Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 164843, 26 pages
doi:10.1155/2011/164843

Research Article

Toeplitz Operators on the Bergman Space of Planar Domains with Essentially Radial Symbols

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Received 25 February 2011; Revised 5 June 2011; Accepted 6 June 2011

Academic Editor: B. N. Mandal

We study the problem of the boundedness and compactness of T_ϕ when $\phi \in L^2(\Omega)$ and Ω is a planar domain. We find a necessary and sufficient condition while imposing a condition that generalizes the notion of radial symbol on the disk. We also analyze the relationship between the boundary behavior of the Berezin transform and the compactness of T_ϕ.

1. Introduction

Let Ω be a bounded multiply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves γ_j $(j = 1, 2, \ldots, n)$ where γ_j are positively oriented with respect to Ω and $\gamma_j \cap \gamma_i = \emptyset$ if $i \neq j$. We also assume that γ_1 is the boundary of the unbounded component of $\mathbb{C} \setminus \Omega$. Let Ω_1 be the bounded component of $\mathbb{C} \setminus \gamma_1$, and Ω_j $(j = 2, \ldots, n)$ the unbounded component of $\mathbb{C} \setminus \gamma_j$, respectively, so that $\Omega = \cap_{j=1}^{n} \Omega_j$.

For $d\nu = (1/\pi)dx\,dy$, we consider the usual L^2-space $L^2(\Omega) = L^2(\Omega, d\nu)$. The Bergman space $L^2_a(\Omega, d\nu)$, consisting of all holomorphic functions which are L^2-integrable, is a closed subspace of $L^2(\Omega, d\nu)$ with the inner product given by

$$\langle f, g \rangle = \int_\Omega f(z)\overline{g(z)}\,d\nu(z) \tag{1.1}$$

for $f, g \in L^2(\Omega, d\nu)$. The Bergman projection is the orthogonal projection

$$P : L^2(\Omega, d\nu) \longrightarrow L^2_a(\Omega, d\nu). \tag{1.2}$$

It is well-known that for any $f \in L^2(\Omega, d\nu)$, we have

$$Pf(w) = \int_\Omega f(z) K^\Omega(z,w)\, d\nu(z), \tag{1.3}$$

where K^Ω is the Bergman reproducing kernel of Ω. For $\varphi \in L^\infty(\Omega, d\nu)$, the Toeplitz operator $T_\varphi : L_a^2(\Omega, d\nu) \to L_a^2(\Omega, d\nu)$ is defined by $T_\varphi = PM_\varphi$, where M_φ is the standard multiplication operator. A simple calculation shows that

$$T_\varphi f(z) = \int_\Omega \varphi(w) f(w) K^\Omega(w,z)\, d\nu(w). \tag{1.4}$$

For square-integrable symbols, the Toeplitz operator is densely defined but is not necessarily bounded; therefore, the problem of finding necessary and sufficient conditions on the function $\varphi \in L^2(\Omega, d\nu)$ for the Toeplitz operators T_φ to be bounded or compact is a natural one, and it has been studied by many authors. Several important results have been established when the symbol has special geometric properties. In fact, in the context of radial symbols on the disk, many papers have been written with quite surprising results (see [1] of Grudsky and Vasilevski, [2] of Zorboska, and [3] of Korenblum and Zhu) showing that operators with unbounded radial symbols can have a very rich structure. In fact, in the case of a continuous symbol, the compactness of the Toeplitz operators depends only on the behavior of the symbol on the boundary of the disk and this is similar to what happens in the Hardy space case, even though in the case of Bergman space, the Toeplitz operator with continuous radial symbol is a compact perturbation of a scalar operator and in the Hardy space case a Toeplitz operator with radial symbol is just a scalar operator. In the case of unbounded radial symbols, a pivotal role is played by the fact that in the Bergman space setting, contrary to the Hardy space setting, there is an additional direction that Grudsky and Vasileski term as *inside the domain direction*: symbols that are nice with respect to the circular direction may have very complicated behavior in the radial direction. Of course, in the context of arbitrary planar domains, it is not possible to use the notion of radial symbol. We go around this difficulty by making two simple observations. To start, it is necessary to notice that the structure of the Bergman kernel suggests that there is in any planar domain an internal region that we can neglect when we are interested in boundedness and compactness of Toeplitz operators with square integrable symbols, therefore the *inside the domain direction* counts up to a certain point. The second observation consists in exploiting the geometry of the domain and conformal equivalence in order to partially recover the notion of radial symbol. For these reasons, we study the problem for planar domains when the Toeplitz operator symbols have an almost-radial behavior and, for this class, we give a necessary and sufficient condition for boundedness and compactness. We also address the problem of the characterization of compactness by using the Berezin transform. In fact, under a growth condition for the almost-radial symbol, we show that the Berezin transform vanishes to the boundary if and only if the operator is compact.

The paper is organized as follows. In Section 2, we describe the setting where we work, give the relevant definitions, and state our main result. In Section 3, we collect results about the Bergman kernel for a planar domain and the structure of $L_a^2(\Omega, d\nu)$. In Section 4, we prove the main result and study several important consequences.

2. Preliminaries

Let Ω be the bounded multiply-connected domain given at the beginning of Section 1, that is, $\Omega = \cap_{j=1}^{n}\Omega_j$, where Ω_1 is the bounded component of $\mathbb{C} \setminus \gamma_1$, and Ω_j $(j = 2,\dots,n)$ is the unbounded component of $\mathbb{C} \setminus \gamma_j$. We use the symbol Δ to indicate the punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Let Γ be any one of the domains Ω, Δ, Ω_j $(j = 2,\dots,n)$.

We call $K^{\Gamma}(z,w)$ the reproducing kernel of Γ and we use the symbol $k^{\Gamma}(z,w)$ to indicate the normalized reproducing kernel, that is, $k^{\Gamma}(z,w) = K^{\Gamma}(z,w)/K^{\Gamma}(w,w)^{1/2}$.

For any $A \in \mathcal{B}(L_a^2(\Gamma, d\nu))$, we define $\widetilde{A}$, the Berezin transform of A, by

$$\widetilde{A}(w) = \langle Ak_w^{\Gamma}, k_w^{\Gamma} \rangle = \int_{\Gamma} Ak_w^{\Gamma}(z)\overline{k_w^{\Gamma}(z)}d\nu(z), \tag{2.1}$$

where $k_w^{\Gamma}(\cdot) = K^{\Gamma}(\cdot,w)K^{\Gamma}(w,w)^{-1/2}$.

If $\varphi \in L^{\infty}(\Gamma)$, then we indicate with the symbol $\widetilde{\varphi}$ the Berezin transform of the associated Toeplitz operator T_{φ}, and we have

$$\widetilde{\varphi}(w) = \int_{\Gamma} \varphi(z)\left|k_w^{\Gamma}(z)\right|^2 d\nu(z). \tag{2.2}$$

We remind the reader that it is well known that $\widetilde{A} \in C_b^{\infty}(\Gamma)$, and we have $\|\widetilde{A}\|_{\infty} \le \|A\|_{\mathcal{B}(L^2(\Omega))}$. It is possible, in the case of bounded symbols, to give a characterization of compactness using the Berezin transform (see [4, 5]).

We remind the reader that any Ω bounded multiply-connected domain in the complex plane $\mathbb{C}$, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves γ_j $(j = 1,2,\dots,n)$, is conformally equivalent to a canonical bounded multiply-connected domain whose boundary consists of finitely many circles (see [6]). This means that it is possible to find a conformally equivalent domain $D = \cap_{i=1}^{n} D_i$ where $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $D_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2,\dots,n$. Here $a_j \in D_1$ and $0 < r_j < 1$ with $|a_j - a_k| > r_j + r_k$ if $j \ne k$ and $1 - |a_j| > r_j$. Before we state the main results of this paper we need to give a few definitions.

Definition 2.1. Let $\Omega = \cap_{i=1}^{n}\Omega_i$ be a canonical bounded multiply-connected domain. We say that the set of $n+1$ functions $\mathfrak{P} = \{p_0, p_1, \dots, p_n\}$ is a ∂-partition for Ω if

(1) for every $j = 0,1,\dots,n$, $p_j : \Omega \to [0,1]$ is a Lipschitz, C^{∞}-function,

(2) for every $j = 2,\dots,n$, there exists an open set $W_j \subset \Omega$ and an $\epsilon_j > 0$ such that $U_{\epsilon_j} = \{\zeta \in \Omega : r_j < |\zeta - a_j| < r_j + \epsilon_j\}$, and the support of p_j is contained in W_j and

$$p_j(\zeta) = 1, \quad \forall \zeta \in U_{\epsilon_j}, \tag{2.3}$$

(3) for $j = 1$, there exists an open set $W_1 \subset \Omega$ and an $\epsilon_1 > 0$ such that $U_{\epsilon_1} = \{\zeta \in \Omega : 1 - \epsilon_1 < |\zeta| < 1\}$ and the support of p_1 is contained in W_1 and

$$p_1(\zeta) = 1, \quad \forall \zeta \in U_{\epsilon_1}, \tag{2.4}$$

(4) for every $j, k = 1, \ldots, n$, $W_j \cap W_k = \emptyset$, the set $\Omega \setminus (\bigcup_{j=1}^{n} W_j)$ is not empty and the function

$$
\begin{aligned}
&p_0(\zeta) = 1, \quad \forall \zeta \in \left(\bigcup_{j=1}^{n} W_j\right)^c \cap \Omega, \\
&p_0(\zeta) = 0, \quad \forall \zeta \in U_{\epsilon_k}, \quad k = 1, \ldots, n,
\end{aligned}
\tag{2.5}
$$

(5) for any $\zeta \in \Omega$, the following equation:

$$
\sum_{k=0}^{n} p_k(\zeta) = 1 \tag{2.6}
$$

holds.

