{\mu(((0 * y) * (x * y)) * (0 * x)), \mu(x * y)\} \\
&= \vee\{\mu(((0 * (x * y)) * y) * (0 * x)), \mu(x * y)\} \\
&= \vee\{\mu((((0 * x) * (0 * y)) * (0 * x)) * y), \mu(x * y)\} \\
&= \vee\{\mu((0 * (0 * y)) * y), \mu(x * y)\} \\
&= \vee\{\mu(0), \mu(x * y)\} = \mu(x * y)
\end{aligned} \tag{3.14}$$

for all $x, y \in X$. □

Using the $\mathcal{N}$-transfer principle, we have a characterization of an $\mathcal{N}$-closed ideal.

Theorem 3.10. *For an $\mathcal{N}$-structure (X, μ), the following are equivalent:*

(1) *(X, μ) is an $\mathcal{N}$-closed ideal of X.*

(2) *(for all $t \in [-1, 0]$)$(C(\mu; t) \in \mathcal{I}(X) \cup \{\emptyset\})$.*

Consider two subsets of X as follows:

$$D_1 := \{x \in X \mid 0 * x = 0\}, \qquad D_2 := \{x \in X \mid 0 * (0 * x) = x\}. \tag{3.15}$$

Since D_1 and D_2 are a closed ideal and a subalgebra, respectively, the following theorems are direct results of the $\mathcal{N}$-transfer principle.

Theorem 3.11. *Let (X, μ) be an $\mathcal{N}$-structure in which μ is given by*

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D_1, \\ \beta & \text{otherwise} \end{cases} \tag{3.16}$$

for all $x \in X$ where $\alpha < \beta$. Then (X, μ) is an $\mathcal{N}$-closed ideal of X.

Theorem 3.12. *Let (X, μ) be an $\mathcal{N}$-structure in which μ is given by*

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D_2, \\ \beta & \text{otherwise} \end{cases} \tag{3.17}$$

for all $x \in X$ where $\alpha < \beta$. Then (X, μ) is an $\mathcal{N}$-subalgebra of X.

We provide a condition for an $\mathcal{N}$-subalgebra to be an $\mathcal{N}$-closed ideal.

Theorem 3.13. *Let (X, μ) be an $\mathcal{N}$-subalgebra of X in which μ satisfies*

$$(\forall x, y \in X) \quad (\mu(y * x) \geq \mu(x * y)). \tag{3.18}$$

Then (X, μ) is an $\mathcal{N}$-closed ideal of X.

Proof. Taking $x = 0$ in (3.18) induces $\mu(0 * y) \leq \mu(y * 0) = \mu(y)$ for all $y \in X$. Using (a1), (3.18), (H1), (H3), and (3.5), we have

$$\begin{aligned} \mu(y) &= \mu(y * 0) \leq \mu(0 * y) \\ &= \mu((x * x) * y) = \mu((x * y) * x) \\ &\leq \vee\{\mu(x * y), \mu(x)\} \leq \vee\{\mu(y * x), \mu(x)\} \end{aligned} \tag{3.19}$$

for all $x, y \in X$. Therefore (X, μ) is an $\mathcal{N}$-closed ideal of X. □

For any $\mathcal{N}$-structure (X, μ) and any element $w \in X$, we consider the set

$$X_w := \{x \in X \mid \mu(x) \leq \mu(w)\}. \tag{3.20}$$

Then X_w is nonempty subset of X.

