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Research Article

On the Complex and Real Hessian Polynomials

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We study some realization problems related to the Hessian polynomials. In particular, we solve the Hessian curve realization problem for degrees zero, one, two, and three and the Hessian polynomial realization problem for degrees zero, one, and two.

1. Introduction

A topic which has been of interest since the XIX century is the study of the parabolic curve of smooth surfaces in real three-dimensional space, as shown in the works of Gauss, Darboux, Salmon [1], Kergosien and Thom [2], Arnold [3], among others.

The parabolic curve of the graph of a smooth function, $f : \mathbb{R}^2 \to \mathbb{R}$, is the set $\{(p, f(p)) \in \mathbb{R}^2 \times \mathbb{R}$: Hess$f(p) = 0\}$, where Hess $f := f_{xx}f_{yy} - f_{xy}^2$. In this case, *the Hessian curve of* f, Hess $f(x,y) = 0$, is a plane curve which is the projection of the parabolic curve into the xy-plane along the z-axis. When f is a polynomial of degree n in two variables, Hess f is a polynomial of degree at most $2n-4$. Therefore, the Hessian curve is an algebraic plane curve. In this setting there are two natural realization problems related to the Hessian of a polynomial.

(1) *Hessian curve realization problem.* Given an algebraic plane curve $g(x,y) = 0$ in $\mathbb{K}^2$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we ask: When is $g(x,y) = 0$ the Hessian curve of a polynomial $f \in \mathbb{K}[x,y]$?

(2) *Hessian polynomial realization problem.* Given $g \in \mathbb{K}[x,y]$ we ask: When does $f \in \mathbb{K}[x,y]$ exist such that Hess $f = g$? If such f exists and $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$), then g is called *a complex Hessian polynomial* (*real Hessian polynomial*). We remark that Arnold (see [4]) calls *Hessian topology problem* to the study of the problem 1 in the real case.

Note that problem 2 contains problem 1. Moreover, in the real case, problem 2 is a global realization problem of a smooth function such as the Gaussian curvature function. In [5], Arnold studies this problem locally.

This work is devoted to problems 1 and 2 for complex and real case of degree less or equal to three. It is divided in two parts. In the first we give the results according to the degree of the polynomials g. In the second part, we give the proofs and some other results such as a geometric interpretation of Corollary 2.8.

2. Notation and Results

For each nonnegative integer number n, we define $\mathbb{A}_n^{\mathbb{K}} := \{f \in \mathbb{K}[x,y] \mid \deg(f) \leq n\}$. Let us consider the *Hessian map*

$$H_n^{\mathbb{K}} : \mathbb{A}_n^{\mathbb{K}} \longrightarrow \mathbb{A}_{2n-4}^{\mathbb{K}}, \text{ given by } f \longmapsto \operatorname{Hess} f. \tag{2.1}$$

We will say that the Hessian map $H_n^{\mathbb{K}}$ is *complex* or *real* if $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, respectively. We remark that $\dim(\mathbb{A}_n^{\mathbb{K}}) \geq \dim(\mathbb{A}_{2n-4}^{\mathbb{K}})$ if $2 \leq n \leq 4$ and $\dim(\mathbb{A}_n^{\mathbb{K}}) < \dim(\mathbb{A}_{2n-4}^{\mathbb{K}})$ if $n \geq 5$. In particular, for $n \geq 4$, the image, $H_n^{\mathbb{K}}(\mathbb{A}_n^{\mathbb{K}})$, is a connected subset of codimension at least three in $\mathbb{A}_{2n-4}^{\mathbb{K}}$. This means that, in "general", a polynomial $g \in \mathbb{A}_{2n-4}^{\mathbb{K}}$ is not a Hessian polynomial if $n \geq 4$ under $H_n^{\mathbb{K}}$.

In diagram form we have

$$\begin{array}{c} \mathbb{A}_2^{\mathbb{K}} \subsetneq \mathbb{A}_3^{\mathbb{K}} \subsetneq \mathbb{A}_4^{\mathbb{K}} \subsetneq \cdots \subsetneq \mathbb{A}_n^{\mathbb{K}} \cdots \\ \downarrow_{H_2^{\mathbb{K}}} \downarrow_{H_3^{\mathbb{K}}} \downarrow_{H_4^{\mathbb{K}}} \cdots \downarrow_{H_n^{\mathbb{K}}} \cdots \\ \mathbb{A}_0^{\mathbb{K}} \subsetneq \mathbb{A}_2^{\mathbb{K}} \subsetneq \mathbb{A}_4^{\mathbb{K}} \subsetneq \cdots \subsetneq \mathbb{A}_{2n-4}^{\mathbb{K}} \cdots \end{array} \tag{2.2}$$

In virtue of the previous remarks, we introduce *the fiber of* $g \in \mathbb{A}_{2n-4}^{\mathbb{K}}$ *under* $H_n^{\mathbb{K}}$ as the set

$$\left(H_n^{\mathbb{K}}\right)^{-1}(g) := \left\{f \in \mathbb{A}_n^{\mathbb{K}} \mid H_n^{\mathbb{K}}(f) = g\right\}. \tag{2.3}$$

Even when we consider the fibers, $(H_n^{\mathbb{K}})^{-1}(g)$, for different values of n, we are interested in knowing if the fiber $(H_n^{\mathbb{K}})^{-1}(g)$ is not empty when n satisfies $0 \leq 2n - 4 - \deg(g) \leq 1$. If the fiber $(H_n^{\mathbb{K}})^{-1}(g)$ is empty, the next problem is to see if the fibers $(H_s^{\mathbb{K}})^{-1}(g)$ are not empty for $s \geq n+1$ (this problem will not be studied in this work). Another way to study the Hessian polynomial realization problem is by describing the set of all polynomials $f \in \mathbb{A}_r^{\mathbb{K}}$, with $r \geq n = (m+4)/2$, such that $H_n^{\mathbb{K}}(f) = g$ whenever $g \in \mathbb{A}_m^{\mathbb{K}}$.

Remark 2.1. Let $g(x,y) = \sum_{i+j=0}^{2n-4} b_{ij} x^i y^j$ be a polynomial in $\mathbb{A}_{2n-4}^{\mathbb{K}}$. There exists a polynomial $f(x,y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j$ in $(H_n^{\mathbb{K}})^{-1}(g)$ if and only if f satisfies the following system of $(2n-3)(n-1)$ equations:

$$h_{ij}(a_{rs}) = b_{ij}, \quad 0 \leq i+j \leq 2n-4, \; 2 \leq r+s \leq n, \tag{2.4}$$

where $h_{ij}(a_{rs})$ are the coefficients of the Hessian polynomial $H_n^{\mathbb{K}}(f)$. They are also quadratic polynomials in the variables a_{rs}.

It is important to note that the computations for solving the system (2.4) are generally very complicated.

In the following results we describe the sets of polynomials of a given degree which are Hessian and those which are not Hessian under a specific Hessian map.

2.1. Degree of g Equal to Zero

Proposition 2.2. *For each $g \in \mathbb{A}_0^{\mathbb{C}}$, the set $(H_2^{\mathbb{C}})^{-1}(g)$ is a quadric in $\mathbb{A}_2^{\mathbb{C}}$ given by $a_{11}^2 = 4a_{20}a_{02} - g$. Moreover, this quadric is singular if and only if $g = 0$.*

Corollary 2.3. *In the complex case every element in $A_0^{\mathbb{C}} = \mathbb{C}$ is a complex Hessian polynomial under $H_2^{\mathbb{C}}$. And, in the real case every element in $A_0^{\mathbb{R}} = \mathbb{R}$ is a real Hessian polynomial under $H_2^{\mathbb{R}}$.*

Proposition 2.4. *For each $g \in \mathbb{A}_0^{\mathbb{C}}$, the set $(H_3^{\mathbb{C}})^{-1}(g)$ is an analytic subvariety in $\mathbb{A}_3^{\mathbb{C}}$ which is given by the union of connected analytic subvarieties which are parametrized by the following.*

(1) $F_1^{\pm} : \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^3 \to \mathbb{A}_3^{\mathbb{C}}; (t_1, t_2, t_3, t_4, t_5, t_6) \mapsto (a_{30} = t_2^2/3t_3, a_{21} = t_2, a_{12} = t_3, a_{03} = t_3^2/3t_2, a_{20} = t_1, a_{11} = (2t_3t_1 \pm t_2\sqrt{-g})/t_2, a_{02} = t_3(t_3t_1 \pm t_2\sqrt{-g})/t_2^2, a_{10} = t_4, a_{01} = t_5, a_{00} = t_6)$.

(2) $F_2 : \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^3 \to \mathbb{A}_3^{\mathbb{C}}; (t_1, t_2, t_3, t_4, t_5) \mapsto (a_{30} = 0, a_{21} = 0, a_{12} = 0, a_{03} = 0, a_{20} = (t_1^2 + g)/4t_2, a_{11} = t_1, a_{02} = t_2, a_{10} = t_3, a_{01} = t_4, a_{00} = t_5)$.

(3) $F_3^{\pm} : \mathbb{C}^5 \to \mathbb{A}_3^{\mathbb{C}}; (t_1, t_2, t_3, t_4, t_5) \mapsto (a_{30} = t_1, a_{21} = 0, a_{12} = 0, a_{03} = 0, a_{20} = t_2, a_{11} = \pm\sqrt{-g}, a_{02} = 0, a_{10} = t_3, a_{01} = t_4, a_{00} = t_5)$.

(4) $F_4^{\pm} : \mathbb{C}^5 \to \mathbb{A}_3^{\mathbb{C}}; (t_1, t_2, t_3, t_4, t_5) \mapsto (a_{30} = 0, a_{21} = 0, a_{12} = 0, a_{03} = t_1, a_{20} = 0, a_{11} = \pm\sqrt{-g}, a_{02} = t_2, a_{10} = t_3, a_{01} = t_4, a_{00} = t_5)$.