We need to point out two facts about the definition above: (i) that near each connected component of the boundary there is only one function which is different from zero (note that this implies that the function must be equal to 1), and (ii) far away from the boundary only the function p_0 is different from zero.

Definition 2.2. A function $\varphi : \Omega = \cap_{i=1}^{n} \Omega_i \to \mathbb{C}$ is said to be essentially radial if there exists a conformally equivalent canonical bounded domain $D = \cap_{i=1}^{n} D_i$, such that if the map $\Theta : \Omega \to D$ is the conformal mapping from Ω onto D, then

(1) for every $k = 2, \ldots, n$ and for some $\epsilon_k > 0$, we have

$$
\varphi \circ \Theta^{-1}(z) = \varphi \circ \Theta^{-1}(|z - a_k|), \tag{2.7}
$$

when $z \in U_{\epsilon_k} = \{\zeta \in \Omega : r_k < |\zeta - a_k| < r_k + \epsilon_k\}$,

(2) for $k = 1$ and for some $\epsilon_1 > 0$, we have

$$
\varphi \circ \Theta^{-1}(z) = \varphi \circ \Theta^{-1}(|z|), \tag{2.8}
$$

when $z \in U_{\epsilon_1} = \{\zeta \in \Omega : 1 - \epsilon_1 < |\zeta| < 1\}$.

The reader should note that in the case where it is necessary to stress the use of a specific conformal equivalence, we will say that the map φ is essentially radial *via* $\Theta : \cap_{\ell=1}^{n} \Omega_\ell \to \cap_{\ell=1}^{n} D_\ell$.

Before we proceed, the reader should notice that the definition, in the case of the disk, just says that, when we are near to the boundary, the values depend only on the distance from the center of the disk, so the function is essentially radial. In the general case, to formalize the fact that the values depend essentially on the distance from the boundary, we can simplify our analysis if we use the fact that this type of domain is conformally equivalent to a canonical bounded multiply-connected domain whose boundary consists of finitely many circles. For this type of domain the idea of essentially radial symbol is quite natural. For this reason, we use this simple geometric intuition to give the general definition.

Before we state the main result, we stress that in what follows, when we are working with a general multiply-connected domain and we have a conformal equivalence $\Theta : \cap_{\ell=1}^{n}\Omega_\ell \rightarrow \cap_{\ell=1}^{n}D_\ell$, we always assume that the ∂-partition is given on $\cap_{\ell=1}^{n}D_\ell$ and transferred to $\cap_{\ell=1}^{n}\Omega_\ell$ through Θ in the natural way.

At this point, we can state the main result.

Theorem 2.3. *Let $\varphi \in L^2(\Omega)$ be an essentially radial function via $\Theta : \cap_{\ell=1}^{n}\Omega_\ell \rightarrow \cap_{\ell=1}^{n}D_\ell$, if one defines $\varphi_j = \varphi \cdot p_j$, where $j = 1, \ldots, n$ and $\mathfrak{P}$ is a ∂-partition for Ω, then the following are equivalent:*

(1) *the operator*

$$T_\varphi : L_a^2(\Omega, d\nu) \longrightarrow L_a^2(\Omega, d\nu) \tag{2.9}$$

is bounded (compact).

(2) *for any $j = 1, \ldots, n$ the sequences $\gamma_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}_{m\in\mathbb{N}}$ are in $\ell_\infty(\mathbb{Z}_+)(c_0(\mathbb{Z}_+))$ where, by definition, if $j = 2, \ldots, n$,*

$$\gamma_{\varphi_j}(m) = r_j \int_{r_j}^{\infty} \varphi_j \circ \Theta^{-1}\left(r_j^{(2m+1)/2(m+1)} s^{1/2(m+1)} + a_j\right)\frac{1}{s^2}ds, \quad \forall m \in \mathbb{Z}_+, \tag{2.10}$$

and if $j = 1$

$$\gamma_{\varphi_1}(m) = \int_0^1 \varphi_1 \circ \Theta^{-1}\left(s^{1/2(m+1)}\right)ds, \quad \forall m \in \mathbb{Z}_+. \tag{2.11}$$

3. The Structure of $L_a^2(\Omega)$ and Some Estimates about the Bergman Kernel

From now on, we will assume that $\Omega = \cap_{j=1}^{n}\Omega_j$ where $\Omega_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\Omega_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2, \ldots, n$. Here, $a_j \in \Omega_1$ and $0 < r_j < 1$ with $|a_j - a_k| > r_j + r_k$ if $j \neq k$ and $1 - |a_j| > r_j$. We will indicate with the symbol $\Delta_{0,1}$ the punctured disk $\Omega_1 \setminus \{0\}$.

With the symbols $K^{\Omega_j}(z,w),\ K^{\Omega}(z,w),\ K^{\Delta}(z,w)$, we denote the Bergman kernel on $\Omega_j,\ \Omega$, and Δ, respectively.

In order to gain more information about the kernel of a planar domain, it is important to remind the reader that for the the punctured disk $\Delta_{0,1}$ and the disk Ω_1, we have $L_a^p(\Delta_{0,1}) = L_a^p(\Omega_1)$, if $p \geq 2$, and, for any $(z,w) \in \Delta^2,\ K^{\Delta}(z,w) = K^{\Omega_1}(z,w)$ (see [7, 8]). This fact has an important and simple consequence. In fact, if we consider $\Delta_{a,r} = \{z \in \mathbb{C} : 0 < |z - a| < r\}$ and $O_{a,r} = \{z \in \mathbb{C} : |z - a| > r\}$, we can conclude that

$$K^{O_{a,r}}(z,w) = \frac{r^2}{\left(r^2 - (z-a)\cdot\overline{(w-a)}\right)^2}, \quad \forall (z,w) \in O_{a,r} \times O_{a,r}. \tag{3.1}$$

To see this, we use the well-known fact that the reproducing kernel of the unit disk is given by $(1 - z\overline{w})^{-2}$, therefore we have

$$K^{\Delta_{0,1}}(z,w) = \frac{1}{(1 - z \cdot \overline{w})^2}, \quad \forall (z,w) \in \Delta_{0,1} \times \Delta_{0,1}. \tag{3.2}$$

This implies, by conformal mapping, that the reproducing kernel of $\Delta_{a,r}$ is

$$K^{\Delta_{a,r}}(z,w) = \frac{r^2}{\left(r^2 - (z-a) \cdot \overline{(w-a)}\right)^2}, \quad \forall (z,w) \in \Delta_{a,r} \times \Delta_{a,r}. \tag{3.3}$$

Now, we define $\varphi : \Delta_{a,r} \to O_{a,r}$ by

$$\varphi(z) = (z-a)^{-1} r^2 + a, \tag{3.4}$$

and we use the well-known fact that the Bergman kernels of $\Delta_{a,r}$ and $\varphi(\Delta_{a,r}) = O_{a,r}$ are related via

$$K^{O_{a,r}}(\varphi(z), \varphi(w)) \varphi'(z) \overline{\varphi'(w)} = K^{\Delta_{a,r}}(z,w) \tag{3.5}$$

to obtain that

$$K^{O_{a,r}}(z,w) = \frac{r^2}{\left(r^2 - (z-a) \cdot \overline{(w-a)}\right)^2}, \quad \forall (z,w) \in O_{a,r} \times O_{a,r}. \tag{3.6}$$

Since $\Omega_1 = O_{0,1}$ and, for $j = 2, \dots, n$, $O_{a_j, r_j} = \Omega_j$, then the last equation implies that

$$\begin{aligned} K^{\Omega_1}(z,w) &= \frac{1}{(1 - z \cdot \overline{w})^2}, \\ K^{\Omega_j}(z,w) &= \frac{r_j^2}{\left(r_j^2 - (z-a_j) \cdot \overline{(w-a_j)}\right)^2} \end{aligned} \tag{3.7}$$

if $j = 2, \dots, n$.

We also note that if we define

$$E^{\Omega}(z,w) = K^{\Omega}(z,w) - \sum_{j=1}^{n} K^{\Omega_j}(z,w), \tag{3.8}$$

we can prove the following.

Lemma 3.1. *(1) E^{Ω} is conjugate symmetric about z and w. For each $w \in \Omega$, $E^{\Omega}(\cdot, w)$ is conjugate analytic on Ω and $E^{\Omega} \in C^{\infty}(\overline{\Omega} \times \Omega)$.*

(2) There are neighborhoods U_j of $\partial\Omega_j$ $(j = 1, \ldots, n)$ and a constant $C > 0$ such that $U_j \cap U_k$ is empty if $j \neq k$ and

$$\left|K^{\Omega}(z,w) - K^{\Omega_j}(z,w)\right| < C, \tag{3.9}$$

for $z \in \Omega$ and $w \in U_j$.

(3) $E^{\Omega} \in L^{\infty}(\Omega \times \Omega)$.

Proof. (a) Since the Bergman kernels K^{Ω} and K^{Ω_j} have these properties (see [9]), by the definition of E^{Ω}, we get (1).

(b) The proof is given in [7, 8].

(c) Using the fact that

$$\begin{gathered} K^{\Omega_1}(z,w) = \frac{1}{(1 - z \cdot \overline{w})^2}, \\ K^{\Omega_j}(z,w) = \frac{r_j^2}{\left(r_j^2 - (z - a_j) \cdot \overline{(w - a_j)}\right)^2}, \end{gathered} \tag{3.10}$$

for $j = 2, \ldots, n$ and (1) and (2), we get (3). □

We observe that we can choose $R_j > r_j$ for $j = 2, \ldots, n$ and $R_1 < 1$ such that $G_j = \{z : r_j < |z - a_j| < R_j\}$ $(j = 2, \ldots, n)$ and $G_1 = \{z : R_1 < |z| < 1\}$, then we have $\overline{G_j} \subset U_j$, where U_j is the same as in Lemma 3.1. We also have the following.