Theorem 3.14. *If an $\mathcal{N}$-structure (X, μ) is an $\mathcal{N}$-closed ideal of X, then X_w is a closed ideal of X for all $w \in X$.*

Proof. If $x \in X_w$, then $\mu(x) \leq \mu(w)$ which implies from (3.6) that $\mu(0 * x) \leq \mu(x) \leq \mu(w)$. Thus $0 * x \in X_w$. Let $x, y \in X$ be such that $y \in X_w$ and $x * y \in X_w$. Then $\mu(y) \leq \mu(w)$ and $\mu(x * y) \leq \mu(w)$. Using (3.6), we have

$$\mu(x) \leq \vee\{\mu(x * y), \mu(y)\} \leq \mu(w), \quad \text{i.e., } x \in X_w. \tag{3.21}$$

Therefore X_w is a closed ideal of X. □

Proposition 3.15. *Let (X, μ) be an $\mathcal{N}$-structure such that X_w is a closed ideal of X for all $w \in X$. Then (X, μ) satisfies the following assertion:*

$$\mu(x) \geq \vee\{\mu(y * z), \mu(z)\} \Longrightarrow \mu(x) \geq \mu(y) \tag{3.22}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $\mu(x) \geq \vee\{\mu(y * z), \mu(z)\}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is a closed ideal of X, it follows that $y \in X_x$ so that $\mu(y) \leq \mu(x)$. This completes the proof. □

Theorem 3.16. *If an $\mathcal{N}$-structure (X, μ) satisfies (3.22) and $\mu(0 * x) \leq \mu(x)$ for all $x \in X$, then X_w is a closed ideal of X for all $w \in X$.*

Proof. For each $w \in X$, let $x, y \in X$ be such that $x * y \in X_w$ and $y \in X_w$. Then $\mu(x * y) \leq \mu(w)$ and $\mu(y) \leq \mu(w)$, which imply that $\vee\{\mu(x * y), \mu(y)\} \leq \mu(w)$. It follows from (3.22) that $\mu(x) \leq \mu(w)$ so that $x \in X_w$. If $x \in X_w$, then $\mu(0 * x) \leq \mu(x) \leq \mu(w)$ by assumption. Hence $0 * x \in X_w$. Therefore X_w is a closed ideal of X. □

Theorem 3.17. *Given an $\mathcal{N}$-structure (X, μ), let (X, μ^*) be an $\mathcal{N}$-structure in which μ^* is defined by*

$$\mu^*(x) = \wedge\{t \in [-1, 0] \mid x \in \langle C(\mu; t)\rangle\} \tag{3.23}$$

for all $x \in X$. Then (X, μ^*) is the greatest $\mathcal{N}$-closed ideal of X such that $(X, \mu^*) \subseteq (X, \mu)$, where $\langle C(\mu; t)\rangle$ is a closed ideal of X generated by $C(\mu; t)$.

Proof. For any $s \in \mathrm{Im}(\mu^*)$, let $s_n = s + (1/n)$ for any $n \in \mathbb{N}$. Let $x \in C(\mu^*; s)$. Then $\mu^*(x) \leq s$, and so

$$\wedge\{t \in [-1, 0] \mid x \in \langle C(\mu; t)\rangle\} \leq s < s + \frac{1}{n} = s_n \tag{3.24}$$

for all $n \in \mathbb{N}$. Hence there exists $t^* \in \{t \in [-1, 0] \mid x \in \langle C(\mu; t)\rangle\}$ such that $t^* < s_n$. Thus $C(\mu; t^*) \subseteq C(\mu; s_n)$, and so $x \in \langle C(\mu; t^*)\rangle \subseteq \langle C(\mu; s_n)\rangle$ for all $n \in \mathbb{N}$. Consequently

$x \in \bigcap_{n\in\mathbb{N}}\langle C(\mu; s_n)\rangle$, . On the other hand, if $x \in \bigcap_{n\in\mathbb{N}}\langle C(\mu; s_n)\rangle$, , then $s_n \in \{t \in [-1,0] \mid x \in \langle C(\mu; t)\rangle\}$ for any $n \in \mathbb{N}$. Therefore

$$\mu^*(x) = \wedge\{t \in [-1,0] \mid x \in \langle C(\mu; t)\rangle\} \leq s_n = s + \frac{1}{n} \tag{3.25}$$