2.2. Degree of g Equal to One

Proposition 2.5. *For each $g(x, y) = b_{10}x + b_{01}y + b_{00} \in \mathbb{A}_2^{\mathbb{C}}$ of degree one, the set $(H_3^{\mathbb{C}})^{-1}(g)$ is an analytic subvariety in $\mathbb{A}_3^{\mathbb{C}}$ which is given by the union of analytic subvarieties parametrized by the following.*

(1) *For* $b_{10}b_{01} = 0$, $F_1 : \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^3 \setminus \{4b_{01}^2t_1^2 - 4b_{10}b_{01}t_1t_2 + b_{10}^2t_2^2 + b_{00}b_{10}^2 = 0\} \to \mathbb{A}_3^{\mathbb{C}}; (t_1, t_2, t_3, t_4, t_5) \mapsto (a_{30} = (1/3)(b_{10}^3t_1/(4b_{01}^2t_1^2 - 4b_{10}b_{01}t_1t_2 + b_{10}^2t_2^2 + b_{00}b_{10}^2)), a_{21} = b_{01}b_{10}^2t_1/(4b_{01}^2t_1^2 - 4b_{10}b_{01}t_1t_2 + b_{10}^2t_2^2 + b_{00}b_{10}^2), a_{12} = b_{01}^2b_{10}t_1/(4b_{01}^2t_1^2 - 4b_{10}b_{01}t_1t_2 + b_{10}^2t_2^2 + b_{00}b_{10}^2), a_{03} = b_{01}^3t_1/3(4b_{01}^2t_1^2 - 4b_{10}b_{01}t_1t_2 + b_{10}^2t_2^2 + b_{00}b_{10}^2), a_{20} = t_1, a_{11} = t_2, a_{02} = (t_2^2 + b_{00})/4t_1, a_{10} = t_3, a_{01} = t_4, a_{00} = t_5)$.

(2) *For* $b_{10}b_{01} \neq 0$, $F_2^{\pm} : \mathbb{C}^* \times \mathbb{C}^3 \to \mathbb{A}_3^{\mathbb{C}}; (t_1, t_2, t_3, t_4) \mapsto (a_{30} = (1/3)(t_1b_{10}/b_{01}), a_{21} = t_1, a_{12} = b_{01}t_1/b_{10}, a_{03} = (1/3)(b_{01}^2t_1/b_{10}^2), a_{20} = 0, a_{11} = \pm\sqrt{-b_{00}}, a_{02} = (1/4)(b_{01}(\pm 4t_1\sqrt{-b_{00}} + b_{10})/b_{10}t_1), a_{10} = t_2, a_{01} = t_3, a_{00} = t_4)$.

From this proposition we have the following.

Corollary 2.6. *Every complex polynomial of degree one is a complex Hessian polynomial under $H_3^{\mathbb{C}}$. And every real polynomial of degree one is a real Hessian polynomial under $H_3^{\mathbb{R}}$.*

2.3. Degree of g Equal to Two

Let $g_1, g_2 \in \mathbb{K}[x, y]$. We say that g_1 *is in the orbit of* g_2 if they are equivalent by an affine transformation of the plane $\mathbb{K}^2$.

Theorem 2.7. *Complex Case*

(1) *The complex polynomials of degree two*

$$\boxed{y^2 \mid x - y^2 \mid y^2 - x^2 - k,\ k \in \mathbb{C}} \tag{2.5}$$

are complex Hessian polynomials under $H_3^{\mathbb{C}}$*. Moreover, a polynomial* $g \in \mathbb{A}_2^{\mathbb{C}}$ *of degree two is a complex Hessian polynomial under* $H_3^{\mathbb{C}}$ *if and only if it belongs to the orbit of one of those polynomials.*

(2) *The complex polynomials,* $y^2 + r \in \mathbb{A}_2^{\mathbb{C}}$*, where* $r \in \mathbb{C}^*$*, are not complex Hessian polynomials under* $H_3^{\mathbb{C}}$*. Moreover, a polynomial* $g \in \mathbb{A}_2^{\mathbb{C}}$ *of degree two is not a complex Hessian polynomial under* $H_3^{\mathbb{C}}$ *if and only if it belongs to the orbit of one of those polynomials.*

Real case

(1) *The real polynomials of degree two*

$$\boxed{-y^2 \mid x - y^2 \mid y^2 - x^2 - r_1,\ r_1 > 0 \mid -y^2 - x^2 + r_2,\ r_2 > 0} \tag{2.6}$$

are real Hessian polynomials under $H_3^{\mathbb{R}}$*. Moreover, a polynomial* $g \in \mathbb{A}_2^{\mathbb{R}}$ *of degree two is a real Hessian polynomial under* $H_3^{\mathbb{R}}$ *if and only if it belongs to the orbit of one of those polynomials.*

(2) *The real polynomials of degree two*

y^2	$y^2 - x^2 - r_3,\ r_3 \le 0$	$y^2 + x^2 - r_4,\ r_4 \in \mathbb{R}$	$-y^2 - x^2 + r_5,\ r_5 \le 0$
$y^2 + x$	$y^2 - r_6,\ r_6 \in \mathbb{R}^*$	$-y^2 - r_7,\ r_7 \in \mathbb{R}^*$	

(2.7)

are not real Hessian polynomials under $H_3^{\mathbb{R}}$*. Moreover, a polynomial* $g \in \mathbb{A}_2^{\mathbb{R}}$ *of degree two is not a real Hessian polynomial under* $H_3^{\mathbb{R}}$ *if and only if it belongs to the orbit of one of those polynomials.*

It is well known that the complex affine classification of conics is given by the normal forms: $y^2 = x$ (parabola), $y^2 = x^2 + 1$ (general conic), $y^2 = x^2$ (line pair), $y^2 = 1$ (parallel lines), and finally $y^2 = 0$ (double line). Therefore, we have the following corollary.

Corollary 2.8. *All the complex affine conics, except the parallel lines, are complex Hessian curves of polynomials in* $\mathbb{A}_3^{\mathbb{C}}$*.*

In the next section we will give a geometric proof of this corollary.

Proposition 2.9. *Let* $g(x,y) = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{00}$ *be a polynomial in* $\mathbb{A}_2^{\mathbb{C}}$ *with* $b_{20}(4b_{20}b_{02} - b_{11}^2) \neq 0$. *Then the set* $(H_3^{\mathbb{C}})^{-1}(g)$ *is an analytic subvariety on* $\mathbb{A}_3^{\mathbb{C}}$ *which is the union of analytic subvarieties parametrized by the following.*

(1) $F_1^{\pm} : \mathbb{C}^4 \to \mathbb{A}_3^{\mathbb{C}}$; $(t_1,t_2,t_3,t_4) \mapsto (a_{30} = t_1, a_{21} = (1/2)(3b_{11}t_1 \pm \sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3}/b_{20}), a_{12} = (-6t_1b_{02}b_{20} + 3b_{11}t_1 \pm \sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3})/2b_{20}^2$, $a_{03} = (-1/6b_{20}^2)(\pm b_{02}\sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3} + 6b_{11}t_1b_{02} - (3b_{11}t_1 \pm \sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3})b_{11}^2/b_{20} + 3b_{11}b_{02}t_1)$, $a_{20} = b_{20}\sqrt{b_{00}/4b_{20}b_{02} - b_{11}^2}$, $a_{11} = b_{11}\sqrt{b_{00}/4b_{20}b_{02} - b_{11}^2}$, $a_{02} = b_{02}\sqrt{b_{00}/(4b_{20}b_{02} - b_{11}^2)}$, $a_{10} = t_2$, $a_{01} = t_3$, $a_{00} = t_4)$.

(2) $F_2^{\pm}$: $\mathbb{C}^4 \to \mathbb{A}_3^{\mathbb{C}}$; $(t_1,t_2,t_3,t_4) \mapsto (a_{30} = t_1, a_{21} = (3b_{11}t_1 \pm \sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3})/2b_{20}$, $a_{12} = (3b_{11}t_1 - 6t_1b_{02}b_{20} \pm \sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3})/2b_{20}^2$, $a_{03} = (-1/6b_{20}^2)(\pm b_{02}\sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3} + 3b_{02}b_{11}t_1 + 6b_{11}t_1b_{02} - (3b_{11}^3t_1 \pm b_{11}^2\sqrt{9t_1^2b_{11}^2 - 36t_1^2b_{02}b_{20} - b_{20}^3})/b_{20})$, $a_{20} = -b_{20}\sqrt{b_{00}/4b_{20}b_{02} - b_{11}^2}$, $a_{11} = -b_{11}\sqrt{b_{00}/4b_{20}b_{02} - b_{11}^2}$, $a_{02} = -b_{02}\sqrt{b_{00}/4b_{20}b_{02} - b_{11}^2}$, $a_{10} = t_2$, $a_{01} = t_3$, $a_{00} = t_4)$.

From the study of the previous fibers, a natural question arises. What is the relation between the set of critical points of $H_n^{\mathbb{C}}$ and the set of singular Hessian curves defined by polynomials in $\mathbb{A}_{2n-4}^{\mathbb{C}}$?

We define $\mathbb{A}_{nR}^{\mathbb{C}} := \{f \in \mathbb{C}[x,y] \mid 2 \le \deg(f) \le n\}$. Let us describe the relation between the set of critical points of $H_{3R}^{\mathbb{C}} : \mathbb{A}_{3R}^{\mathbb{C}} \to \mathbb{A}_2^{\mathbb{C}}$ and the set of polynomials in $\mathbb{A}_2^{\mathbb{C}}$ such that they define singular Hessian curves.

Proposition 2.10. *If* $f \in \mathbb{A}_{3R}^{\mathbb{C}}$ *is a critical point of the map* $H_{3R}^{\mathbb{C}} : \mathbb{A}_{3R}^{\mathbb{C}} \to \mathbb{A}_2^{\mathbb{C}}$, *then the Hessian curve* $H_{3R}^{\mathbb{C}}(f)(x,y) = 0$ *is singular or it has degree one.*

For the general case, we have the following conjecture.