Lemma 3.2. *There are constants $\mathfrak{D} > 0$ and $\mathcal{M} > 0$ such that*

(1) *for any $(z,w) \in G_i \times \Omega \cup \Omega \times G_i$, one has*

$$\begin{gathered} \left|K^{\Omega}(z,w)\right| < D\left|K^{\Omega_j}(z,w)\right|, \\ \left|K^{\Omega_j}(z,w)\right| < \left|K^{\Omega}(z,w)\right| + \mathcal{M}, \end{gathered} \tag{3.11}$$

(2) *for any $z \in \Omega$, one has $K^{\Omega_j}(z,z) < K^{\Omega}(z,z)$.*

Proof. By the explicit formula of the Bergman kernels K^{Ω_i}, there are constants C_i and M_i such that

$$\left|K^{\Omega_i}(z,w)\right| \geq C_i, \tag{3.12}$$

for $(z,w) \in (G_i \times \Omega) \cup (\Omega \times G_i)$ and

$$\left|K^{\Omega_i}(z,w)\right| \leq M_i \tag{3.13}$$

if $(z, w) \notin G_i \times G_i$ for $i = 1, 2, \ldots, n$. From the last Lemma, it follows that

$$\begin{aligned}
\left|K^{\Omega}(z,w)\right| &\le \left|K^{\Omega_i}(z,\mathrm{w})\right| + C \le \left(1 + \frac{C}{C_i}\right)\left|K^{\Omega_i}(z,w)\right|, \\
\left|K^{\Omega_i}(z,w)\right| &\le \left|K^{\Omega}(z,w)\right| + \left|E^{\Omega}(z,w)\right| + \sum_{j \ne i}\left|K^{\Omega_j}(z,w)\right| \\
&< \left|K^{\Omega}(z,w)\right| + \|E^{\Omega}\|_{\infty} + \sum_{i \ne j} M_j,
\end{aligned} \tag{3.14}$$

whenever $(z, w) \in (G_i \times \Omega) \cup (\Omega \times G_i)$. If we call $\mathfrak{D}$ the biggest number among $\{1 + C/C_j\}$ and we let $\mathcal{M} = \|E^{\Omega}\|_{\infty} + \sum_{j=1}^{n} M_j$, then we get the first claimed estimate. The proof of (2) can be found in [8, 10]. □

It is clear from what we wrote so far that we put a strong emphasis on the fact that the domain under analysis Ω is actually the intersection of other domains, that is, $\Omega = \cap_{j=1}^{n} \Omega_j$. This also suggests that we should look for a representation of the elements of $L_a^2(\Omega)$ that reflects this fact. For this reason, we give the following.

Definition 3.3. Given $\Omega = \cap_{j=1}^{n} \Omega_j$ with $\Omega_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\Omega_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$, for any $f \in L_a^2(\Omega)$, we define $n+1$ functions $P_0 f, P_1 f, P_2 f, \ldots, P_n f$ as follows: if $z \in \Omega$, then we set, for $j = 1$,

$$P_1 f(z) = \frac{1}{2\pi i} \cdot \int_{\widehat{\gamma}_1} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{3.15}$$

for $j = 2, 3, \ldots, n$,

$$P_j f = \frac{1}{2\pi i} \cdot \int_{\widehat{\gamma}_j} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \cdot \int_{\widehat{\gamma}_j} f(\zeta) d\zeta, \tag{3.16}$$

and for $j = 0$,

$$P_0 f = \sum_{j=2}^{n} \left(\frac{1}{2\pi i} \cdot \int_{\widehat{\gamma}_j} f(\zeta) d\zeta \right) \frac{1}{z - a_j}, \tag{3.17}$$

where $\widehat{\gamma}_j$ $(j = 1, \ldots, n)$ are the circles which center at a_j $(a_1 = 0)$ and lie in G_j (see Lemma 3.2), respectively, so that z is exterior to $\widehat{\gamma}_j$ $(j = 2, \ldots, n)$ and interior to $\widehat{\gamma}_1$.

It is important that the reader notices that the Cauchy theorem implies that our definition is independent from how we choose $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_n$. Moreover, it is important to notice that the domains of the functions $P_2 f, \ldots, P_n f$ are actually the sets $\Omega_2, \ldots, \Omega_n$. In the next Lemma, we give more information about this representation.

Lemma 3.4. *For $f \in L_a^2(\Omega)$, one can write it uniquely as*

$$f(z) = \sum_{j=1}^{n} (P_j f)(z) + (P_0 f)(z), \tag{3.18}$$

with $P_j f \in L_a^2(\Omega_j)$, $P_0 f \in L_a^2(\Omega) \cap C^\infty(\overline{\Omega})$, $P_k(P_j f) = 0$ if $j \neq k$, and moreover, there exists a constant M_1 such that, for $j = 0, 1, \ldots, n$, one has

$$\|P_j f\|_{\Omega} \leq \|P_j f\|_{\Omega_j} \leq M_1 \|f\|_{\Omega}. \tag{3.19}$$

In particular, if $f \in L_a^2(\Omega_i)$, then $P_i f = f$ and

$$\|f\|_{\Omega_i} \leq M_1 \|f\|_{\Omega}, \tag{3.20}$$

for $i = 1, \ldots, n$.

Proof. Let f be any function analytic on Ω. For any $z \in \Omega$, let γ_i $(i = 1, \ldots, n)$ be the circles which center at a_i $(a_1 = 0)$ and lie in G_i, respectively, so that z is exterior to γ_i $(i = 2, \ldots, n)$ and interior to γ_1. Using Cauchy's Formula, we can write

$$f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \cdot \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{3.21}$$

Let

$$f_j(z) = \frac{1}{2\pi i} \cdot \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{3.22}$$

By Cauchy's Formula, the value $f_j(z)$ does not depend on the choice of γ_j if $1 \leq j \leq n$ and $f(z) = \sum_j^n f_j(z)$. Of course, each f_j is well defined for all $z \in \Omega_j$ and analytic in Ω_j. In addition, if $j \neq 1$, we have that $f_j(z) \to 0$ as $|z| \to \infty$. Writing the Laurent expansion at a_j of f_j, we have

$$f_1(z) = \sum_{k=0}^{\infty} \alpha_{1,k} z^k, \tag{3.23}$$

and, for $j \neq 1$,

$$f_j(z) = \sum_{k=-1}^{-\infty} \alpha_{j,k} \left(z - a_j\right)^k, \tag{3.24}$$

and these series converge to f_j uniformly and absolutely on any compact subset of Ω_j, respectively. We remark that the coefficients are given by the following formula:

$$\alpha_{j,k} = \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(\zeta)}{\left(\zeta - a_j\right)^{k+1}} d\zeta, \tag{3.25}$$

where $k \geq 0$ if $j = 1$ and $k \leq -1$ if $j \neq 1$ and $\gamma_j \subset G_j$, $1 \leq j \leq n$. Moreover, if f is holomorphic in some Ω_j and $f(z) \to 0$ as $|z| \to \infty$ when $i \neq 1$, then $\alpha_{jk} = 0$ for all $j \neq i$ by Cauchy's theorem and, therefore, $f_j = 0$.

Now, we define $P_1 f = f_1$ and

$$P_j f(z) = \sum_{k=-2}^{-\infty} \alpha_{jk}\left(z - a_j\right)^k, \tag{3.26}$$

for $j = 2, 3, \ldots, n$ and

$$P_0 f(z) = \sum_{j=2}^{n} \alpha_{j,-1}\left(z - a_j\right)^{-1}, \tag{3.27}$$

then $f(z) = \sum_{i=0}^{n} P_i f(z)$ for all $z \in \Omega$ and $P_k(P_j f) = 0$ if $0 \neq k \neq j \neq 0$ as we have proved above.

We claim that $f \in L_a^2(\Omega)$ implies that $P_i f \in L_a^2(\Omega_j)$ for $j = 1, 2, \ldots, n$, respectively. Indeed, since each annulus G_j is contained in Ω, $f \in L_a^2(\Omega)$ implies that f is an element of $L_a^2(G_i)$ for all $i = 1, 2, \ldots, n$.

