

*Research Article*

# On a New Summation Formula for ${}_2\psi_2$ Basic Bilateral Hypergeometric Series and Its Applications

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We have obtained a new summation formula for  ${}_2\psi_2$  bilateral basic hypergeometric series by the method of parameter augmentation and demonstrated its various uses leading to some development of  $q$ -gamma, and  $q$ -beta function identities.

## 1. Introduction

The summation formulae for hypergeometric series form a very interesting and useful component of the theory of (basic) hypergeometric series. The  $q$ -binomial theorem of Cauchy [1] is perhaps the first identity in the class of the summation formulae, which can be stated as

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} z^k = \frac{(az)_{\infty}}{(z)_{\infty}}, \quad |z| < 1, \quad |q| < 1, \quad (1.1)$$

where

$$(a)_{\infty} := (a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k), \quad (1.2)$$
$$(a)_k := (a; q)_k := \frac{(a)_{\infty}}{(aq^k)_{\infty}}, \quad k \text{ is an integer.}$$

For more details about the  $q$ -binomial theorem and about the identities which fall in this sequel, one may refer to the book by Gasper and Rahman [2]. Another famous identity in the sequel is the Ramanujan's  ${}_1\psi_1$  summation formula [3]

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty}}{(z)_{\infty} (b)_{\infty} (b/az)_{\infty} (q/a)_{\infty}}, \quad \left| \frac{b}{a} \right| < |z| < 1, \quad |q| < 1. \quad (1.3)$$

There are a number of proofs of the  ${}_1\psi_1$  summation formula (1.3) in the literature. For more details, one refers to the book by Berndt [4] and a recent paper of Johnson [5].

In this paper, we derive a new summation formula for  ${}_2\psi_2$  basic bilateral hypergeometric series using the  ${}_1\psi_1$  summation formula (1.3) by the method of parameter augmentation. We then use the formula to derive the  $q$ -analogue of Gauss summation formula and to obtain a number of etafunction,  $q$ -gamma, and  $q$ -beta function identities, which complement the works of Bhargava and Somashekara [6], Bhargava et al. [7], Somashekara and Mamta [8], Srivastava [9], and Bhargava and Adiga [10].

First, we recall that  $q$ -difference operator and the  $q$ -shift operator are defined by

$$D_q f(a) = \frac{f(a) - f(aq)}{a}, \quad \zeta(f(a)) = f(aq), \quad (1.4)$$

respectively. In [11], Chen and Liu have constructed an operator  $\theta$  as

$$\theta = \zeta^{-1} D_q, \quad (1.5)$$

and thereby they defined the operator  $E(b\theta)$  as

$$E(b\theta) = \sum_{k=0}^{\infty} \frac{(b\theta)^k q^{k(k-1)/2}}{(q; q)_k}. \quad (1.6)$$

Then, we have the following operator identities [12, Theorem 1]:

$$\begin{aligned} E(b\theta)\{(at; q)_{\infty}\} &= (at, bt; q)_{\infty}, \\ E(b\theta)\{(as, at; q)_{\infty}\} &= \frac{(as, at, bs, bt; q)_{\infty}}{(abst/q; q)_{\infty}}, \quad \left| \frac{abst}{q} \right| < 1. \end{aligned} \quad (1.7)$$

Further, the Dedekind etafunction is defined by

$$\eta(\tau) := e^{\pi i \tau / 12} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}) := q^{1/24} (q; q)_{\infty}, \tag{1.8}$$

where  $q = e^{2\pi i \tau}$ , and  $\text{Im}(\tau) > 0$ .