Conjecture 2.11. *If* $f \in \mathbb{A}_{nR}$ *is a critical point of the map* $H_{nR}^{\mathbb{C}} : \mathbb{A}_{nR}^{\mathbb{C}} \to \mathbb{A}_{2n-4}^{\mathbb{C}}$, *then its Hessian curve,* Hess $f(x,y) = 0$, *is singular or has degree less than* $2n-4$.

2.4. Degree of g Equal to Three

The following theorem is one of the most important of this paper.

Theorem 2.12. *Complex Case*

(1) *The curves defined by Table 1 of complex cubic curves (see [6]) are complex Hessian curves under* $H_4^{\mathbb{C}}$.

(2) *The curves defined by Table 2 of complex cubic curves are not complex Hessian curves under* $H_4^{\mathbb{C}}$.

Table 1: Complex Hessian cubic curves.

$xy^2 = bx^2 + cx + d,\ b, c, d \in \mathbb{C}$	$xy^2 + ey = bx^2 + cx + d,\ e \in \mathbb{C}^*,\ b, c, d \in \mathbb{C}$
$y^2 = ax^3 + bx^2 + cx + d,\ a \in \mathbb{C}^*,\ b, c, d \in \mathbb{C}$	

Table 2: Complex cubic curves which are not complex Hessian curves.

$xy^2 + ey = ax^3 + bx^2 + cx + d,\ a \in \mathbb{C}^*,\ b, c, d, e \in \mathbb{C}$	$xy = ax^3 + bx^2 + cx + d,\ a \in \mathbb{C}^*,\ b, c, d \in \mathbb{C}$
$y = ax^3 + bx^2 + cx + d,\ a \in \mathbb{C}^*,\ b, c, d \in \mathbb{C}$	$x^3 + bx^2 + cx + d = 0,\ b, c, d \in \mathbb{C}$

Real Case

(1) *The normal form of curves (see [7]) defined in Table 3 is a real Hessian curve under* $H_4^{\mathbb{R}}$*. Moreover, a curve* $g \in \mathbb{A}_3^{\mathbb{R}}$ *of degree three is a real Hessian curve under* $H_4^{\mathbb{R}}$ *if and only if its polynomial belongs to the orbit of one of those polynomials.*

(2) *The normal form of curves defined by the polynomials in Table 4 is not a real Hessian polynomial under* $H_4^{\mathbb{R}}$*. Moreover, a curve* $g \in \mathbb{A}_3^{\mathbb{R}}$ *of degree three is not a real Hessian curve under* $H_4^{\mathbb{R}}$ *if and only if its polynomial belongs to the orbit of one of those polynomials.*

3. Proofs

Let us consider the complex Hessian map $H_n^{\mathbb{C}} : \mathbb{A}_n^{\mathbb{C}} \to \mathbb{A}_{2n-4}^{\mathbb{C}}\ f \mapsto \operatorname{Hess} f$ and recall that the fiber of $g \in \mathbb{A}_{2n-4}$ under $H_n^{\mathbb{C}}$ is the set $(H_n^{\mathbb{C}})^{-1}(g) := \{f \in \mathbb{A}_n^{\mathbb{C}} \mid H_n^{\mathbb{C}}(f) = g\}$.

Proof of the Proposition 2.2. Let $f(x,y) = \sum_{i+j=0}^{2} a_{ij}x^i y^j \in \mathbb{A}_2^{\mathbb{C}}$. A direct calculus shows that $f_{xx} = 2a_{20}$, $f_{xy} = a_{11}$, and $f_{yy} = 2a_{02}$. Therefore,

$$H_2^{\mathbb{C}}(f) = 4a_{20}a_{02} - a_{11}^2. \tag{3.1}$$

For each $g \in \mathbb{A}_0^{\mathbb{C}}$, consider $S(g) = \{(a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}) : a_{11}^2 = 4a_{20}a_{02} - g\}$. To show that $(H_2^{\mathbb{C}})^{-1}(g) = S(g)$ it is enough to show $(H_2^{\mathbb{C}})^{-1}(g) \subset S$ because a direct substitution shows $S \subset (H_2^{\mathbb{C}})^{-1}(g)$. Therefore, let us consider $f \in (H_2^{\mathbb{C}})^{-1}(g)$, that is, $H_2^{\mathbb{C}}(f) = 4a_{20}a_{02} - a_{11}^2 = g$. Hence $a_{11}^2 = 4a_{20}a_{02} - g$ and the first part of the claim is done. Finally, the derivative of $H_2^{\mathbb{C}}$ is given by

$$DH_2^{\mathbb{C}}(f) = (0, 0, 0, 4a_{02}, -2a_{11}, 4a_{20}). \tag{3.2}$$

Therefore, we conclude the proof of the result. □

Proof of the Corollary 2.3. The polynomial $f(x,y) = x^2 + (g/4)y^2$ satisfies that $H_2^{\mathbb{K}}(f) = g$ in complex or real case. □

Table 3: Real Hessian polynomials.

$x^2y+y^2+x+a_1y+a_2=0,\ a_1,a_2\in\mathbb{R}$	$x^2y+y^2+y+a_3=0,\ a_3\in(-\infty,0)\cup(0,1/4)$
$x^2y+y^2-y+a_4=0,\ a_4<-3/4$	$x^2y+y^2-1=0$
$x^2y+3y=0$	$x^2y-3y=0$
$x^2y=0$	$x^3+a_5x-y^2-1=0,\ a_5\in\mathbb{R},$
$x^3-y^2+a_6x+1=0,\ a_6\in\mathbb{R}$	$x^3-y^2+x=0$
$x^3-y^2-x=0$	$x^3-y^2=0$

Table 4: Real polynomials which are not Hessian polynomials.

$xy^2-x(x-3)^2+b_1x+b_2y-b_3=0,\ b_1,b_2,b_3\in\mathbb{R}$	$xy^2-x^3+b_4x+2y-b_5=0,\ b_5\in\mathbb{R}$
$xy^2-x^3+b_6x-1=0,\ b_6\in\mathbb{R}$	$xy^2-x^3+x=0$
$xy^2-x^3-x=0$	$xy^2-x^3=0$
$x^3+xy^2-6y^2+b_7x+b_8y+b_9=0,\ b_7,b_8,b_9\in\mathbb{R}$	$x^3+xy^2+b_{10}x+3y+b_{11}=0, b_{10},b_{11}\in\mathbb{R}$
$x^3+xy^2+b_{12}x+1=0,\ b_{12}\in\mathbb{R}$	$x^3+xy^2+3x=0$
$x^3+xy^2-3x=0$	$x^3+xy^2=0$
$x^2y+y^2+y+c=0,\ c\in\{0\}\cup[1/4,\infty)$	$x^2y+y^2-y+d=0,\ d\geq-3/4$
$x^2y+y^2+1=0$	$x^2y+y^2=0$
$x^2y+3x+3y+b_{13}=0,\ b_{13}\in\mathbb{R}$	$x^2y+3y+1=0$
$x^2y+3x-3y+b_{14}=0,\ b_{14}\in\mathbb{R}$	$x^2y-3y+1=0$
$x^2y+3x+1=0$	$x^2y+3x=0$
$x^2y+1=0$	$x^3-xy+1=0$
$x^3-xy=0$	$x^3-y=0$
$x^3+b_{15}x+1=0,\ b_{15}\in\mathbb{R}$	$x^3-3x=0$
$x^3+3x=0$	$x^3=0$

Lemma 3.1. *If* $f(x,y)=\sum_{i+j=0}^{3}a_{ij}x^iy^j\in\mathbb{A}_3^{\mathbb{C}}$, *then the map* $H_3^{\mathbb{C}}:A_3^{\mathbb{C}}\to A_2^{\mathbb{C}}$ *is*

$$H_3^{\mathbb{C}}(f)=b_{20}x^2+b_{11}xy+b_{02}y^2+b_{10}x+b_{01}y+b_{00}, \tag{3.3}$$

where the b_{rs} *coefficients satisfy the following system of quadratic equations:*

$$\begin{aligned}
b_{20}&=-4a_{21}^2+12a_{30}a_{12},\\
b_{11}&=-4a_{21}a_{12}+36a_{30}a_{03},\\
b_{02}&=-4a_{12}^2+12a_{21}a_{03},\\
b_{10}&=12a_{30}a_{02}+4a_{20}a_{12}-4a_{21}a_{11},\\
b_{01}&=12a_{20}a_{03}+4a_{21}a_{02}-4a_{12}a_{11},\\
b_{00}&=-a_{11}^2+4a_{20}a_{02}.
\end{aligned} \tag{3.4}$$

Proof of the Proposition 2.4. For $g \in \mathbb{A}_0^{\mathbb{C}}$ we have that, by Lemma 3.1, $H_3^{\mathbb{C}}(f) = g$ is equivalent to the following system of equations:

$$0 = -4a_{21}^2 + 12a_{30}a_{12}, \tag{3.5}$$

$$0 = -4a_{21}a_{12} + 36a_{30}a_{03}, \tag{3.6}$$

$$0 = -4a_{12}^2 + 12a_{21}a_{03}, \tag{3.7}$$

$$0 = 12a_{30}a_{02} + 4a_{20}a_{12} - 4a_{21}a_{11}, \tag{3.8}$$

$$0 = 12a_{20}a_{03} + 4a_{21}a_{02} - 4a_{12}a_{11}, \tag{3.9}$$

$$g = -a_{11}^2 + 4a_{20}a_{02}. \tag{3.10}$$

Let S_g be the set obtained by the union of the image of parametrizations $F_1, \ldots, F_4$. To prove that $(H_3^{\mathbb{C}})^{-1}(g) = S_g$, it is enough to show that $(H_3^{\mathbb{C}})^{-1}(g) \subset S_g$ because a direct substitution shows that $S_g \subset (H_3^{\mathbb{C}})^{-1}(g)$. Now, to prove $(H_3^{\mathbb{C}})^{-1}(g) \subset S_g$ we will consider two cases: Case 1 is when $a_{12} \neq 0$ and Case 2 is when $a_{12} = 0$.

