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Research Article

Fekete-Szegö Problem for a New Class of Analytic Functions

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We consider the Fekete-Szegö problem with complex parameter μ for the class $R_{\gamma}^{\tau}(\phi)$ of analytic functions.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathbb{U}$. A function $f(z)$ in $\mathcal{A}$ is said to be in class $\mathcal{S}^*$ of starlike functions of order zero in $\mathbb{U}$, if $\Re(zf'(z)/f(z)) > 0$ for $z \in \mathbb{U}$. Let $\mathcal{K}$ denote the class of all functions $f \in \mathcal{A}$ that are convex. Further, f is convex if and only if $zf'(z)$ is star-like. A function $f \in \mathcal{A}$ is said to be close-to-convex with respect to a fixed star-like function $g \in \mathcal{S}^*$ if and only if $\Re(zf'(z)/g(z)) > 0$ for $z \in \mathbb{U}$. Let $\mathcal{C}$ denote of all such close-to-convex functions [1].

Fekete and Szegö proved a noticeable result that the estimate

$$\left|a_3 - \lambda a_2^2\right| \leqq 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right) \tag{1.2}$$

holds for any normalized univalent function $f(z)$ of the form (1.1) in the open unit disk $\mathbb{U}$ and for $0 \leqq \lambda \leqq 1$. This inequality is sharp for each λ (see [2]). The coefficient functional

$$\phi_\lambda(f) = a_3 - \lambda a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\lambda}{2}\left[f''(0)\right]^2\right), \tag{1.3}$$

on normalized analytic functions f in the unit disk represents various geometric quantities, for example, when $\lambda = 1$, $\phi_\lambda(f) = a_3 - a_2^2$, becomes $S_f(0)/6$, where S_f denote the Schwarzian derivative $(f'''/f')' - (f''/f')^2/2$ of locally univalent functions f in $\mathbb{U}$. In literature, there exists a large number of results about inequalities for $\phi_\lambda(f)$ corresponding to various subclasses of $\mathcal{S}$. The problem of maximising the absolute value of the functional $\phi_\lambda(f)$ is called the Fekete-Szegö problem; see [2]. In [3], Koepf solved the Fekete-Szegö problem for close-to-convex functions and the largest real number λ for which $\phi_\lambda(f)$ is maximised by the Koebe function $z/(1-z)^2$ is $\lambda = 1/3$, and later in [4] (see also [5]), this result was generalized for functions that are close-to-convex of order β.

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\mathbb{U}$ onto a star-like region with respect to 1 which is symmetric with respect to the real axis. Let $\mathcal{S}^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathbb{U}), \tag{1.4}$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \quad (z \in \mathbb{U}), \tag{1.5}$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [6]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$.

Motivated by the class $R_\lambda^\tau(\beta)$ in paper [7], we introduce the following class.

Definition 1.1. Let $0 \leqq \gamma \leqq 1$, $\tau \in \mathbb{C} \setminus \{0\}$. A function $f \in \mathcal{A}$ is in th class $R_\gamma^\tau(\phi)$, if

$$1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) \prec \phi(z) \quad (z \in \mathbb{U}), \tag{1.6}$$

where $\phi(z)$ is defined the same as above.

If we set

$$\phi(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leqq B < A \leqq 1;\ z \in \mathbb{U}), \tag{1.7}$$

in (1.6), we get

$$R_\gamma^\tau\left(\frac{1 + Az}{1 + Bz}\right) = R_\gamma^\tau(A, B) = \left\{ f \in \mathcal{A} : \left| \frac{f'(z) + \gamma z f''(z) - 1}{\tau(A - B) - B\left(f'(z) + \gamma z f''(z) - 1\right)} \right| < 1 \right\}, \tag{1.8}$$

which is again a new class. We list few particular cases of this class discussed in the literature

(1) $R_\gamma^\tau(1-2\beta,-1) = R_\gamma^\tau(\beta)$ for $0 \leqq \beta < 1$, $\tau = \mathbb{C} \setminus \{0\}$ was discussed recently by Swaminathan [7].

(2) The class $R_\gamma^\tau(1-2\beta,-1)$ for $\tau = e^{i\eta}\cos\eta$, where $-\pi/2 < \eta < \pi/2$ is considered in [8] (see also [9]).

(3) The class $R_1^\tau(0,-1)$ with $\tau = e^{i\eta}\cos\eta$ was considered in [10] with reference to the univalencey of partial sums.