For any fixed i, note that $P_j f$ $(0 \neq j \neq i)$ and $P_0 f - \alpha_{j,-1} \cdot (z - a_j)^{-1}$ are analytic on $\overline{G_i} \cup (\mathbb{C}/\Omega_i)$ and $\lim_{|z| \to \infty} P_j f(z) = 0$ for $j \neq 1$. Expanding them as Laurent series, it follows that:

(1) if $i = 1$, then $P_j f = \sum_{k=1}^{+\infty} \beta_{jk}/z^k$ for $j \neq 1$,

(2) if $i \neq 1$, then

$$P_j f(z) = \sum_{k=0}^{+\infty} \beta_{jk}(z - a_i)^k, \tag{3.28}$$

for $0 \neq j \neq i$ and

$$P_0 f(z) = \sum_{k=0}^{+\infty} \beta_{0k}(z - a_i)^k + \frac{\alpha_{i,-1}}{z - a_i}. \tag{3.29}$$

It is obvious that, in any case, these series converge uniformly and absolutely on $\overline{G_i}$. Observing that each G_i is an annulus at a_i, we have, by direct computation, that

$$\langle f, f \rangle_{G_i} \geq \langle P_i f, P_i f \rangle_{G_i} + |\alpha_{i,-1}|^2 (\ln R_i - \ln r_i) \tag{3.30}$$

if $i \neq 1$ and

$$\langle f, f \rangle_{G_1} \geq \langle P_1 f, P_1 f \rangle_{G_1}. \tag{3.31}$$

Therefore, for any $i = 1, \ldots, n$, there exists a constant M' such that

$$\|P_i f\|_{G_i} \leq \|f\|_{G_i} \leq \|f\|_{\Omega}, \tag{$*$}$$

$$|\alpha_{i,-1}| \leq M' \cdot \|f\|_{\Omega}. \tag{$**$}$$

From the definition of $P_j f$, we derive

$$\begin{aligned} \|P_1 f\|^2_{G_1} &= \sum_0^{+\infty} \frac{|\alpha_{1k}|^2 \left(1 - R_1^{2k+2}\right)}{k+1}, \\ \|P_i f\|^2_{G_i} &= \sum_{k=-2}^{-\infty} \frac{|\alpha|^2_{ik} \left(r_i^{2k+2} - R_i^{2k+2}\right)}{k+1}, \end{aligned} \tag{3.32}$$

for $i = 2, \ldots, n$. The convergence of these series is guaranteed by the conditions $(*)$ and $(**)$. Since $R_1 < 1$ and $r_i < R_i$, it follows that $P_i f \in L^2_a(\Omega_i)$ and

$$\begin{aligned} \|P_1 f\|^2_{\Omega_1} &= \sum_0^{+\infty} \frac{|\alpha_{1k}|^2}{k+1}, \\ \|P_i f\|^2_{\Omega_i} &= \sum_{k=-2}^{-\infty} \frac{|\alpha_{1k}|^2 r_i^{2k+2}}{k+1}, \end{aligned} \tag{3.33}$$

for $i = 2, \ldots, n$. Comparing the expression of $\|P_i f\|_{\Omega_i}$ with the expression of $\|P_i f\|_{G_i}$, it follows that $\|P_i f\|_{\Omega_i} < M \cdot \|P_i f\|_{G_i}$ for some constant M for $i = 1, \ldots, n$. Hence, $\|P_i f\|_{\Omega_i} < M \cdot \|P_i f\|_{\Omega}$. Moreover, if we define $M'' = \mathrm{Max}\{\|(z - a_i)^{-1}\|_{\Omega}\}$, from the inequalities $\|P_i f\|_{G_i} \leq \|f\|_{G_i} \leq \|f\|_{\Omega}$ and $|\alpha_{i,-1}| \leq M' \cdot \|f\|_{\Omega}$ and the definition of P_0, it follows that $\|P_0 f\|_{\Omega} \leq n \cdot M' \cdot M'' \cdot \|f\|_{\Omega}$.

If $f \in L^2_a(\Omega_i)$ for some $i \in \{1, 2, \ldots, n\}$, note that $\lim f(z) = 0$ as $|z| \to \infty$ for $i \neq 1$, then $f(z) = P_i f(z) + \alpha_{i,-1}(z - a_i)^{-1}$ if $i \neq 1$ and $P_1 f = f$ if $i = 1$. For $i \neq 1$, since $f \in L^2_a(\Omega_i) \subset L^2_a(\Omega)$ implies that $P_i f \in L^2_a(\Omega_i)$, then $\alpha_{i,-1} \cdot (z - a_i)^{-1} \in L^2_a(\Omega_i)$. We must have $\alpha_{i,-1} = 0$ and, consequently, $P_0 f = 0$. Hence, in any case, $f \in L^2_a(\Omega_i)$ implies $f = P_i f$ and $P_j f = 0$ if $i \neq j$, and this remark completes our proof. □

Lemma 3.5. *If $\{f_n\}$ is a bounded sequence in $L^2_a(\Omega)$ and $f_n \to 0$ weakly in $L^2_a(\Omega)$, then $P_j f_n \to 0$ weakly on $L^2_a(\Omega_j)$ for $j = 1, \ldots, n$ and $P_0 f_n \to 0$ uniformly on Ω.*

Proof. By the previous Lemma, we know that the linear transformations $\{P_j\}$ are bounded operators, then $f_n \to 0$ weakly in $L^2_a(\Omega)$ implies that $P_j f_n \to 0$ weakly on $L^2_a(\Omega_j)$ for $j =$

$1, \ldots, n$. For the same reason, $P_0 f_n \to 0$ weakly in $L^2_a(\Omega)$ and then $P_0 f_n(\zeta) \to 0$ for any $\zeta \in \Omega$. Since

$$P_0 f_m = \sum_{i=2}^{n} \frac{\alpha_{i,-1}(m)}{(\zeta - a_i)}, \tag{3.34}$$

by the estimates given in the last lemma, we have that $|\alpha_{i,-1}(m)| < M\|f_m\|_\Omega$. The boundedness of $\{\|f_m\|_\Omega\}$ implies that the family of continuous functions $\{P_0 f_m\}$ is uniformly bounded and equicontinuous on $\overline{\Omega}$, then, by Arzela-Ascoli's Theorem, we have that $P_0 f_m \to 0$ uniformly on Ω. □

4. Canonical Multiply-Connected Domains and Essentially Radial Symbols

In this section, we investigate, with the help of the results established in the previous section, necessary and sufficient conditions on the essentially radial function $\varphi \in L^2(\Omega, d\nu)$ for the Toeplitz operator T_φ to be bounded or compact.

Before we state the next Theorem, we remind the reader that

$$K^\Omega(\zeta, z) = E^\Omega(\zeta, z) + \sum_{\ell=1}^{n} K^\Omega_\ell(\zeta, z), \tag{4.1}$$

where $E^\Omega \in L^\infty(\Omega \times \Omega)$ and, for all $\ell = 1, \ldots, n$, we have

$$K^\Omega_\ell(\zeta, z) = K^{\Omega_\ell}(\zeta, z), \quad \forall \zeta, z \in \Omega \times \Omega, \tag{4.2}$$

where K^{Ω_ℓ} is the reproducing kernel of Ω_ℓ. If we use the symbol K^Ω_0 to indicate E^Ω, we can write

$$K^\Omega(\zeta, z) = \sum_{\ell=0}^{n} K^\Omega_\ell(\zeta, z). \tag{4.3}$$

We also remind the reader that if $I : L^2_a(\Omega) \to L^2_a(\Omega)$ is the identity operator, then

$$I = \sum_{\ell=0}^{n} P_\ell, \tag{4.4}$$

where $P_\ell : L^2_a(\Omega) \to L^2_a(\Omega)$ is a bounded operator for all $\ell = 0, 1, \ldots, n$ with $P_\ell f \in L^2_a(\Omega_\ell)$ if $\ell = 1, \ldots, n$ and $P_0 f \in C^\infty(\overline{\Omega})$ and $P_k P_\ell = 0$ if $k \neq \ell$ (see Lemma 3.4).

In order to make our notation a little simpler, when we use a kernel operator we will denote it by the name of its kernel function. For example, the Bergman projection will be denoted by the symbol K^Ω.

We are now in a position to prove the following result.

Lemma 4.1. *Let $\varphi \in L^2(D)$ be an essentially radial function where $D = \cap_{j=1}^{n} D_j$ with $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $D_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2, \ldots, n$. If one defines $\varphi_j = \varphi \cdot p_j$ where $j = 1, \ldots, n$ and $\mathfrak{P} = \{p_0, p_1, \ldots, p_n\}$ is a ∂-partition for D, then the following are equivalent:*

(1) *the operator*

$$T_{\varphi} : L_a^2(D, d\nu) \longrightarrow L_a^2(D, d\nu) \tag{4.5}$$

is bounded (compact);

(2) *for any $j = 1, \ldots, n$, the operators*

$$T_{\varphi_j} : L_a^2(D_j, d\nu) \longrightarrow L_a^2(D_j, d\nu) \tag{4.6}$$

are bounded (compact).

Proof. Let $\{p_0, p_1, \ldots, p_n\}$ be a partition of the unit on $D = \cap_{j=1}^{n} D_j$, which is a canonical domain. Now, we notice that for all $f \in L^2(D)$ and for all $w \in D$, we have the following:

$$\begin{aligned}
T_{\varphi} f(w) &= \int_D \varphi(z) f(z) K^D(z, w) dv(z) \\
&= \sum_{j=0}^{n} \int_D \varphi(z) f(z) K^{D_j}(z, w) dv(z) \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} \int_D \varphi(z) p_k(z) f(z) K^{D_j}(z, w) dv(z) \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n} T_{jk} f(w),
\end{aligned} \tag{4.7}$$

where, by definition, we have

$$T_{jk} f(w) = \int_D \varphi(z) p_k(z) K^{D_j}(z, w) f(z) dv(w) dv(z). \tag{4.8}$$

□

Claim 1. The operator T_{j0} is Hilbert-Schmidt for any $j = 0, 1, \ldots, n$.

Proof. We observe that, by definition, we have

$$T_{j0} f(w) = \int_D \varphi(z) p_0(z) K^{D_j}(z, w) f(z) dv(z), \tag{4.9}$$

therefore, if we define

$$\mathcal{I}_1 = \iint_D \left| \varphi(z) p_0(z) K^{D_j}(z, w) \right|^2 dv(z) dv(w), \tag{4.10}$$

we have

$$\begin{aligned}
\mathcal{I}_1 &= \int_D \left|\varphi(z)p_0(z)\right|^2 \left(\int_D \left|K^{D_j}(z,w)\right|^2 dv(w)\right) dv(z) \\
&\leq \int_D \left|\varphi(z)p_0(z)\right|^2 \left|K^{D_j}(z,z)\right| dv(z) \\
&\leq \left(\operatorname*{Max}_{z\in\operatorname{supp}(p_0)} \left|p_0(z)\right|^2 K^{D_j}(z,z)\right) \int_D \left|\varphi_j(z)\right|^2 dv(z) \\
&\leq \left(\operatorname*{Max}_{z\in\operatorname{supp}(p_0)} \left|p_0(z)\right|^2 K^{D_j}(z,z)\right) \cdot \|\varphi\|_{D,2}^2 \\
&< \infty.
\end{aligned} \tag{4.11}$$

This implies that for any $t = 0, 1, \ldots, n$, T_{t0} is Hilbert-Schmidt. Therefore, the operator

$$\sum_{t=0}^{n} T_{t0} \tag{4.12}$$

is Hilbert-Schmidt, and this completes the proof of the claim. □

Claim 2. The operator T_{0k} is Hilbert-Schmidt for any $k = 0, 1, \ldots, n$.