Case 1. In this case, from a direct substitution we obtain $a_{30} = a_{21}^2/3a_{12}$, $a_{03} = a_{12}^2/3a_{21}$, $a_{12} \in \mathbb{C}^*$, and (3.8), (3.9), and (3.10). To solve these equations we will assume the following.

Subcase 1.1. $a_{20} = 0$. In this case we obtain $a_{11} = \pm\sqrt{-g}$; $a_{02} = \pm a_{12}a_{21}\sqrt{-g}/a_{21}^2$. All this values together are contained in the set whose parametrization is F_1.

Subcase 1.2. $a_{20} \neq 0$. This case will be subdivided in two subcases.

(1) $a_{02} = 0$. First, we obtain a_{11} from (3.10). Later, from a substitution of a_{11} together with the value of a_{30} in (3.8) we get $a_{20}a_{12} \pm a_{21}\sqrt{-g} = 0$. This equation implies $a_{11} = 2a_{20}a_{12} \pm a_{21}\sqrt{-g}/a_{21}$. All these values together are contained in the set whose parametrization is F_1.

(2) $a_{02} \neq 0$. In this case, from (3.8) we obtain

$$a_{11} = \frac{1}{a_{21}a_{12}}\left(a_{21}^2 a_{02} + a_{20}a_{12}^2\right). \tag{3.11}$$

A substitution of a_{11} in (3.10) gives us the quadratic equation in the a_{20} variable:

$$a_{21}^4 a_{02}^2 - 2a_{21}^2 a_{12}^2 a_{02} + \left(a_{20}^2 a_{12}^4 + g a_{21}^2 a_{12}^2\right) = 0. \tag{3.12}$$

Solving this quadratic equation we get $a_{02} = a_{12}(a_{20}a_{12} \pm a_{21}\sqrt{-g})/a_{21}^2$. Finally, from a substitution of a_{02} in (3.11) we get $a_{11} = (2a_{20}a_{12} \pm a_{21}\sqrt{-g})/a_{21}$. From these values we obtain the parametrization F_1.

Case 2. From a direct calculus we obtain $a_{21} = 0$ and the equations:

$$\begin{aligned} 0 &= a_{30}a_{03}, \\ 0 &= a_{30}a_{02}, \\ 0 &= a_{20}a_{03}, \\ g &= -a_{11}^2 + 4a_{20}a_{02}. \end{aligned} \tag{3.13}$$

To solve these four equations we will consider two cases.

Subcase 2.1. $a_{02} = 0$. From a direct substitution we get $a_{11} = \pm\sqrt{-g}$ and the two equations:

$$\begin{aligned} 0 &= a_{30}a_{03}, \\ 0 &= a_{30}a_{02}. \end{aligned} \tag{3.14}$$

To solve these two equations we will consider two subcases.

(1) $a_{03} = 0$. We obtain $a_{30} \in \mathbb{C}$ and $a_{20} \in \mathbb{C}$. From all of these values we get the parametrization F_3.

(2) $a_{03} \neq 0$. We obtain $a_{30} = a_{20} = 0$. From all these values we get the parametrization F_4 when $a_{02} = 0$.

Subcase 2.2. $a_{02} \neq 0$. We obtain $a_{20} = (a_{11}^2 + g)/4a_{02}$ and the three equations:

$$\begin{aligned} 0 &= a_{30}a_{03}, \\ 0 &= a_{30}a_{02}, \\ 0 &= a_{20}a_{03}. \end{aligned} \tag{3.15}$$

To solve these three equations we will consider two subcases.

(1) $a_{03} = 0$. We obtain $a_{30} = 0$ and then the parametrization F_2.

(2) $a_{03} \neq 0$. We obtain $a_{30} = 0$, $a_{20} = 0$, as well as $a_{11} = \pm\sqrt{-g}$. All these values together are included in the parametrization F_4 when $a_{02} \in \mathbb{C}^*$. Therefore, we have obtained all parametrizations in the proposition and the proof is done.

□

Let $g_1, g_2 \in \mathbb{K}[x, y]$. We say that g_1 *is in the orbit of* g_2 (or g_2 *is in the orbit of* g_1) if they are equivalent by an affine transformation of the plane $\mathbb{K}^2$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

Remark 3.2. If g_1 is in the orbit of g, then g_1 is a Hessian polynomial if and only if g is a Hessian polynomial. This remark is due to the equality $\operatorname{Hess}((1/\det T)(f \circ T)) = (\operatorname{Hess} f) \circ T$, where $T \in \operatorname{Aff}(2, \mathbb{K})$, and $\operatorname{Hess} f = g$.

Proof of Corollary 2.6. Note that the polynomial $g(x,y) = x \in \mathbb{A}_2^{\mathbb{C}}$ is a complex Hessian polynomial under $H_3^{\mathbb{C}}$ because $f(x,y) = (1/12)x^3 + y^2 \in (H_3^{\mathbb{C}})^{-1}(g)$. On the other hand, every complex polynomial of degree one is in the orbit of g. By Remark 3.2 we have that every complex polynomial of degree one is a Hessian polynomial. The real case is analogous. □

Proof of Proposition 2.5. Let $g(x,y) = b_{10}x + b_{01}y + b_{00}$ be a polynomial of degree one in $\mathbb{A}_2^{\mathbb{C}}$ with constant term. Then the expression $H_3^{\mathbb{C}}(f) = g$ is equivalent, by Lemma 3.1, to the system of equations:

$$0 = -a_{21}^2 + 3a_{30}a_{12}, \tag{3.16}$$

$$0 = -a_{21}a_{12} + 9a_{30}a_{03}, \tag{3.17}$$

$$0 = -a_{12}^2 + 3a_{21}a_{03}, \tag{3.18}$$

$$b_{10} = 12a_{30}a_{02} + 4a_{20}a_{12} - 4a_{21}a_{11}, \tag{3.19}$$

$$b_{01} = 12a_{20}a_{03} + 4a_{21}a_{02} - 4a_{12}a_{11}, \tag{3.20}$$

$$b_{00} = -a_{11}^2 + 4a_{20}a_{02}. \tag{3.21}$$

Denote by S_g the union of the images of F_1 and F_2. We shall prove that $(H_3^{\mathbb{C}})^{-1}(g) = S_g$. After some calculus it is proved that $S_g \subset (H_3^{\mathbb{C}})^{-1}(g)$. Now, we shall prove that $(H_3^{\mathbb{C}})^{-1}(g) \subset S_g$.

Suppose that $a_{21}, a_{12} \neq 0$. Multiply (3.19) by a_{21} and (3.20) by a_{12} and subtract the two obtained equations. It gives

$$4a_{20}a_{12}^2 - 4a_{02}a_{21}^2 + 12a_{30}a_{02}a_{12} - 12a_{03}a_{20}a_{21} = b_{10}a_{12} - b_{01}a_{21}. \tag{3.22}$$

From (3.16) and (3.18) we obtain $a_{21}^2 = 3a_{30}a_{12}$ and $a_{12}^2 = 9a_{30}a_{03}$, respectively, which we insert in (3.22) to obtain

$$a_{12}b_{10} = b_{01}a_{21}. \tag{3.23}$$

From (3.23) we obtain $a_{12} = b_{01}a_{21}/b_{10}$ if $b_{10} \neq 0$ and we insert it in (3.16) to obtain

$$a_{30} = \frac{b_{10}a_{21}}{3b_{01}}. \tag{3.24}$$

Analogous, we insert it in (3.18) to obtain

$$a_{03} = \frac{b_{01}^2 a_{21}}{3b_{10}^2}. \tag{3.25}$$

When we put (3.23), (3.24), and (3.25) in (3.17), it fulfills identically.

Case 1. Suppose that $a_{20} \neq 0$. From (3.21) it has

$$a_{02} = \frac{b_{00} + a_{11}^2}{4a_{20}}. \tag{3.26}$$

From (3.19)

$$a_{12} = \frac{4a_{21}a_{11} - 12a_{30}a_{02} + b_{10}}{4a_{20}}. \tag{3.27}$$

From (3.23) we obtain a_{21} and we replace it in (3.27). We replace also a_{30} from (3.24) and a_{02} to obtain

$$12a_{20}^2 b_{01}^2 a_{12} = 12a_{11}a_{20}a_{12}b_{10}b_{01} - 3a_{12}b_{10}^2\left(b_{00} + a_{11}^2\right) + 3a_{20}b_{10}b_{01}^2. \tag{3.28}$$

We associate the terms containing a_{12}

$$a_{12} = \frac{a_{20}b_{10}b_{01}^2}{4a_{20}^2 b_{01}^2 - 4a_{11}a_{20}b_{10}b_{01} + b_{10}^2 b_{00} + b_{10}^2 a_{11}^2}. \tag{3.29}$$

We replace this last expression of a_{12} in (3.23) and we obtain

$$a_{21} = \frac{a_{20}b_{10}^2 b_{01}}{4a_{20}^2 b_{01}^2 - 4a_{11}a_{20}b_{10}b_{01} + b_{10}^2 b_{00} + b_{10}^2 a_{11}^2}. \tag{3.30}$$

We replace the expression of a_{21} in (3.24) and (3.25) and we have

$$\begin{aligned} a_{30} &= \frac{a_{20}b_{10}^3}{3\left(4a_{20}^2 b_{01}^2 - 4a_{11}a_{20}b_{10}b_{01} + b_{10}^2 b_{00} + b_{10}^2 a_{11}^2\right)}, \\ a_{03} &= \frac{a_{20}b_{01}^3}{3\left(4a_{20}^2 b_{01}^2 - 4a_{11}a_{20}b_{10}b_{01} + b_{10}^2 b_{00} + b_{10}^2 a_{11}^2\right)}. \end{aligned} \tag{3.31}$$

Note that those last four expressions are in the image of F_1.