Jackson [13] defined the  $q$ -analogue of the gamma function by

$$\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1 - q)^{1-x}, \quad 0 < q < 1. \tag{1.9}$$

In his paper on the  $q$ -gamma and  $q$ -beta function, Askey [14] has obtained  $q$ -analogues of several classical results about the gamma function. Further, he has given the definition for  $q$ -beta function as

$$B_q(x, y) = (1 - q) \sum_{n=0}^{\infty} q^{nx} \frac{(q^{n+1})_{\infty}}{(q^{x+y})_{\infty}}. \tag{1.10}$$

In fact, he has shown that

$$B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{1.11}$$

In Section 2, we prove our main result. In Section 3, we deduce the well-known  $q$ -analogue of the Gauss summation formula and some etafunction,  $q$ -gamma, and  $q$ -beta function identities.

## 2. Main Result

**Theorem 2.1.** *If  $0 < |z| < 1$ ,  $|q| < 1$ , then*

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k (bc/azq)_k}{(b)_k (c)_k} z^k = \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty} (c/a)_{\infty} (bc/azq)_{\infty}}{(z)_{\infty} (b)_{\infty} (c)_{\infty} (b/az)_{\infty} (c/az)_{\infty} (q/a)_{\infty}}. \tag{2.1}$$

*Proof.* Ramanujan’s  ${}_1\psi_1$  summation formula (1.3) can be written as

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} z^k + \sum_{k=1}^{\infty} \frac{(q/b)_k}{(q/a)_k} \left(\frac{b}{az}\right)^k = \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty}}{(z)_{\infty} (b)_{\infty} (b/az)_{\infty} (q/a)_{\infty}}. \tag{2.2}$$

This is the same as

$$\begin{aligned} & \sum_{k=0}^{\infty} (a)_k z^k \left\{ (bq^k)_{\infty} \left( \frac{b}{az} \right)_{\infty} \right\} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{(q/a)_k} \left( \frac{1}{az} \right)^k \left\{ (bq^{-k})_{\infty} \left( \frac{b}{az} \right)_{\infty} \right\} \\ &= \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty}}{(z)_{\infty} (q/a)_{\infty}} \left\{ \left( \frac{b}{a} \right)_{\infty} \right\}. \end{aligned} \quad (2.3)$$

On applying  $E(c\theta)$  to both sides with respect to  $b$ , we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} (a)_k z^k \left\{ \frac{(bq^k)_{\infty} (b/az)_{\infty} (cq^k)_{\infty} (c/az)_{\infty}}{(bcq^k/azq)_{\infty}} \right\} \\ &+ \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{(q/a)_k} \left( \frac{1}{az} \right)^k \left\{ \frac{(bq^{-k})_{\infty} (b/az)_{\infty} (cq^{-k})_{\infty} (c/az)_{\infty}}{(bcq^{-k}/azq)_{\infty}} \right\} \\ &= \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty}}{(z)_{\infty} (q/a)_{\infty}} \left\{ \left( \frac{b}{a} \right)_{\infty} \left( \frac{c}{a} \right)_{\infty} \right\}. \end{aligned} \quad (2.4)$$

Multiplying (2.4) throughout by  $\{(bc/azq)_{\infty}/(b)_{\infty}(c)_{\infty}\}$ , we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(a)_k (bc/azq)_k}{(b)_k (c)_k} z^k + \sum_{k=1}^{\infty} \frac{(a)_{-k} (bc/azq)_{-k}}{(b)_{-k} (c)_{-k}} z^{-k} \\ &= \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty} (c/a)_{\infty} (bc/azq)_{\infty}}{(z)_{\infty} (b)_{\infty} (c)_{\infty} (b/az)_{\infty} (c/az)_{\infty} (q/a)_{\infty}}, \end{aligned} \quad (2.5)$$

which yields (2.1). □

### 3. Some Applications of the Main Identity

The following identity is the well-known  $q$ -analogue of the Gauss summation formula.