Proof. We observe that, by definition, we have

$$T_{0k}f(w) = \int_D \varphi(z)p_k(z)K^{D_0}(z,w)f(z)dv(z), \tag{4.13}$$

therefore, if we define

$$\mathcal{I}_2 = \iint_D \left|\varphi(z)p_k(z)K^{D_0}(z,w)\right|^2 dv(z)dv(w), \tag{4.14}$$

we have

$$\begin{aligned}
\mathcal{I}_2 &= \iint_D \left|\varphi(z)p_0(z)\right|^2 \left|K^{D_0}(z,w)\right|^2 dv(w)dv(z) \\
&\leq \left(\operatorname*{Max}_{(z,w)\in D\times D} \left|K^{D_0}(z,w)\right|^2\right) \cdot v(D) \cdot \int_D \left|\varphi(z)p_0(z)\right|^2 dv(z) \\
&\leq \left(\operatorname*{Max}_{(z,w)\in D\times D} \left|K^{D_0}(z,w)\right|^2\right) \cdot v(D) \cdot \|\varphi\|_{D,2}^2 \\
&< \infty.
\end{aligned} \tag{4.15}$$

This implies that for any $t = 0, 1, \ldots, n$, T_{0t} is Hilbert-Schmidt. Therefore, the following

$$\sum_{t=0}^{n} T_{0t} \tag{4.16}$$

is Hilbert-Schmidt, and this completes the proof of the claim. □

Claim 3. The operator T_{ij} is Hilbert-Schmidt if $i \neq j \neq 0$ and $j, i = 1, \ldots, n$.

Proof. We observe that

$$T_{jk} f(w) = \int_D \varphi(z) p_k(z) K^{D_j}(z, w) f(z) dv(w) dv(z). \tag{4.17}$$

To start, we give the following:

$$\mathcal{N}_{ji}(z, w) \overset{\text{def}}{=} \varphi_j(z) \cdot K^{D_i}(z, w). \tag{4.18}$$

We will show that Fubini theorem and the properties of the ∂-partition imply that

$$\iint_D \left| \mathcal{N}_{ji}(z, w) \right|^2 dv(w) dv(z) < \infty. \tag{4.19}$$

In fact, we have

$$\begin{aligned}
\iint_D \left| \mathcal{N}_{ji}(z, w) \right|^2 &= \int_D \left(\int_D \left| \mathcal{N}_{ji}(z, w) \right|^2 dv(w) \right) dv(z) \\
&= \iint_D \left| \varphi_j(z) \right|^2 \left| K^{D_i}(z, w) \right|^2 dv(w) dv(z) \\
&= \int_D \left| \varphi_j(z) \right|^2 \left(\int_D \left| K^{D_i}(z, w) \right|^2 dv(w) \right) dv(z) \\
&= \int_D \left| \varphi_j(z) \right|^2 K^{D_i}(z, z) dv(z) \\
&= \int_D \left| \varphi(z) \right|^2 \left| p_j(z) \right|^2 K^{D_i}(z, z) dv(z) \\
&\leq \left(\operatorname*{Max}_{z \in \operatorname{supp}(p_j)} \left| p_j(z) \right|^2 K^{D_i}(z, z) \right) \cdot \|\varphi\|_{D,2}^2 \\
&< \infty.
\end{aligned} \tag{4.20}$$

Therefore, we can write that

$$T_\varphi = \mathcal{K} + \sum_{\ell=1}^{n} T_{\varphi_{\ell\ell}}, \tag{4.21}$$

where $\mathcal{K}$ is a compact operator.

We also observe that Lemma 3.4 implies that $T_{\varphi_{\ell\ell}} = \sum_{j=0}^{n} T_{\varphi_{\ell\ell}} P_j$, and we prove that the operator $T_{\varphi_{\ell\ell}} P_j$ is compact if $j \neq \ell$ and $j,\ \ell = 1, \ldots, n$. □

Proof. In order to simplify the notation, we define the operator $R_{j,\ell} = T_{\varphi_{\ell\ell}} P_j = K_\ell^D M_{\varphi p_\ell} P_j$. To prove our statement, it is enough to prove that if we take a bounded sequence $\{f_n\}$ in $L^2(D)$ such that $f_n \to 0$ weakly, then we can prove that $\|R_{j,\ell} f_n\|_2 \to 0$. We know that the continuity of P_ℓ implies that $P_j f_k \to 0$ weakly on $H^2(D_l)$, and $\{\|P_j f_k\|_{D_\ell}\}$ is bounded by Lemma 3.5. Since it is a sequence of holomorphic functions, we know that $\{P_j f_k\}$ is uniformly bounded on any compact subset of D_ℓ. Therefore, the sequence $\{P_j f_k\}$ is a normal family of functions. Since $P_j f_k(\zeta) \to 0$ for any $\zeta \in D_j$, then $P_j f_k$ converges uniformly on any compact subset of D_j and consequently on $F = \operatorname{supp}(p_\ell)$. To complete the proof, we remind the reader that if we define the operators $Q_\ell : L^2(D) \to L^2(D)$, for $\ell = 1, 2, \ldots, n$, in this way

$$Q_\ell f(z) = \int_D f(\zeta) \left| K_\ell^D(\zeta, z) \right| dv(\zeta). \tag{4.22}$$

It is possible to prove, with the help of Schur's test (see [11]), that Q_ℓ is a bounded operator (see [5]). Now, we observe that

$$\left| R_{j,\ell} f_k(\zeta) \right| \le \operatorname{Sup}\left\{ \left| P_j f_k(\zeta) \right| : \zeta \in F \right\} \cdot \left| Q_j\left(\left| \mathcal{X}_F \varphi p_s \right| \right)(\zeta) \right|, \tag{4.23}$$

then, by using the fact that Q_ℓ is bounded, we have

$$\left\| R_{j,\ell} f_k \right\|_D \le \operatorname{Sup}\left\{ \left| P_j f_k(\zeta) \right| : \zeta \in F \right\} \cdot M \cdot \left\| \varphi_1 p_s \right\|_{D,2} \longrightarrow 0, \tag{4.24}$$

and this completes the proof of our claim. Notice also that using the same strategy, we can prove that each $T_{\varphi_{\ell\ell}} P_0$ is compact. □

Therefore, we have

$$\begin{aligned} T_\varphi &= \mathcal{K} + \sum_{\ell=1}^{n} T_{\varphi_{\ell\ell}} \\ &= \mathcal{K} + K_1 + \sum_{\ell=1}^{n} T_{\varphi_{\ell\ell}} P_\ell, \end{aligned} \tag{4.25}$$

where $\mathcal{K}, K_1$ are compact operators. Since $P_t^2 = P_t$, $P_t P_s = 0$ and if $j \neq \ell$, then T_φ is bounded (compact) if and only if the operators $T_{\varphi_{\ell\ell}} P_\ell$ are bounded (compact) operators.

Since $P_\ell L_a^2(D) = L_a^2(D_\ell)$, then it follows that the operator $T_{\varphi_{\ell\ell}} P_\ell$ is bounded (compact) if and only if $T_{\varphi_{\ell\ell}}$is bounded (compact).

We are finally, with the help of [1]'s main result, in a position to prove the main result of this paper.

Theorem 4.2. *Let $\varphi \in L^2(D)$ be an essentially radial function where $D = \cap_{j=1}^n D_j$ with $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $D_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2, \ldots, n$. If one defines $\varphi_j = \varphi \cdot p_j$ where $j = 1, \ldots, n$ and $\mathfrak{P} = \{p_0, p_1, \ldots, p_n\}$ is a ∂-partition for D then the following are equivalent:*

(1) *the operator*

$$T_\varphi : L_a^2(D, d\nu) \longrightarrow L_a^2(D, d\nu) \tag{4.26}$$

is bounded (compact).