Case 2. Suppose that $a_{20} = 0$. From (3.21) we obtain

$$a_{11}^2 = -b_{00}. \tag{3.32}$$

From (3.19)

$$a_{02} = \frac{4a_{21}a_{11} + b_{10}}{12a_{30}}. \tag{3.33}$$

We replace the value of a_{30} (obtained from (3.24)) and the expression a_{11} in this last equation to obtain

$$a_{02} = \frac{b_{01}\left(\pm 4a_{21}\sqrt{-b_{00}} + b_{10}\right)}{4a_{21}b_{10}}. \tag{3.34}$$

The expression of a_{12} is obtained from (3.23). The expressions of this case are in the image of F_2. □

Proof of Theorem 2.7. Complex Case

(1) Let $g(x,y) = ax^2 + bxy + cy^2 + dx + ey + h \in \mathbb{A}_2^{\mathbb{C}}$ be a quadratic polynomial. By an affine transformation of the complex plane the polynomial g is equivalent to the prenormal form $P(x,y) = y^2 - Ax^2 - Bx - C$, where $A, B, C \in \mathbb{C}$.

If $A \neq 0$, then P is in the orbit of $y^2 - x^2 + k$, where $k \in \mathbb{C}$. If $A = 0$ and $B \neq 0$, then P is in the orbit of $y^2 + x$. Finally, if $A = 0, B = 0, C \neq 0$, then P is in the orbit of $y^2 - m$, where $m \in \mathbb{C}$.

Note that the polynomial $f(x,y) = (\sqrt{-1}/2)xy^2 \in \mathbb{A}_3^{\mathbb{C}}$ satisfies $H_3^{\mathbb{C}}(f) = y^2$. The polynomial $f(x,y) = xy^2/2 + x^2/2 \in \mathbb{A}_3^{\mathbb{C}}$ verifies $H_3^{\mathbb{C}}(f) = x - y^2$. For each $k \in \mathbb{C}$, the polynomial $f_k(x,y) = y^3/6 + x^2y/2 - (\sqrt{-k}/2)x^2 + (\sqrt{-k}/2)y^2 \in \mathbb{A}_3^{\mathbb{C}}$ fulfills $H_3^{\mathbb{C}}(f_k) = y^2 - x^2 - k$.

(2) To verify that the polynomials $y^2 + r$, where $r \in \mathbb{C}^*$, are not complex Hessian polynomials, we used the computer algebra system Maple 9.5. In particular, the Groebner package with the graded reverse lexicographic monomial order. We obtained a reduced Groebner basis for the system $H_3^{\mathbb{C}}(f) = y^2 + r$. The Groebner basis obtained was 1. So, by the Weak Nullstellensatz Theorem there is no solution to this system of equations.

Real Case

Analogous to the complex case, after a composition with an affine transformation of the real plane, the real quadratic polynomial $g(x,y) = ax^2 + bxy + cy^2 + dx + ey + h$ is in the orbit of one of the normal forms: $y^2 - x^2 - q_1$, $y^2 + x^2 - q_2$, $-y^2 - x^2 - q_3$, $x - y^2$, $y^2 + x$, $y^2 - q_4$, $-y^2 - q_5$, where $q_1, \ldots, q_5 \in \mathbb{R}$. □

Definition 3.3. We say that a *complex polynomial is totally imaginary* if the real part of all its coefficients is zero.

Lemma 3.4. *Let $g \in \mathbb{A}_{2n-4}^{\mathbb{R}}$ be a polynomial with real coefficients. Then g is a real Hessian polynomial if and only if there exists a polynomial f totally imaginary on the set $(H_n^{\mathbb{C}})^{-1}(-g)$.*

(1) By Lemma 3.4 and Proposition 2.9 we have that the polynomials $y^2 - x^2 - r_1, -y^2 - x^2 - r_2$, with $r_1, r_2 > 0$ are real Hessian polynomials. To finish this part we note that the polynomial $f(x,y) = xy^2/2 + x^2/2 \in \mathbb{A}_3^{\mathbb{R}}$ verifies $H_3^{\mathbb{R}}(f) = x - y^2$ and that $H_3^{\mathbb{R}}(xy^2/2) = -y^2$.

(2) By Lemma 3.4 and Proposition 2.9 we have that the polynomials $-x^2+y^2$, $-x^2-y^2$, $y^2-x^2-r_3$, $y^2+x^2-r_4$, $-y^2-x^2+r_5$, with $r_3, r_5 < 0$, $r_4 \in \mathbb{R}$, are not real Hessian polynomials. On the other hand, the polynomials y^2-r_6, $-y^2-r_7$, where $r_6, r_7 \in \mathbb{R}^*$, are not real Hessian polynomials because the complex polynomial y^2+r, $r \in \mathbb{C}^*$ is not complex Hessian polynomial. To show that the polynomials y^2 and y^2+x are not real Hessian polynomials, we use the same method of Groebner basis realized in the complex case.

Proof of Lemma 3.4. $\Rightarrow$) By hypothesis there exists $f(x,y) = \sum_{r+s=0}^{n} a_{rs} x^r y^s \in \mathbb{A}_n^{\mathbb{R}}$ such that $H_n^{\mathbb{R}}(f) = g$. Therefore, consider the totally imaginary polynomial $if(x,y) = \sum_{r+s=0}^{n} i a_{rs}\, x^r y^s$, which satisfies $H_n^{\mathbb{C}}(if) = i^2 H_n^{\mathbb{C}}(f) = -g$.

$\Leftarrow$) By hypothesis there exists f totally imaginary such that $H_n^{\mathbb{C}}(f) = -g$. Therefore, if is a real polynomial such that $H_n^{\mathbb{R}}(if) = i^2 H_n^{\mathbb{C}}(f) = g$. Hence, g is a real Hessian polynomial. □

Proof of Corollary 2.8. Let us consider the map $\psi : \mathbb{A}_3^{\mathbb{C}} \times \mathbb{C}^2 \to \mathbb{C}^3 = \{(w_1, w_2, w_3)\}$ given by

$$(f,p) \longmapsto (f_{xx}(p), f_{xy}(p), f_{yy}(p)). \tag{3.35}$$

If $f(x,y) = \sum_{i+j=0}^{3} a_{ij} x^i y^j \in \mathbb{A}_3^{\mathbb{C}}$ and $p = (x,y)$, then

$$\psi(f,p) = (6a_{30}x + 2a_{21}y + 2a_{20}, 2a_{21}x + 2a_{12}y + 2a_{11}, 2a_{12}x + 6a_{03}y + 2a_{02}). \tag{3.36}$$

For each fixed $f \in \mathbb{A}_3^{\mathbb{C}}$ let us consider $\psi_f : \mathbb{C}^2 \to \mathbb{C}_3$ given by $\psi_f(p) := \psi(f,p)$. □

Now, we are interested to describe the conditions in f under which the image under ψ_f of $\mathbb{C}^2$, $\psi_f(\mathbb{C}^2)$, is not a plane.

Lemma 3.5. *The set $S\psi := \{f \in \mathbb{A}_3^{\mathbb{C}} \mid \psi_f(\mathbb{C}^2)$ is not a plane$\}$ is given by the union of the following sets:*

$$\begin{aligned} CP_1 &:= \left\{ f \in \mathbb{A}_3^{\mathbb{C}} \mid a_{30} = a_{21} = a_{12} = 0 \right\}, \\ CP_2 &:= \left\{ f \in \mathbb{A}_3^{\mathbb{C}} \mid a_{30} \in \mathbb{C}^*, a_{12} = \frac{a_{21}^2}{3a_{30}}, a_{03} = \frac{a_{21}^3}{27a_{30}^2} \right\}. \end{aligned} \tag{3.37}$$

Proof. The Jacobian matrix of ψ_f is

$$\begin{pmatrix} 3a_{30} & a_{21} \\ a_{21} & a_{12} \\ a_{12} & 3a_{03} \end{pmatrix}. \tag{3.38}$$

$\psi_f(\mathbb{C}^2)$ is not a plane if and only if $J\psi_f$ has not maximal rank, which is equivalent to solve the system of equations given by the three minor equal to zero. That is,

$$3a_{12}a_{30} - a_{21}^2 = 0, \qquad 3a_{21}a_{03} - a_{12}^2 = 0, \qquad 9a_{03}a_{30} - a_{21}a_{12} = 0. \tag{3.39}$$

The sets CP_1 and CP_2 are obtained by solving the last system of equations. □

Lemma 3.6. *Let* $f(x,y)=\sum_{i+j=0}^{3}a_{ij}x^iy^j$ *be a complex polynomial such that* $f\in S\psi$.

Case 1. Suppose $f\in CP_1$.

(1) If $a_{03}\neq 0$, then $\psi_f(\mathbb{C}^2)$ is a parallel line to the w_3-axis in $\mathbb{C}^3$.

(2) If $a_{03}=0$, then $\psi_f(\mathbb{C}^2)$ is the point $(2a_{20},a_{11},2a_{02})$.

Case 2. If $f\in CP_1$, then $\psi_f(\mathbb{C}^2)$ is the line

$$l_f=\left\{\left(6a_{30}x+2a_{20},2a_{21}x+a_{11},\frac{2a_{21}^2}{3a_{30}}x+2a_{02}\right)\mid x\in\mathbb{C}\right\}. \tag{3.40}$$

Proof. Let $f(x,y)=\sum_{i+j=0}^{3}a_{ij}x^iy^j\in\mathbb{A}_3^{\mathbb{C}}$. In virtue of Lemma 3.5 we have the following cases

Case 1. If $f\in CP_1$, then, by (3.36), $\psi_f(\mathbb{C}^2)=\{(2a_{20},a_{11},6a_{30}y+2a_{02})\in\mathbb{C}^3\mid y\in\mathbb{C}\}$.