**Corollary 3.1** (see [15]). *If  $|q| < 1$ ,  $|\gamma/\alpha\beta| < 1$ , then*

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(q)_k (\gamma)_k} \left( \frac{\gamma}{\alpha\beta} \right)^k = \frac{(\gamma/\alpha)_{\infty} (\gamma/\beta)_{\infty}}{(\gamma)_{\infty} (\gamma/\alpha\beta)_{\infty}}. \quad (3.1)$$

*Proof.* Putting  $a = \alpha$ ,  $b = q$ ,  $c = \gamma$ , and  $z = \gamma/\alpha\beta$  in (2.1), we obtain (3.1). □

**Corollary 3.2.** *If  $|q| < 1$ , then*

$$\sum_{k=-\infty}^{\infty} \frac{(q^2; q^4)_k}{(1 - q^{4k+1})(q^3; q^4)_k} q^k = q^{-1/8} \frac{\eta^7(2\tau)}{\eta^3(\tau)\eta^2(4\tau)}, \tag{3.2}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-q^2; q^2)_k}{(1 + q^{2k+1})^2(-q; q^2)_k} q^k = 2q^{-3/4} \frac{\eta^6(4\tau)}{\eta^3(2\tau)}, \tag{3.3}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-q; q^2)_k(-q^2; q^2)_k}{(1 - q^{2k+1})^2(q; q^2)_n^2} q^k = 2q^{-3/4} \frac{\eta^8(4\tau)}{\eta^7(2\tau)}, \tag{3.4}$$

$$\sum_{k=-\infty}^{\infty} \frac{(1 + q)(-q; q^2)_k(-q^2; q^2)_k}{(1 - q^{2k+2})(q^2; q^2)_k^2} q^k = \frac{\eta(2\tau)}{\eta^2(\tau)}, \tag{3.5}$$

$$\sum_{k=-\infty}^{\infty} \frac{(q^2; q^6)_k}{(1 - q^{6k+1})(q^5; q^6)_k} q^{3k} = q^{-1/4} \frac{\eta^4(2\tau)}{\eta^2(\tau)}, \tag{3.6}$$

$$\sum_{k=-\infty}^{\infty} \frac{(1 + q^2)^2(-q^2; q^2)_{k-1}(-q^2; q^2)_{k+1}}{(1 - q^{2k+2})^2(q^2; q^2)_k^2} q^k = \frac{\eta^2(4\tau)}{\eta^4(2\tau)}. \tag{3.7}$$

*Proof.* Putting  $a = z = q^{1/4}$ ,  $b = q^{3/4}$ ,  $c = q^{5/4}$ , and then changing  $q$  to  $q^4$  in (2.1), we obtain

$$\sum_{k=-\infty}^{\infty} \frac{(q; q^4)_k(q^2; q^4)_k}{(q; q^4)_{k+1}(q^3; q^4)_k} q^k = \frac{(q^2; q^4)_{\infty}^4 (q^4; q^4)_{\infty}^2}{(q; q^4)_{\infty}^3 (q^3; q^4)_{\infty}^3}. \tag{3.8}$$

Simplifying the right hand side and then using (1.8), we obtain (3.2).

Similarly, putting  $a = -q^{1/2}$ ,  $z = q^{1/2}$ ,  $b = -q^{3/2}$ ,  $c = -q^{3/2}$ , and then changing  $q$  to  $q^2$ , we obtain (3.3). Putting  $a = -q^{1/2}$ ,  $z = q^{1/2}$ ,  $b = c = q^{3/2}$ , and then changing  $q$  to  $q^2$ , we obtain (3.4). Putting  $a = -q^{1/2}$ ,  $z = q^{1/2}$ ,  $b = q$ ,  $c = q^2$ , and then changing  $q$  to  $q^2$ , we obtain (3.5). Putting  $a = q^{1/6}$ ,  $z = q^{1/2}$ ,  $b = q^{7/6}$ ,  $c = q^{5/6}$ , and then changing  $q$  to  $q^6$ , we obtain (3.6). Finally, putting  $a = -1$ ,  $z = q$ ,  $b = c = q^2$ , and then changing  $q$  to  $q^2$ , we obtain (3.7).  $\square$