for all $n \in \mathbb{N}$. Since n is arbitrary, it follows that $\mu^*(x) \leq s$ so that $x \in C(\mu^*; s)$. Thus $C(\mu^*; s) = \bigcap_{n\in\mathbb{N}}\langle C(\mu; s_n)\rangle$, which is a closed ideal of X. Using Theorem 3.10, we conclude that (X, μ^*) is an $\mathcal{N}$-closed ideal of X. For any $x \in X$, let

$$s \in \{t \in [-1,0] \mid x \in C(\mu; t)\}. \tag{3.26}$$

Then $x \in C(\mu; s)$ and thus $x \in \langle C(\mu; s)\rangle$. It follows that

$$s \in \{t \in [-1,0] \mid x \in \langle C(\mu; t)\rangle\} \tag{3.27}$$

so that $\{t \in [-1,0] \mid x \in C(\mu; t)\} \subseteq \{t \in [-1,0] \mid x \in \langle C(\mu; t)\rangle\}$. Hence

$$\begin{aligned} \mu(x) &= \wedge\{t \in [-1,0] \mid x \in C(\mu; t)\} \\ &\geq \wedge\{t \in [-1,0] \mid x \in \langle C(\mu; t)\rangle\} \\ &= \mu^*(x), \end{aligned} \tag{3.28}$$

and so $(X, \mu^*) \subseteq (X, \mu)$. Finally, let (X, ν) be an $\mathcal{N}$-closed ideal of X such that $(X, \nu) \subseteq (X, \mu)$. Let $x \in X$. If $\mu^*(x) = 0$, then clearly $\nu(x) \leq \mu^*(x)$. Assume that $\mu^*(x) = s \neq 0$. Then $x \in C(\mu^*; s) = \bigcap_{n\in\mathbb{N}}\langle C(\mu; s_n)\rangle$, and so $x \in \langle C(\mu; s_n)\rangle$ for all $n \in \mathbb{N}$. It follows that $\nu(x) \leq \mu(x) \leq s_n = s + (1/n)$ for all $n \in \mathbb{N}$ so that $\nu(x) \leq s = \mu^*(x)$ since n is arbitrary. This shows that $(X, \nu) \subseteq (X, \mu^*)$. This completes the proof. □

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable suggestions. The first author was supported by the fund of sabbatical year program (2009), Gyeongsang National University.

References

[1] Q. P. Hu and X. Li, "On BCH-algebras," *Mathematics Seminar Notes. Kobe University*, vol. 11, no. 2, part 2, pp. 313–320, 1983.
[2] Q. P. Hu and X. Li, "On proper BCH-algebras," *Mathematica Japonica*, vol. 30, no. 4, pp. 659–661, 1985.
[3] B. Ahmad, "On classification of BCH-algebras," *Mathematica Japonica*, vol. 35, no. 5, pp. 801–804, 1990.
[4] W. A. Dudek and J. Thomys, "On decompositions of BCH-algebras," *Mathematica Japonica*, vol. 35, no. 6, pp. 1131–1138, 1990.
[5] Y. B. Jun, K. J. Lee, and S. Z. Song, "$\mathcal{N}$-ideals of BCK/BCI-algebras," *The Journal of Chungcheong Mathematical Society*, vol. 22, pp. 417–437, 2009.
[6] M. A. Chaudhry and H. Fakhar-Ud-Din, "Ideals and filters in BCH-algebras," *Mathematica Japonica*, vol. 44, no. 1, pp. 101–111, 1996.

[7] M. A. Chaudhry, "On BCH-algebras," *Mathematica Japonica*, vol. 36, no. 4, pp. 665–676, 1991.
[8] Y. B. Jun and M. Kondo, "On transfer principle of fuzzy BCK/BCI-algebras," *Scientiae Mathematicae Japonicae*, vol. 59, no. 1, pp. 35–40, 2004.
[9] M. Kondo and W. A. Dudek, "On the transfer principle in fuzzy theory," *Mathware & Soft Computing*, vol. 12, no. 1, pp. 41–55, 2005.