(1) If $a_{03}\neq 0$, then $\psi_f(\mathbb{C}^2)=\{(2a_{20},a_{11},6a_{30}y+2a_{02})\in\mathbb{C}^3\mid y\in\mathbb{C}\}$.

(2) If $a_{03}=0$, then $\psi_f(\mathbb{C}^2)$ is the point $(2a_{20},a_{11},2a_{02})$.

Case 2. If $f\in CP_2$, then, by (3.36),

$$\psi_f\left(\mathbb{C}^2\right)=\left(6a_{30}x+2a_{21}y+2a_{20},2a_{21}x+\frac{2a_{21}^2}{3a_{30}}y+a_{11},\frac{2a_{21}^3}{9a_{30}^2}y+\frac{2a_{21}^2}{3a_{30}}x+2a_{02}\right). \tag{3.41}$$

Let $v_1=(6a_{30},2a_{21},2a_{21}^2/3a_{30})$ be a nonzero vector (by hypothesis) and $v_2=(2a_{21},2a_{21}/3a_{30},2a_{21}^3/9a_{30}^2)$. Note that $v_2=(a_{21}/3a_{30})v_1$. Therefore, the set l_f is the same set of $\psi_f(\mathbb{C}^2)$. □

Let us consider the cone $C=\{(w_1,w_2,w_3)\in\mathbb{C}^3\mid w_1w_3-w_2^2=0\}$. We shall describe the set $\psi_f(\mathbb{C}^2)\cap C$ when $f\in S\psi$.

Lemma 3.7. *Let* $f\in S\psi$.

(1) *If* $\psi_f(\mathbb{C}^2)$ *is a parallel line to the* w_3*-axis in* $\mathbb{C}^3$*, then*

(a) $\psi_f(\mathbb{C}^2)\cap C$ *is a parallel line to the* w_3*-axis in* $\mathbb{C}^3$ *whenever* $a_{20}=0$;

(b) $\psi_f(\mathbb{C}^2)\cap C$ *is the point* $(2a_{20},a_{11},(a_{11}^2-4a_{20}a_{02})/2a_{20}+2a_{20})$ *whenever* $a_{20}\neq 0$.

(2) *If* $\psi_f(\mathbb{C}^2)$ *is the point* $(2a_{20},a_{11},2a_{02})$ *and* $4a_{20}a_{02}-a_{11}^2=0$*, then* $\psi_f(\mathbb{C}^2)\cap C$ *is the point* $(2a_{20},a_{11},2a_{02})$.

(3) *If* $\psi_f(\mathbb{C}^2)$ *is the line* l_f*, then*

(a) $\psi_f(\mathbb{C}^2)\cap C$ *is the point* $(6a_{30}x_0+2a_{20},2a_{21}x_0+a_{11},(2a_{21}^2/3a_{30})x_0+2a_{02})$ *if* $\alpha=(1/3a_{30})(36a_{30}^2a_{02}+4a_{20}a_{21}^2-12a_{21}a_{11}a_{30})\neq 0$*, where* $x_0=(1/\alpha)(a_{11}^2-4a_{20}a_{02})$;

(b) $\psi_f(\mathbb{C}^2)\cap C$ *is the line* l_f *whenever* $\alpha=0$.

Proof. In virtue of Lemma 3.6 we have the following cases.

(1) If $\psi_f(\mathbb{C}^2)\cap C$ is a parallel line to the w_3-axis, then $\psi_f(\mathbb{C}^2)\cap C = \{(2a_{20}, a_{11}, 6a_{03}y + 2a_{02}) \in \mathbb{C}^3 \mid y \in \mathbb{C}, 2a_{20}(6a_{03}y + 2a_{02}) - a_{11}^2 = 0\}$.

(a) If $a_{20} = 0$, then $\psi_f(\mathbb{C}^2)\cap C$ is a parallel line to the w_3-axis.

(b) If $a_{20} \neq 0$, then $y = (a_{11}^2 - 4a_{20}a_{02})/12a_{20}6a_{03}$ and $\psi_f(\mathbb{C}^2)\cap C$ is a point.

(2) If $\psi_f(\mathbb{C}^2)\cap C$ is the point $(2a_{20}, a_{11}, 2a_{02})$, then $\psi_f(\mathbb{C}^2)\cap C = (2a_{20}, a_{11}, 2a_{02})$ if $4a_{20}a_{02} - a_{11}^2 = 0$.

(3) If $\psi_f(\mathbb{C}^2)\cap C$ is the line l_f, then $\psi_f(\mathbb{C}^2)\cap C = l_f \cap \{(6a_{30}x + 2a_{20})((2a_{21}^2/3a_{30})x + 2a_{20}) - (2a_{21}x + a_{11})^2 = 0\}$. It means that

$$\alpha x + 4a_{20}a_{02} - a_{11}^2 = 0, \quad \text{where } \alpha = \frac{36a_{30}^2a_{02} + 4a_{20}a_{21}^2 - 12a_{21}a_{11}a_{30}}{3a_{30}}. \tag{3.42}$$

Therefore, if $\alpha = 0$, then $4a_{20}a_{02} - a_{11}^2 = 0$ and we obtain a line on C. If $\alpha \neq 0$, then $\psi_f(\mathbb{C}^2)\cap C$ is a point. □

With this lemma we finish the proof of corollary.

Proof of Proposition 2.9. Let $g(x, y) = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{00} \in \mathbb{A}_2^{\mathbb{C}}$. Then the expression $H_3^{\mathbb{C}}(f) = g$ is equivalent, by Lemma 3.1, to the system of equations:

$$\begin{aligned}
b_{02} &= -4a_{21}^2 + 12a_{30}a_{12},\\
b_{11} &= -4a_{21}a_{12} + 36a_{30}a_{03},\\
b_{02} &= -4a_{12}^2 + 12a_{21}a_{03},\\
0 &= 12a_{30}a_{02} + 4a_{20}a_{12} - 4a_{21}a_{11},\\
0 &= 12a_{20}a_{03} + 4a_{21}a_{02} - 4a_{12}a_{11},\\
b_{00} &= -a_{11}^2 + 4a_{20}a_{02}.
\end{aligned} \tag{3.43}$$

Let S be the union of the images of F_1 and F_2. A direct substitution shows that $S \subset (H_3^{\mathbb{C}})^{-1}(g)$. Therefore, to finish the proof it is enough to show that $(H_3^{\mathbb{C}})^{-1}(g) \subset S$. To check this last sentence we have used the computer algebra system Maple 9.5. □

Proof of the Theorem 2.12. Real Case

We will proof that the plane curve $g(x, y) = 0$, where $g(x, y) = xy^2 - x^3 + x$, is not a real Hessian curve (the other cases are analogous). To do that, we will prove that there are no real cuartic polynomial f in $(H_4^{\mathbb{C}})^{-1}(g)$; that is, there are no real polynomial satisfying the system of equations:

$$24a_{40}a_{22} - 9a_{31}^2 = 0, \tag{3.44}$$

$$72a_{40}a_{13} - 12a_{31}a_{22} = 0, \tag{3.45}$$

$$144a_{40}a_{04} + 18a_{31}a_{13} - 12a_{22}^2 = 0, \tag{3.46}$$

$$72a_{31}a_{04} - 12a_{22}a_{13} = 0, \tag{3.47}$$

$$24a_{04}a_{22} - 9a_{13}^2 = 0, \tag{3.48}$$

$$12a_{30}a_{22} + 24a_{40}a_{12} - 12a_{31}a_{21} = -1, \tag{3.49}$$

$$36a_{30}a_{13} + 72a_{40}a_{03} - 12a_{21}a_{22} = 0, \tag{3.50}$$

$$36a_{03}a_{31} + 72a_{04}a_{30} - 12a_{12}a_{22} = 1, \tag{3.51}$$

$$12a_{03}a_{22} + 24a_{04}a_{21} - 12a_{13}a_{12} = 0, \tag{3.52}$$

$$24a_{40}a_{02} + 4a_{20}a_{22} + 12a_{30}a_{12} - 6a_{31}a_{11} - 4a_{21}^2 = 0, \tag{3.53}$$

$$12a_{31}a_{02} + 12a_{20}a_{13} + 36a_{30}a_{03} - 4a_{21}a_{12} - 8a_{11}a_{22} = 0, \tag{3.54}$$

$$24a_{04}a_{20} + 4a_{02}a_{22} + 12a_{03}a_{21} - 6a_{13}a_{11} - 4a_{12}^2 = 0, \tag{3.55}$$

$$12a_{30}a_{02} + 4a_{20}a_{12} - 4a_{21}a_{11} = 1, \tag{3.56}$$

$$12a_{03}a_{20} + 4a_{02}a_{21} - 4a_{12}a_{11} = 0, \tag{3.57}$$

$$4a_{20}a_{02} - a_{11}^2 = 0. \tag{3.58}$$

A Groebner bases for (3.44)–(3.48) with the graded reverse lexicographic monomial order is

$$\begin{gathered} 3a_{13}^2 - 8a_{22}a_{04},\ a_{22}a_{13} - 6a_{31}a_{04},\ 2a_{22}^2 - 3a_{31}a_{13} - 24a_{40}a_{04},\ a_{31}a_{22} - 6a_{40}a_{13}, \\ 3a_{31}^2 - 8a_{40}a_{22},\quad a_{13}a_{31}a_{04} - 16a_{40}a_{04}^2,\ a_{31}a_{40}a_{13} - 16a_{40}^2a_{04}. \end{gathered} \tag{3.59}$$

The set of common zeroes of these last 7 polynomials is the union of the sets

$$\{a_{40} = a_{40},\ a_{31} = 0,\ a_{22} = 0,\ a_{13} = 0,\ a_{04} = 0\}, \tag{3.60}$$

$$\left\{a_{40} = \frac{a_{13}^4}{256\ a_{04}^3},\ a_{31} = \frac{a_{13}^3}{16a_{04}^2},\ a_{22} = \frac{3a_{13}^2}{8a_{04}},\ a_{13} = a_{13},\ a_{04} \neq 0\right\}. \tag{3.61}$$