(2) *for any* $j = 1, \ldots, n$, *the sequences* $\gamma_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}_{m\in\mathbb{N}}$ *are in* $\ell_\infty(\mathbb{Z}_+)(c_0(\mathbb{Z}_+))$ *where, by definition, if* $j = 2, \ldots, n$

$$\gamma_{\varphi_j}(m) = r_j \int_{r_j}^{\infty} \varphi_j\left(r_j^{(2m+1)/2(m+1)} s^{1/2(m+1)} + a_j\right)\frac{1}{s^2}ds \quad \forall m \in \mathbb{Z}_+, \tag{4.27}$$

and for $j = 1$,

$$\gamma_{\varphi_1}(m) = \int_0^1 \varphi_1\left(s^{1/2(m+1)}\right)ds, \quad \forall m \in \mathbb{Z}_+. \tag{4.28}$$

Proof. In the previous theorem, we proved that the operator under examination is bounded (compact) if and only if for any $j = 1, \ldots, n$ the operators

$$T_{\varphi_j} : L^2(D_j, d\nu) \longrightarrow L_a^2(D_j, d\nu) \tag{4.29}$$

are bounded (compact). If $j = 2, \ldots, n$, we observe that if we consider the following sets $\Delta_{0,1} = \{z \in \mathbb{C} : 0 < |z - a| < 1\}$ and $\Delta_{a_j,r_j} = \{z \in \mathbb{C} : 0 < |z - a_j| < r_j\}$ and the following maps

$$\Delta_{0,1} \xrightarrow{\alpha} \Delta_{a_j,r_j} \xrightarrow{\beta} D_j, \tag{4.30}$$

where $\alpha(z) = a_j + r_j z$ and $\beta(w) = (w - a_j)^{-1} r_j^2 + a_j$ and we use Proposition 1.1 in [8], we can claim that

$$T_{\varphi_j} = V_{\beta\circ\alpha}^{-1} T_{\varphi_j\circ\beta\circ\alpha} V_{\beta\circ\alpha}, \tag{4.31}$$

where $V_{\beta\circ\alpha} : L^2(\Delta_{0,1}) \rightarrow L^2(D_j)$ is an isomorphism of Hilbert spaces. Therefore, T_{φ_j} is bounded (compact) if and only if $T_{\varphi_j\circ\beta\circ\alpha}$ is bounded (compact). We also know that this, in turn, is equivalent to the fact that the sequence

$$\gamma_{\varphi_j} = \left\{\gamma_{\varphi_j}(m)\right\}_{m\in\mathbb{N}} \tag{4.32}$$

is in $\ell_\infty(\mathbb{Z}_+)(c_0(\mathbb{Z}_+))$, where

$$\gamma_{\varphi_j}(m) = \int_0^1 \varphi_j \circ \beta \circ \alpha\left(r^{1/2(m+1)}\right)dr, \quad \forall m \in \mathbb{Z}_+. \tag{4.33}$$

To complete the proof, we observe that since φ_j is radial and $\beta \circ \alpha(r) = r^{-1} r_j + a_j$ then, after a change of variable, we can rewrite the last integral, and therefore the formula

$$\gamma_{\varphi_j}(m) = r_j \int_{r_j}^{\infty} \varphi_j \left(r_j^{(2m+1)/2(m+1)} s^{1/2(m+1)} + a_j \right) \frac{1}{s^2} ds, \quad \forall m \in \mathbb{Z}_+ \tag{4.34}$$

must hold for any $j = 2, \ldots, n$. The case $j = 1$ is immediate. □

Now, we can prove the following.

Theorem 4.3. *Let $\varphi \in L^2(\Omega)$ be an essentially radial function via the conformal equivalence $\Theta : \Omega \to D$, define $\varphi_j = \varphi \cdot p_j$ where $j = 1, \ldots, n$ and $\mathfrak{P}$ is a ∂-partition for Ω, then the following conditions are equivalent:*

(1) *the operator*

$$T_\varphi : L_a^2(\Omega, dv) \longrightarrow L_a^2(\Omega, dv) \tag{4.35}$$

is bounded (compact);

(2) *for any $j = 1, \ldots, n$, the sequences $\gamma_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}_{m \in \mathbb{N}}$ are in $\ell_\infty(\mathbb{Z}_+)(c_0(\mathbb{Z}_+))$ where, by definition, if $j = 2, \ldots, n$*

$$\gamma_{\varphi_j}(m) = r_j \int_{r_j}^{\infty} \varphi_j \circ \Theta^{-1} \left(r_j^{(2m+1)/2(m+1)} s^{1/2(m+1)} + a_j \right) \frac{1}{s^2} ds, \quad \forall m \in \mathbb{Z}_+, \tag{4.36}$$

and for $j = 1$

$$\gamma_{\varphi_1}(m) = \int_0^1 \varphi_1 \circ \Theta^{-1} \left(s^{1/2(m+1)} \right) ds, \quad \forall m \in \mathbb{Z}_+. \tag{4.37}$$

Proof. We know that Ω is a regular domain, and therefore if Θ is a conformal mapping from Ω onto D then the Bergman kernels of Ω and $\Theta(\Omega) = D$, are related via $K^D(\Theta(z), \Theta(w))\Theta'(z)\overline{\Theta'(w)} = K^\Omega(z, w)$, and the operator $V_\Theta f = \Theta' \cdot f \circ \Theta$ is an isometry from $L^2(D)$ onto$L^2(\Omega)$ (see Proposition 1.1 in [8]). In particular, we have $V_\Theta P^D = P^\Omega V_\Theta$ and this implies that $V_\Theta T_\varphi = T_{\varphi \circ \Theta^{-1}} V_\Theta$. Therefore, the operator T_φ is bounded (compact) if and only if the operator $T_{\varphi \circ \Theta^{-1}} : L^2(D, dv) \to L_a^2(D, dv)$ is bounded (compact). In the previous theorem we proved that the operator in exam is bounded (compact) if and only if for any $j = 1, \ldots, n$ the operators

$$T_{\varphi_j \circ \Theta^{-1}} : L_a^2(D_j, dv) \longrightarrow L_a^2(D_j, dv) \tag{4.38}$$

are bounded (compact). Hence, we can conclude that the operator is bounded (compact) if and only if for any $j = 1, \ldots, n$ the sequences $\gamma_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}_{m \in \mathbb{N}}$ are in $\ell_\infty(\mathbb{Z}_+)(c_0(\mathbb{Z}_+))$ where, by definition, if $j = 2, \ldots, n$, we have

$$\gamma_{\varphi_j}(m) = r_j \int_{r_j}^{\infty} \varphi_j \circ \Theta^{-1} \left(r_j^{(2m+1)/2(m+1)} s^{1/2(m+1)} + a_j \right) \frac{1}{s^2} ds, \quad \forall m \in \mathbb{Z}_+, \tag{4.39}$$

and for $j = 1$,

$$\gamma_{\varphi_1}(m) = \int_0^1 \varphi_1 \circ \Theta^{-1}\left(s^{1/2(m+1)}\right) ds, \quad \forall m \in \mathbb{Z}_+, \tag{4.40}$$

and this completes the proof. □

We now introduce a set of functions that will allow us to further explore the structure of Toeplitz operators with radial-like symbols. For $j = 2, \ldots, n$, we define

$$B_{\varphi_j}(s) = r_j \int_{r_j}^{s} \varphi_j \circ \Theta^{-1}\left(r_j^{1/2} x^{1/2} + a_j\right) \frac{1}{x^2} dx, \tag{4.41}$$

and for $j = 1$, we set

$$B_{\varphi_1}(s) = \int_s^1 \varphi_1 \circ \Theta^{-1}\left(x^{1/2}\right) dx. \tag{4.42}$$

We obtain the following useful theorem.

Theorem 4.4. *Let $\varphi \in L^2(\Omega)$ be an essentially radial function via the conformal equivalence $\Theta : \Omega \longrightarrow D$. If one defines $\varphi_j = \varphi \cdot p_j$ where $j = 1, \ldots, n$ and $\mathfrak{P}$ is a ∂-partition for Ω, then for the operator $T_\varphi : L_a^2(\Omega, d\nu) \longrightarrow L_a^2(\Omega, d\nu)$ the following hold true:*

(1) *if for any $j = 1, \ldots, n$*

$$\left|B_{\varphi_j}(s)\right| = O\left(r_j - s\right) \quad \text{as } s \longrightarrow r_j, \tag{4.43}$$

then T_φ is bounded;

(2) *if for any $j = 1, \ldots, n$*

$$\left|B_{\varphi_j}(s)\right| = o\left(r_j - s\right) \quad \text{as } s \longrightarrow r_j, \tag{4.44}$$

then T_φ is compact.

Proof. To prove the first, we observe that our main theorem implies that the boundedness (compactness) of the operator is equivalent to the fact that for any $j = 1, \ldots, n$ the sequences $\gamma_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}_{m \in \mathbb{N}}$ are in $\ell_\infty(\mathbb{Z}_+)(c_0(\mathbb{Z}_+))$ where, by definition, if $j = 2, \ldots, n$,

$$\gamma_{\varphi_j}(m) = r_j \int_{r_j}^{\infty} \varphi_j \circ \Theta^{-1}\left(r_j^{(2m+1)/2(m+1)} s^{1/2(m+1)} + a_j\right) \frac{1}{s^2} ds \quad \forall m \in \mathbb{Z}_+, \tag{4.45}$$

and for $j = 1$

$$\gamma_{\varphi_j}(m) = \int_0^1 \varphi_1 \circ \Theta^{-1}\left(s^{1/2(m+1)}\right) ds \quad \forall m \in \mathbb{Z}_+, \tag{4.46}$$

and, in virtue of [1]'s main result, it is true that $\gamma_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}_{m\in\mathbb{N}}$ are in $\ell_\infty(\mathbb{Z}_+)$ if for any $j = 1, \ldots, n$,

$$\left|B_{\varphi_j}(s)\right| = O(r_j - s) \quad \text{as } s \longrightarrow r_j, \tag{4.47}$$

and $\gamma_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}_{m\in\mathbb{N}}$ are in $c_0(\mathbb{Z}_+)$) if for any $j = 1, \ldots, n$

$$\left|B_{\varphi_j}(s)\right| = o(r_j - s) \quad \text{as } s \longrightarrow r_j. \tag{4.48}$$

□

It is also useful to observe that in the case of a positive symbol, we can prove that the condition above is necessary and sufficient. In fact (see [1]), we have the following.

Theorem 4.5. *Let $\varphi \in L^2(\Omega)$ be an essentially radial function via the conformal equivalence $\Theta : \Omega \to D$. If we define $\varphi_j = \varphi \cdot p_j$ where $j = 1, \ldots, n$ and $\mathfrak{P}$ is a ∂-partition for Ω and if $\varphi \geq 0$ a.e. in Ω, then for the operator $T_\varphi : L_a^2(\Omega, d\nu) \to L_a^2(\Omega, d\nu)$, the following hold true:*

(1) *T_φ is bounded if and only if*

$$\left|B_{\varphi_j}(s)\right| = O(r_j - s) \quad \textit{as } s \longrightarrow r_j, \tag{4.49}$$

for any $j = 1, \ldots, n$,

(2) *T_φ is compact if and only if*

$$\left|B_{\varphi_j}(s)\right| = o(r_j - s) \quad \textit{as } s \longrightarrow r_j, \tag{4.50}$$

for any $j = 1, \ldots, n$.