**Corollary 3.3.** *If  $0 < q < 1$ ,  $0 < x, y < 1$ , and  $0 < x + y < 1$ , then*

$$B_q^3(x, y) = \frac{\Gamma_q(1 - x + y)\Gamma_q(x - y)(1 - q)^2}{(1 - q^y)^2} \sum_{k=-\infty}^{\infty} \frac{(q^{1-x})_k (q^{x+y})_k}{(q^{1+y})_k^2} q^{ky}. \tag{3.9}$$

*Proof.* Putting  $a = q^{1-x}$ ,  $z = q^y$ , and  $b = c = q^{1+y}$  in (2.1), we get

$$\sum_{k=-\infty}^{\infty} \frac{(q^{1-x})_k (q^{x+y})_k}{(q^{1+y})_k^2} q^{ky} = \frac{(q^{1-x+y})_{\infty} (q)_{\infty} (q^{x-y})_{\infty}}{(q^y)_{\infty} (q^{1+y})_{\infty} (q^{1+y})_{\infty}} \left( \frac{(q^{x+y})_{\infty}}{(q^x)_{\infty}} \right)^3. \tag{3.10}$$

On using (1.9), (1.10), and (1.11), we obtain (3.9).  $\square$

**Corollary 3.4.** *If  $0 < q < 1$ ,  $0 < x, y < 1$ , and  $1 < x + y < 2$ , then*

$$B_q^2(x, y) = \frac{\Gamma_q(y-x)\Gamma_q(x-y+1)\Gamma_q(x+y-1)}{\Gamma_q(y)\Gamma_q(1+x)} \sum_{k=-\infty}^{\infty} \frac{(q^{-x})_k (q^{x+y-1})_k}{(q^y)_k^2} q^{ky}. \quad (3.11)$$

*Proof.* Putting  $a = q^{-x}$ ,  $z = b = c = q^y$ , in (2.1), and then using (1.9), (1.10), and (1.11), we obtain (3.11).  $\square$

**Corollary 3.5.** *If  $0 < x, y < 1$ , and  $0 < x + y < 1$ , then*

$$B^3(x, y) = \frac{\Gamma(1-x+y)\Gamma(x-y)}{y^2} \left[ \sum_{k=0}^{\infty} \frac{(1-x)_k (x+y)_k}{(1+y)_k^2} + \sum_{k=1}^{\infty} \frac{(-y)_k^2}{(x)_k (1-x-y)_k} \right]. \quad (3.12)$$

*Proof.* Letting  $q \rightarrow 1$  in (3.9), we obtain (3.12).  $\square$

**Corollary 3.6.** *If  $0 < x, y < 1$ , and  $1 < x + y < 2$ , then*

$$B^2(x, y) = \frac{\Gamma(y-x)\Gamma(x-y+1)\Gamma(x+y-1)}{\Gamma(y)\Gamma(1+x)} \times \left[ \sum_{k=0}^{\infty} \frac{(-x)_k (x+y-1)_k}{(y)_k^2} + \sum_{k=1}^{\infty} \frac{(1-y)_k^2}{(1+x)_k (2-x-y)_k} \right]. \quad (3.13)$$

*Proof.* Letting  $q \rightarrow 1$  in (3.11), we obtain (3.13).  $\square$

**Corollary 3.7.** *If  $0 < q < 1$ ,  $0 < x, y < 1$ , and  $0 < x + y < 1$ , then*

$$B_q(x, y) = \Gamma_q(y)\Gamma_q(1-y) \sum_{k=0}^{\infty} \frac{(q^{1-x-y})_k (q^y)_k}{(q)_k^2} q^{kx}. \quad (3.14)$$

*Proof.* Putting  $a = q^{1-x-y}$ ,  $z = q^x$ , and  $b = c = q$  in (2.1), we obtain (3.14).  $\square$

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