On the one hand, replacing the solution (3.60) in (3.49)–(3.58), particularly in (3.56) we obtain $0 = 1$. On the other hand, replacing the solution (3.61) in (3.49)–(3.58), we obtain the system

$$
\begin{gathered}
4a_{20}a_{02} - a_{11}^2 = 0, \\
12a_{30}a_{02} + 4a_{20}a_{12} - 4a_{21}a_{11} - 1 = 0, \\
-4a_{12}a_{11} + 4a_{21}a_{02} + 12a_{20}a_{03} = 0, \\
12a_{30}a_{12} + \frac{3a_{13}^4a_{02}}{32a_{04}^3} - 4a_{21}^2 - \frac{3a_{13}^3a_{11}}{8a_{04}^2} + \frac{3a_{20}a_{13}^2}{2a_{04}} = 0, \\
36a_{30}a_{03} + 12a_{20}a_{13} - \frac{3a_{13}^2a_{11}}{a_{04}} - 4a_{21}a_{12} + \frac{3a_{13}^3a_{02}}{4a_{04}^2} = 0, \\
-6a_{13}a_{11} + 24a_{20}a_{04} + \frac{3a_{13}^2a_{02}}{2a_{04}} + 12a_{21}a_{03} - 4a_{12}^2 = 0, \\
\frac{-3a_{13}^3a_{21}}{4a_{04}^2} + \frac{9a_{30}a_{13}^2}{2a_{04}} + \frac{3a_{13}^4a_{12}}{32a_{04}^3} + 1 = 0, \\
\frac{9a_{13}^4a_{03}}{32a_{04}^3} + 36a_{30}a_{13} - \frac{9a_{21}a_{13}^2}{2a_{04}} = 0, \\
\frac{9a_{13}^3a_{03}}{4a_{04}^2} - \frac{9a_{13}^2a_{12}}{2a_{04}} + 72a_{30}a_{04} - 1 = 0, \\
-12a_{13}a_{12} + \frac{9a_{13}^2a_{03}}{2a_{04}} + 24a_{21}a_{04} = 0.
\end{gathered}
\tag{3.62}
$$

From the last equation we obtain that $a_{21} = -a_{13}(-8a_{12}a_{04} + 3a_{13}a_{03})/16a_{04}^2$ and from the sixth equation,

$$
a_{20} = -\frac{6a_{13}^2a_{02}a_{04} - 24a_{13}a_{11}a_{04}^2 + 24a_{13}a_{03}a_{12}a_{04} - 9a_{13}^2a_{03}^2 - 16a_{12}^2a_{04}^2}{96a_{04}^3}. \tag{3.63}
$$

When we replace these expressions in the last four equations of the last system (we do not write the other equations because we do not need it):

$$
\frac{-18a_{13}^4a_{12}a_{04} + 9a_{13}^5a_{03} + 288a_{30}a_{13}^2a_{04}^3 + 64a_{04}^4}{64a_{04}^4} = 0, \tag{3.64}
$$

$$
\frac{9a_{13}(32a_{30}a_{04}^3 - 2a_{12}a_{13}^2a_{04} + a_{13}^3a_{03})}{8a_{04}^3} = 0, \tag{3.65}
$$

$$\frac{9a_{13}^3a_{03} - 18a_{12}a_{13}^2a_{04} + 288a_{30}a_{04}^3 - 4a_{04}^2}{4a_{04}^2} = 0, \tag{3.66}$$

$$0 = 0. \tag{3.67}$$

From (3.65) we have that $a_{13} = 0$ or $32a_{30}a_{04}^3 - 2a_{12}a_{13}^2a_{04} + a_{13}^3a_{03} = 0$. If $a_{13} = 0$, then (3.64) becomes $-1 = 0$.

If $32a_{30}a_{04}^3 - 2a_{12}a_{13}^2a_{04} + a_{13}^3a_{03} = 0$, then $a_{30} = (2a_{12}a_{13}^2a_{04} - a_{13}^3a_{03})/32a_{04}^3$. Substituing a_{30} in (3.66), we obtain $-1 = 0$. □

Lemma 3.8. *The set, $Cv(3)$, of critical points of the map $H_{3R}^{\mathbb{C}} : A_{3R}^{\mathbb{C}} \to A_2^{\mathbb{C}}$ is the union of the following six sets:*

$$\begin{gathered}
S_1 = \left\{ a_{02} = \frac{9a_{03}a_{11}a_{30} - 6a_{03}a_{21}a_{20} - a_{12}a_{11}a_{21} + 2a_{12}^2a_{20}}{2\left(3a_{12}a_{30} - a_{21}^2\right)}, 3a_{12}a_{30} - a_{21}^2 \neq 0 \right\}, \\
S_2 = \{a_{11} = 0, a_{21} = 0, a_{12} = 0\}, \\
S_3 = \left\{ a_{11} = \frac{2a_{12}a_{20}}{a_{21}}, a_{30} = \frac{a_{21}^2}{3a_{12}}, a_{21}a_{12} \neq 0 \right\}, \\
S_4 = \{a_{30} = 0, a_{21} = 0, a_{12} = 0\}, \\
S_5 = \{a_{20} = 0, a_{30} = 0, a_{21} = 0\}, \\
S_6 = \left\{ a_{12} = \frac{a_{21}^2}{3a_{30}}, a_{03} = \frac{a_{21}^3}{27a_{30}^2}, a_{30} \neq 0 \right\}.
\end{gathered} \tag{3.68}$$

Proof. Let $f(x, y) = \sum_{i+j=2}^{3} a_{ij}x^iy^j \in \mathbb{A}_{3R}^{\mathbb{C}}$. The Jacobian matrix of $H_{3R}^{\mathbb{C}}$ is,

$$J\left(H_{3R}^{\mathbb{C}}\right) = \begin{pmatrix}
2a_{02} & -a_{11} & 2a_{20} & 0 & 0 & 0 & 0 \\
6a_{03} & -2a_{12} & 2a_{21} & 0 & 2a_{02} & -2a_{11} & 6a_{20} \\
2a_{12} & -2a_{21} & 6a_{30} & 6a_{02} & -2a_{11} & 2a_{20} & 0 \\
0 & 0 & 0 & 0 & 6a_{03} & -4a_{12} & 6a_{21} \\
0 & 0 & 0 & 18a_{03} & -2a_{12} & -2a_{21} & 18a_{30} \\
0 & 0 & 0 & 6a_{12} & -4a_{21} & 6a_{30} & 0
\end{pmatrix}. \tag{3.69}$$

Let us denote by $M_k, k = 1, \ldots, 7$, the 6×6 matrix obtained from $J(H_{3R}^{\mathbb{C}})$ by deleting the column $7 - k + 1$. On the other hand, f is a critical point of $H_{3R}^{\mathbb{C}}$ if and only if $\det(M_k) = 0$ for all $k = 1, \ldots, 7$, that is, if and only if the following system of seven equations is satisfied:

$$\begin{gathered}
\left(-9a_{21}a_{12}a_{03} + 27a_{30}a_{03}^2 + 2a_{12}^3\right)(F) = 0, \\
\left(a_{12}^2 a_{21} - 6a_{21}^2 a_{03} + 9a_{12}a_{30}a_{03}\right)(F) = 0, \\
\left(a_{21}^2 a_{12} - 6a_{12}^2 a_{30} + 9a_{21}a_{30}a_{03}\right)(F) = 0, \\
\left(2a_{21}^3 - 9a_{12}a_{21}a_{30} + 27a_{30}^2 a_{03}\right)(F) = 0, \\
\left(-3a_{03}a_{21}a_{11} + a_{12}^2 a_{11} - a_{21}a_{12}a_{02} + 9a_{03}a_{30}a_{02}\right)(F) = 0, \\
\left(3a_{02}a_{12}a_{30} + a_{12}^2 a_{20} - 3a_{03}a_{21}a_{20} - a_{21}^2 a_{02}\right)(F) = 0, \\
\left(a_{11}a_{21}^2 - a_{12}a_{21}a_{20} + 9a_{20}a_{30}a_{03} - 3a_{12}a_{11}a_{30}\right)(F) = 0,
\end{gathered} \tag{3.70}$$

where

$$F = -9a_{03}a_{11}a_{30} + 6a_{03}a_{21}a_{20} + 6a_{02}a_{12}a_{30} - 2a_{21}^2 a_{02} + a_{12}a_{11}a_{21} - 2a_{12}^2 a_{20}. \tag{3.71}$$

Therefore $f(x, y) = \sum_{i+j=2}^{3} a_{ij} x^i y^j$ satisfies the previous system of seven equations if and only if $f \in S = \bigcup_{k=1}^{6} S_k$, where S is the union of the six solutions S_j. Therefore, the proof is done. □

Lemma 3.9. *Consider $f \in \mathbb{A}_3^{\mathbb{C}}$ and $g = H_3(f)$. The curve $g = 0$ is singular if and only if*

$$\begin{gathered}
-2a_{12}^2 a_{20} + 6a_{12}a_{02}a_{30} + 6a_{21}a_{03}a_{20} + a_{21}a_{12}a_{11} - 2a_{21}^2 a_{02} - 9a_{03}a_{11}a_{30} = 0, \\
-a_{21}^2 a_{12}^2 - 18a_{21}a_{12}a_{30}a_{03} + 27a_{30}^2 a_{03}^2 + 4a_{12}^3 a_{30} + 4a_{21}^3 a_{03} \neq 0,
\end{gathered} \tag{3.72}$$

or,

$$\begin{aligned}
&- 9a_{12}a_{21}a_{03}a_{20} - a_{21}a_{11}a_{12}^2 - a_{12}a_{21}^2 a_{02} + 27a_{30}a_{03}^2 a_{20} + 2a_{20}a_{12}^3 + 1.8 \\
&+ 6a_{21}^2 a_{03}a_{11} - 9a_{30}a_{21}a_{03}a_{02} + 6a_{30}a_{02}a_{12}^2 - 9a_{30}a_{03}a_{11}a_{12} = 01, \\
&- a_{21}^2 a_{12}^2 - 18a_{21}a_{12}a_{30}a_{03} + 27a_{30}^2 a_{03}^2 + 4a_{12}^3 a_{30} + 4a_{21}^3 a_{03} = 0.2.1,
\end{aligned} \tag{3.73}$$