Proof. The proof is an immediate consequence of Theorem 3.5 in [1] and the theorem above. □

There are a few useful observations that we can make at this point. If the Toeplitz operator $T_\varphi : L_a^2(\Omega, d\nu) \to L_a^2(\Omega, d\nu)$ has an essentially radial positive symbol $\varphi \geq 0$ such that for some $\ell = 1, \ldots, n$, the following

$$\lim_{\delta\to 0}\left(\inf_{\operatorname{dist}(z,\partial\Omega_\ell)<\delta} \varphi(z)\right) = \infty \tag{4.51}$$

holds, then the operator T_φ is unbounded. Moreover, if T_φ is bounded and the symbol is an unbounded essentially radial function, then it must be true that around any $\partial\Omega_\ell$, the symbol has an oscillating behavior.

In order to present an application, we consider a family of examples. Let us consider the case where $\Omega = \cap_{j=1}^{n}\Omega_j$ with $\Omega_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\Omega_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2, \dots, n$. Let $\varphi \in L^2(\Omega)$ be a function that can be written in the following way:

$$\varphi = \prod_{\ell=1}^{n} \varphi(\ell), \tag{4.52}$$

where, for any $\ell = 1, 2, \dots, n$, $\varphi(\ell)$ is radial, that is, $\varphi(\ell) = \varphi(\ell)(|z - a_\ell|)$ and satisfies

$$\inf_{|z-a_\ell|>r_\ell+\epsilon_\ell} \varphi(\ell)(|z-a_\ell|) = m_\ell > 0, \qquad \sup_{|z-a_\ell|>r_\ell+\epsilon_\ell} \varphi(\ell)(|z-a_\ell|) = M_\ell < \infty \tag{4.53}$$

if $\ell = 2, \dots, n$ and

$$\inf_{|z|<1-\epsilon_1} \varphi(1)(|z|) = m_1 > 0, \qquad \sup_{|z-a_\ell|<1-\epsilon_1} \varphi(1)(|z|) = M_1 < \infty. \tag{4.54}$$

if $\ell = 1$. As a consequence of our results, we can conclude that

(1) T_φ is bounded if there exists a constant C_1 such that for any $j = 2, \dots, n$,

$$\begin{gathered} \limsup_{s\to r_\ell} \left| \frac{r_\ell}{s - r_\ell} \int_{r_\ell}^{s} \varphi(\ell)\left(r_\ell^{1/2}x^{1/2} + a_\ell\right)\frac{1}{x^2}dx \right| < C_1, \\ \limsup_{s\to 1} \left| \frac{1}{1-s} \int_{s}^{1} \varphi(\ell)\left(x^{1/2}\right)dx \right| < C_1, \end{gathered} \tag{4.55}$$

for any $j = 1$,

(2) T_φ is compact if for any $j = 2, \dots, n$

$$\begin{gathered} \lim_{s\to r_\ell} \frac{r_\ell}{s - r_\ell} \int_{r_\ell}^{s} \varphi(\ell)\left(r_\ell^{1/2}x^{1/2} + a_\ell\right)\frac{1}{x^2}dx = 0, \\ \lim_{s\to 1} \frac{1}{1-s} \int_{s}^{1} \varphi(\ell)\left(x^{1/2}\right)dx = 0, \end{gathered} \tag{4.56}$$

for $j = 1$.

It is also possible to show that the sufficient conditions may fail, but the operator is still bounded or even compact. In fact, we can show that given any planar bounded multiply-connected domain Ω, whose boundary $\partial\Omega$ consists of finitely many simple closed smooth analytic curves, there exist unbounded functions $\varphi \in L^2(\Omega)$ such that T_φ is compact even when the sufficient conditions are not satisfied. To prove this claim, we observe that for the domain Ω there exists a conformally equivalent domain $D = \cap_{i=1}^{n} D_i$ where $D_1 = \{z \in \mathbb{C} :$

$|z| < 1\}$ and $D_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2, \ldots, n$ where $a_j \in D_1$ and $0 < r_j < 1$ with $|a_j - a_k| > r_j + r_k$ if $j \neq k$ and $1 - |a_j| > r_j$. If we denote with the symbol

$$\Psi : \Omega \longrightarrow D \tag{4.57}$$

the conformal equivalence between Ω and D, then we can define, on Ω, the map

$$\varphi_{\mathbf{u},\mathbf{v}} = \prod_{\ell=1}^{n} \varphi_{u_\ell, v_\ell}(\ell), \tag{4.58}$$

where, for any ℓ, we have

$$\begin{aligned} \varphi_{u_\ell,v_\ell}(\ell)(z) = -(1-u_\ell)\left(1 - \operatorname{dist}(\Psi(z), \partial D_\ell)^2\right)^{-u_\ell} \sin\left(1 - \operatorname{dist}(\Psi(z), \partial D_\ell)^2\right)^{-v_\ell} \\ + v_\ell\left(1 - \operatorname{dist}(\Psi(z), \partial D_\ell)^2\right)^{-v_\ell-u_\ell} \cos\left(1 - \operatorname{dist}(\Psi(z), \partial D_\ell)^2\right)^{-v_\ell}, \end{aligned} \tag{4.59}$$

where $b_\ell, a_\ell \in (0, \infty)$. It is very easy to see that if we denote with

$$Q_\ell = \left\{(u_\ell, v_\ell) \in (0, \infty)^2 \mid v_\ell + u_\ell < 1, u_\ell < v_\ell\right\}, \tag{4.60}$$

then on the set of parameters $Q_1 \times Q_2 \cdots \times Q_n$, the operator $T_{\varphi_{\mathbf{a},\mathbf{b}}}$ is bounded and compact.

In the last part of this paper, we concentrate on the relationship between compact operators and the Berezin transform. We remind the reader that given a Toeplitz operator for any T_ϕ on L_a^2, we define $\widetilde{T_\phi}$, the Berezin transform of T_ϕ, by

$$\widetilde{\phi}(w) = \langle T_\phi k_w, k_w \rangle = \int_\Omega \phi(z) |k_w(z)|^2 d\nu(z), \tag{4.61}$$

where $k_w(\cdot) = K(\cdot, w) K(w, w)^{-1/2}$. It is quite simple to show that if an operator $A \in \mathcal{B}(L_a^2(\Gamma, d\nu))$ is compact, then $\widetilde{A}$, the Berezin transform of A, must vanish at the boundary. However, it is possible to show (see [12]) that there are bounded operators which are not compact but whose Berezin transforms vanish at the boundary. In a beautiful paper, Axler and Zheng have proved (see [4]) that if D is the disk, $S = \sum_j^m \prod_k^{m_j} T_{\varphi_{i,k}}$, where $\varphi_{i,k} \in L^\infty(D)$, then S is compact if and only if its Berezin transform vanishes at the boundary of the disk. Their fundamental result has been extended in several directions, in particular when Ω is a general smoothly bounded multiply-connected planar domain [5].

So far, we have characterized the boundedness and compactness of the operator T_φ with the help of the sequences $\boldsymbol{\gamma}_{\varphi_j} = \{\gamma_{\varphi_j}(m)\}$. However, we did not so far try to characterize the compactness in terms of the Berezin transform. In the next theorem, under a certain condition, we will show that the Berezin transform characterization of compactness still holds in this context.

Before we state and prove the next result, we would like to say a few words about the intuition behind it. In the case of the disk, it is possible to show that when the operator is radial then its Berezin transform has a very special form. In fact, if $\varphi : \mathbb{D} \to \mathbb{C}$ is radial then $\widetilde{T_\varphi}(z) = (1-|z|^2)^2 \sum (n+1)\langle T_\varphi e_n, e_n\rangle |z|^{2n}$. Therefore to show that the vanishing of the Berezin Transform implies compactness is equivalent, given that T_φ is diagonal, to show that $\lim_{|z|\to 1}(1-|z|^2)^2 \sum (n+1)\langle T_\varphi e_n, e_n\rangle |z|^{2n} = 0$ implies $\lim_{n\to\infty}\langle T_\varphi e_n, e_n\rangle = 0$. Korenblum and Zhu realized this in the their seminal paper [3] and, along this line, more was discovered by Zorboska (see [2]) and Grudsky and Vasilevski (see [1]). In the case of a multiply-connected domain, it is not possible to write things so neatly; however, we can exploit our estimates near the boundary to use similar arguments. This is the content of what we prove in the following.