(4) $f \in R_\gamma^{e^{i\eta}\cos\eta}(1-2\beta,-1)$ whenever $zf'(z) \in P_\gamma^\tau(\beta)$, the class considered in [11].

For geometric aspects of these classes, see the corresponding references. The class $R_\gamma^\tau(A,B)$ is new as the author Swaminathan [7] has introduced class $R_\gamma^\tau(\beta)$ which is subclass of the class $R_\gamma^\tau(A,B)$, in his recent paper. To prove our main result, we need the following lemma.

Lemma 1.2 (see [12, 13]). *If $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$ $(z \in \mathbb{U})$ is a function with positive real part, then for any complex number μ,*

$$\left|c_2 - \mu c_1^2\right| \leqq 2\ \max\{1, |2\mu - 1|\}, \tag{1.9}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}). \tag{1.10}$$

2. Fekete-Szegö Problem

Our main result is the following theorem.

Theorem 2.1. *Let $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, where $\phi(z) \in \mathcal{A}$ with $\phi'(0) > 0$. If $f(z)$ given by* (1.1) *belongs to $R_\gamma^\tau(\phi)(0 \leqq \gamma \leqq 1, \tau \in \mathbb{C} \setminus \{0\},\ z \in \mathbb{U})$, then for any complex number μ*

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{B_1|\tau|}{3(1+2\gamma)} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3\tau\mu B_1(1+2\gamma)}{4(1+\gamma)^2}\right|\right\}. \tag{2.1}$$

The result is sharp.

Proof. If $f(z) \in R_\gamma^\tau(\phi)$, then there exists a Schwarz function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ such that

$$1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) = \phi(w(z)) \quad (z \in \mathbb{U}). \tag{2.2}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots. \tag{2.3}$$

Since $w(z)$ is a Schwarz function, we see that $\Re p_1(z) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) = 1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) = 1 + b_1 z + b_2 z^2 + \cdots. \tag{2.4}$$

In view of (2.2), (2.3), (2.4), we have

$$\begin{aligned} p(z) &= \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \phi\left(\frac{c_1 z + c_2 z^2 + \cdots}{2 + c_1 z + c_2 z^2 + \cdots}\right) \\ &= \phi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right) \\ &= 1 + B_1\frac{1}{2}c_1 z + B_1\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + B_2\frac{1}{4}c_1^2 z^2 + \cdots. \end{aligned} \tag{2.5}$$

Thus,

$$b_1 = \frac{1}{2}B_1 c_1; \quad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2. \tag{2.6}$$

From (2.4), we obtain

$$a_2 = \frac{B_1 c_1 \tau}{4(1+\gamma)}; \quad a_3 = \frac{\tau}{6(1+2\gamma)}\left[B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}B_2 c_1^2\right]. \tag{2.7}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{B_1 \tau}{6(1+2\gamma)}\left(c_2 - \nu c_1^2\right), \tag{2.8}$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{3\tau\mu B_1(1+2\gamma)}{4(1+\gamma)^2}\right). \tag{2.9}$$

Our result now is followed by an application of Lemma 1.2. Also, by the application of Lemma 1.2 equality in (2.1) is obtained when

$$p_1(z) = \frac{1+z^2}{1-z^2} \quad \text{or} \quad p_1(z) = \frac{1+z}{1-z} \tag{2.10}$$

but

$$p(z) = 1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) = \phi\left(\frac{p_{1(z)-1}}{p_{1(z)+1}}\right). \tag{2.11}$$

Putting value of $p_1(z)$ we get the desired results. □

For class $R_\gamma^\tau(A, B)$,

$$\begin{aligned}\phi(z) &= \frac{1 + Az}{1 + Bz} = (1 + Az)(1 + Bz)^{-1} \quad (z \in \mathbb{U}) \\ &= 1 + (A - B)z - \left(AB - B^2\right)z^2 + \cdots .\end{aligned} \tag{2.12}$$

Thus, putting $B_1 = A - B$ and $B_2 = -B(A - B)$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *If $f(z)$ given by* (1.1) *belongs to $R_\gamma^\tau(A, B)$, then*

$$\left|a_3 - \mu a_2^2\right| \leqq \frac{(A - B)|\tau|}{3(1 + 2\gamma)} \max\left\{1, \left|B + \frac{3\tau\mu(A - B)(1 + 2\gamma)}{4(1 + \gamma)^2}\right|\right\}. \tag{2.13}$$

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