Proof. Suppose $f(x,y)=\sum_{r+s=0}^{3} a_{rs}x^r y^s$; then

$$\begin{aligned} H_3(f) = {} & -a_{11}^2 + 4a_{20}a_{02} + (12a_{30}a_{02} + 4a_{20}a_{12} - 4a_{21}a_{11})x \\ & + (12a_{20}a_{03} - 4a_{12}a_{11} + 4a_{21}a_{02})y + \left(-4a_{21}^2 + 12a_{30}a_{12}\right)x^2 \\ & + (-4a_{21}a_{12} + 36a_{30}a_{03})yx + \left(-4a_{12}^2 + 12a_{21}a_{03}\right)y^2. \end{aligned} \tag{3.74}$$

The curve $g = H_3^{\mathbb{C}}(f)$ is singular if and only if the system $g_x(p) = g_y(p) = g(p) = 0$ has a solution for some p in $\mathbb{C}^2$. The system formed by $g_x = 0, g_y = 0$ is

$$\begin{aligned} & \left(6a_{30}a_{12} - 2a_{21}^2\right)x + (9a_{30}a_{03} - a_{21}a_{12})y + 3a_{30}a_{02} + a_{20}a_{12} - a_{21}a_{11} = 0, \\ & (9a_{30}a_{03} - a_{21}a_{12})x + \left(6a_{21}a_{03} - 2a_{12}^2\right)y + 3a_{20}a_{03} - a_{12}a_{11} + a_{21}a_{02} = 0. \end{aligned} \tag{3.75}$$

The proof concludes by analyzing the system when its determinant is distinct from zero and when it is zero. □

Proof of Proposition 2.10. Let us consider the sets $S_k, k = 1, \ldots, 6$, of Lemma 3.8. If $f \in S_1$, then

$$\begin{aligned} H_3^{\mathbb{C}}(f) = {} & \frac{1}{3a_{12}a_{30} - a_{21}^2} \\ & \times \Big[x^2\left(36a_{30}^2a_{12}^2 + 4a_{21}^4 - 24a_{30}a_{12}a_{21}^2\right) \\ & \quad + xy\left(108a_{30}^2a_{03}a_{12} + 36a_{30}a_{03}a_{21}^2 + 4a_{21}^3a_{12} - 12a_{21}a_{12}^2a_{30}\right) \\ & \quad + y^2\left(4a_{12}^2a_{21}^2 - 12a_{12}^3a_{30} + 36a_{21}a_{03}a_{12}a_{30} - 12a_{21}^3a_{03}\right) \\ & \quad + x\big(24a_{30}a_{12}^2a_{20} + 54a_{30}^2a_{03}a_{11} + 4a_{21}^3a_{11} - 4a_{20}a_{12}a_{21}^2 \\ & \qquad -36a_{30}a_{03}a_{21}a_{20} - 18a_{30}a_{12}a_{11}a_{21}\big) \\ & \quad + y\big(-12a_{12}^2a_{11}a_{30} + 18a_{21}a_{03}a_{11}a_{30} + 36a_{20}a_{03}a_{12}a_{30} \\ & \qquad -24a_{21}^2a_{03}a_{20} + 2a_{21}^2a_{12}a_{11} + 4a_{21}a_{12}^2a_{20}\big) \\ & \quad - 3a_{11}^2a_{12}a_{30} - 12a_{03}a_{21}a_{20}^2 + 18a_{20}a_{03}a_{11}a_{30} \\ & \quad -2a_{20}a_{12}a_{11}a_{21} + 4a_{12}^2a_{20}^2 + a_{11}^2a_{21}^2\Big]. \end{aligned} \tag{3.76}$$

Denote by h the polynomial $H_3^{\mathbb{C}}(f)$. Then,

$$
\begin{aligned}
hx &= \frac{1}{3a_{12}a_{30}-a_{21}^2} \\
&\quad \times \Big[x\Big(72a_{30}^2a_{12}^2-48a_{30}a_{12}a_{21}^2+8a_{21}^4\Big) \\
&\qquad + y\Big(108a_{30}^2a_{03}a_{12}+4a_{21}^3a_{12}-36a_{30}a_{03}a_{21}^2-12a_{21}a_{12}^2a_{30}\Big) \\
&\qquad + 4a_{21}^3a_{11}+54a_{30}^2a_{03}a_{11}-4a_{20}a_{12}a_{21}^2+24a_{30}a_{12}^2a_{20} \\
&\qquad -36a_{30}a_{03}a_{21}a_{20}-18a_{30}a_{12}a_{11}a_{21}\Big], \\
hy &= \frac{1}{3a_{12}a_{30}-a_{21}^2} \\
&\quad \times \Big[x\Big(108a_{30}^2a_{03}a_{12}-36a_{30}a_{03}a_{21}^2-12a_{21}a_{12}^2a_{30}+4a_{21}^3a_{12}\Big) \\
&\qquad + y\Big(8a_{12}^2a_{21}^2-24a_{12}^3a_{30}-24a_{21}^3a_{03}+72a_{21}a_{03}a_{12}a_{30}\Big) \\
&\qquad + 4a_{21}a_{12}^2a_{20}-24a_{21}^2a_{03}a_{20}+2a_{21}^2a_{12}a_{11}-12a_{12}^2a_{11}a_{30} \\
&\qquad +18a_{21}a_{03}a_{11}a_{30}+36a_{20}a_{03}a_{12}a_{30}\Big].
\end{aligned}
\tag{3.77}
$$

The point (x,y) where the Hessian curve is singular is

$$
x=\frac{-2a_{20}a_{12}+a_{21}a_{11}}{2\big(3a_{12}a_{30}-a_{21}^2\big)}, \qquad y=-\frac{3a_{11}a_{30}-2a_{21}a_{20}}{2\big(3a_{12}a_{30}-a_{21}^2\big)}. \tag{3.78}
$$

If $f\in S_2$, then $H_3^{\mathbb{C}}(f)=(6a_{30}x+2a_{20})(6a_{03}y+2a_{02})$, $hx=6a_{30}(6a_{03}y+2a_{02})$, and $hy=6(6a_{30}x+2a_{20})a_{03}$.

The point (x,y) where the Hessian curve is singular is $x=-a_{20}/3a_{30}$, $y=-a_{02}/3a_{03}$.

If $f\in S_3$, then

$$
\begin{aligned}
H_3^{\mathbb{C}}(f) &= \frac{-4\big(a_{21}^2x+a_{21}ya_{12}+a_{20}a_{12}\big)\big(y\big(-3a_{21}^2a_{03}+a_{21}a_{12}^2\big)-a_{21}^2a_{02}+a_{20}a_{12}^2\big)}{a_{12}a_{21}^2}, \\
hx &= \frac{-4\big(y\big(-3a_{21}^2a_{03}+a_{21}a_{12}^2\big)-a_{21}^2a_{02}+a_{20}a_{12}^2\big)}{a_{12}}, \\
hy &= -\frac{4\big(y\big(-3a_{21}^2a_{03}+a_{21}ya_{12}^2\big)-a_{21}^2a_{02}+a_{20}a_{12}^2\big)}{a_{21}} \\
&\quad -\frac{4\big(a_{21}^2x+a_{21}ya_{12}+a_{20}a_{12}\big)\big(-3a_{21}^2a_{03}+a_{21}a_{12}^2\big)}{a_{12}a_{21}^2}.
\end{aligned}
\tag{3.79}
$$

The point (x, y) where the Hessian curve is singular is

$$x = \frac{a_{12}(-a_{21}a_{02} + 3a_{20}a_{03})}{a_{21}(-3a_{21}a_{03} + a_{12}^2)}, \qquad y = -\frac{-(-a_{21}^2 a_{02} + a_{20}a_{12}^2)}{a_{21}(-3a_{21}a_{03} + a_{12}^2)}. \tag{3.80}$$

If $-3a_{21}a_{03} + a_{12}^2 = 0$, then $H_3^{\mathbb{C}}(f)$ is of degree one.
If $f \in S_4$, then $H_3^{\mathbb{C}}(f) = 12a_{20}a_{03}y + 4a_{20}a_{02} - a_{11}^2$, $hx = 0$, and $hy = 12a_{20}a_{03}$.
The Hessian curve in this case has degree one.
If $f \in S_5$, then $H_3^{\mathbb{C}}(f) = -(2a_{12}y + a_{11})^2$, $hx = 0$, $hy = -4(2a_{12}y + a_{11})a_{12}$.
The point (x, y) where the Hessian curve is singular is $y = -a_{11}/2a_{12}$.
If $f \in S_6$, then

$$\begin{aligned} H_3^{\mathbb{C}}(f) = &\left(\frac{108a_{30}^3 a_{02} + 12a_{20}a_{21}^2 a_{30} - 36a_{21}a_{30}^2 a_{11}}{9a_{30}^2}\right)x \\ &+ \left(\frac{36a_{21}a_{02}a_{30}^2 + 4a_{20}a_{21}^3 - 12a_{21}^2 a_{11}a_{30}}{9a_{30}^2}\right)y + \frac{36a_{20}a_{02}a_{30}^2 - 9a_{11}^2 a_{30}^2}{9a_{30}^2}. \end{aligned} \tag{3.81}$$

The Hessian curve in this case has degree one. □

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