Theorem 4.6. *Let $\varphi \in L^2(D)$ be an essentially radial function where $D = \cap_{j=1}^n D_j$ with $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $D_j = \{z \in \mathbb{C} : |z-a_j| > r_j\}$ for $j = 2,\dots,n$. If one defines $\varphi_j = \varphi \cdot p_j$ where $j = 1,\dots,n$ and $\mathfrak{P} = \{p_0, p_1, \dots, p_n\}$ is a ∂-partition for D. Let us assume that $\gamma_{\phi_j} = \{\gamma_{\phi_j}(m)\}_{m\in\mathbb{N}}$ is in $\ell_\infty(\mathbb{Z}_+)$ and that there is a constant C_2 such that for $j = 2,\dots,n$,*

$$\sup_{\tau\in[a_j+r_j,\infty)} \left| \varphi_j(\tau) - \frac{\tau - a_j}{\tau - r_j - a_j} \int_{a_j+r_j}^{\tau} \varphi_j(y) \left(\frac{r_j}{(y-a_j)^2} \right) dy \right| < C_2, \tag{4.62}$$

and for $j = 1$

$$\sup_{\tau\in[0,1]} \left| \varphi_1(\tau) - \frac{1}{1-\tau} \int_{\tau}^{1} \varphi_1(s)\, ds \right| < C_2, \tag{4.63}$$

then the operator $T_\varphi : L^2_a(D, d\nu) \to L^2_a(D, d\nu)$ is compact if and only if

$$\lim_{w\to\partial D} \widetilde{T_\varphi}(w) = 0. \tag{4.64}$$

Proof. We know that the operator $T_\varphi : L^2_a(D, d\nu) \to L^2_a(D, d\nu)$ is bounded if and only if for any $j = 1,\dots,n$ the operators $T_{\varphi_j} : L^2_a(D_j, d\nu) \to L^2_a(D_j, d\nu)$ are bounded. Since we assume that $\gamma_{\phi_j} = \{\gamma_{\phi_j}(m)\}_{m\in\mathbb{N}}$ is in $\ell_\infty(\mathbb{Z}_+)$, then we can conclude that T_φ is bounded. As we have done before if we fix $j = 2,\dots,n$, by using $\Delta_{0,1} \xrightarrow{\alpha} \Delta_{a_j,r_j} \xrightarrow{\beta} D_j$ with $\Delta_{0,1} = \{z \in \mathbb{C} : 0 < |z-a| < 1\}$ and $\Delta_{a_j,r_j} = \{z \in \mathbb{C} : 0 < |z-a_j| < r_j\}$, and the maps $\Delta_{0,1} \xrightarrow{\alpha} \Delta_{a_j,r_j} \xrightarrow{\beta} D_j$ where $\alpha(z) = a_j + r_j z$ and $\beta(w) = (w-a_j)^{-1} r_j^2 + a_j$, we can claim that $T_{\varphi_j} = V_{\beta\circ\alpha}^{-1} T_{\varphi_j\circ\beta\circ\alpha} V_{\beta\circ\alpha}$ where $V_{\beta\circ\alpha} : L^2(\Delta_{0,1}) \to L^2(D_j)$ is an isomorphism of Hilbert spaces. Therefore, T_{φ_j} is compact if and only if $T_{\varphi_j\circ\beta\circ\alpha}$ is compact. We also know that this, in turn, is equivalent to the vanishing of the Berezin transform if the function

$$(\varphi\circ\beta\circ\alpha)_{\mathrm{Ber},j}(s) = \varphi_j\circ\beta\circ\alpha(s) - \frac{1}{1-s}\int_s^1 \varphi_j\circ\beta\circ\alpha(t)\, dt \tag{4.65}$$

is bounded. Since φ_j is radial and $\beta \circ \alpha(t) = t^{-1} r_j + a_j$, then, after a change of variable under the sign of integral, we can rewrite the last integral, and therefore we obtain the formula

$$\begin{aligned}(\varphi \circ \beta \circ \alpha)_{\mathrm{Ber},j}(s) &= \varphi_j \circ \beta \circ \alpha(s) - \frac{1}{1-s} \int_{(r_j/s)+a_j}^{a_j+r_j} \varphi_j(y) \left(\frac{-r_j}{(y-a_j)^2} \right) dy \\ &= \varphi_j \circ \beta \circ \alpha(s) - \frac{1}{1-s} \int_{a_j+r_j}^{(r_j/s)+a_j} \varphi_j(y) \left(\frac{r_j}{(y-a_j)^2} \right) dy \\ &= \varphi_j \circ \beta \circ \alpha(s) - \frac{1}{1-s} \int_{a_j+r_j}^{(r_j/s)+a_j} \varphi_j(y) \left(\frac{r_j}{(y-a_j)^2} \right) dy.\end{aligned} \tag{4.66}$$

Moreover, if we define $\tau = s^{-1} r_j + a_j$, we can write

$$\varphi_{\mathrm{Ber},j}(\tau) = \varphi_j(\tau) - \frac{\tau - a_j}{\tau - r_j - a_j} \int_{a_j+r_j}^{\tau} \varphi_j(y) \left(\frac{r_j}{(y-a_j)^2} \right) dy. \tag{4.67}$$

Therefore, if we assume that this function is bounded, we can conclude (see [2]) that from $\lim_{w \to \Delta_{0,1}} \widetilde{T_{\varphi_j \circ \beta \circ \alpha}}(w) = 0$, it follows that $\widetilde{T_{\varphi_j \circ \beta \circ \alpha}}$ is compact for $j = 2, \dots, n$. Therefore, we can infer that $\widetilde{T_{\varphi_j}}$ is compact. We also observe that

$$\lim_{w \to \Delta_{0,1}} \widetilde{T_{\varphi_j \circ \beta \circ \alpha}}(w) = 0 \tag{4.68}$$

if and only if $\lim_{w \to \partial D_j} \widetilde{T_{\varphi_j}}(w) = 0$. To prove this fact, we observe that, by definition, we have

$$\begin{aligned}\widetilde{T_{\varphi_j \circ \beta \circ \alpha}}(z) &= \left\langle T_{\varphi_j \circ \beta \circ \alpha} k_z^{\Delta_{0,1}}, k_z^{\Delta_{0,1}} \right\rangle \\ &= \int_{\Delta_{0,1}} T_{\varphi_j \circ \beta \circ \alpha} k_z^{\Delta_{0,1}}(w) \overline{k_z^{\Delta_{0,1}}(w)} dw,\end{aligned} \tag{4.69}$$

where

$$k_z^{\Delta_{0,1}}(\cdot) = K_z^{\Delta_{0,1}}(\cdot, z) K_z^{\Delta_{0,1}}(z, z)^{-1/2}. \tag{4.70}$$

Let us take $(\beta \circ \alpha)^{-1} : D_j \to \Delta_{0,1}$. Since $(J_{\mathbf{R}}(\beta \circ \alpha)^{-1})(w)$ is $|((\beta \circ \alpha)^{-1})'(w)|^2$ and there exists $\zeta \in D_j$ such that $(\beta \circ \alpha)(z) = \zeta$, we obtain

$$\widetilde{T_{\varphi_j \circ \beta \circ \alpha}}(z) = \int_{D_j} \left(A \left(V_{(\beta \circ \alpha)^{-1}} k^{\Delta_{0,1}}_{(\beta \circ \alpha)^{-1}(\varsigma)} \right) \right) \overline{k^{\Delta_{0,1}}_{(\beta \circ \alpha)^{-1}(\varsigma)} \left(\left(\beta \circ \alpha \right)^{-1} \right) \left(\left(\beta \circ \alpha \right)^{-1} \right)'} dw, \tag{4.71}$$

where $A = (V_{(\beta\circ\alpha)^{-1}} T_{\varphi_j\circ\beta\circ\alpha} V_{(\beta\circ\alpha)})$. Given the relationship between

$$k^{\Delta_{0,1}}_{(\beta\circ\alpha)^{-1}(\zeta)}\left((\beta\circ\alpha)^{-1}(w)\right) \tag{4.72}$$

and $k^{D_j}_{\zeta}(w)$, we obtain

$$\widetilde{T_{\varphi_j\circ\beta\circ\alpha}}(z) = \int_{D_j}\left(T_{\varphi_j}k^{D_j}_{\zeta}\right)(w)\overline{k^{D_j}_{\zeta}(w)}dw. \tag{4.73}$$

Therefore, it follows that $\widetilde{T_{\varphi_j\circ\beta\circ\alpha}}(z) = \widetilde{T_{\varphi_j}}((\beta\circ\alpha)(z))$. The case $j = 1$ is immediate.

Hence, we observe that from what we have proved so far, we can infer with the help of Lemmas 3.1 and 3.2 in Section 2 that, under the stated condition, if

$$\lim_{w\to\partial D}\widetilde{T_{\varphi}}(w) = 0 \tag{4.74}$$

then T_φ is a compact operator. To complete the proof, we observe that the compactness of $T_\varphi : L^2_a(D, dv) \to L^2_a(D, dv)$ implies the vanishing of the Berezin transform since k_w converges weakly and uniformly to zero as $w \to \partial D$. □

Finally, we also observe that as a simple consequence, we obtain the following.

Corollary 4.7. *Let $\varphi \in L^2(\Omega)$ be an essentially radial function via the conformal equivalence $\Theta : \Omega \to D$. If one defines $\varphi_j = \varphi\cdot p_j$ where $j = 1,\dots,n$ and $\mathfrak{P}$ is a ∂-partition for Ω. Let us assume that $\gamma_{\phi_j} = \{\gamma_{\phi_j}(m)\}_{m\in\mathbb{N}}$ is in $\ell_\infty(\mathbb{Z}_+)$ and that there is a constant C_3 such that for $j = 2,\dots,n$,*

$$\sup_{\tau\in[a_j+r_j,\infty)}\left|\varphi_j\circ\Theta(\tau) - \frac{\tau-a_j}{\tau-r_j-a_j}\int_{a_j+r_j}^{\tau}\varphi_j\circ\Theta(y)\left(\frac{r_j}{(y-a_j)^2}\right)dy\right| < C_3, \tag{4.75}$$

and for $j = 1$

$$\sup_{\tau\in[0,1]}\left|\varphi_1\circ\Theta(\tau) - \frac{1}{1-\tau}\int_{\tau}^{1}\varphi_1\circ\Theta(s)ds\right| < C_3, \tag{4.76}$$

then the operator $T_\varphi : L^2_a(\Omega, dv) \to L^2_a(\Omega, dv)$ is compact if and only if

$$\lim_{w\to\partial\Omega}\widetilde{T_{\varphi}}(w) = 0. \tag{4.77}$$

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